# *On lp Norms of Weighted Mean Matrices*

# ON $l^p$ NORMS OF WEIGHTED MEAN MATRICES

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ABSTRACT. We extend a result of Cartlidge on the $l^p$ operator norms of weighted mean matrices using an approach of Knopp.

## 1. Introduction

Suppose throughout that $p \neq 0, \frac{1}{p}+\frac{1}{q}=1$. Let $l^p$ be the Banach space of all complex sequences $\mathbf{a}=(a_n)_{n\geq 1}$ with norm

$$||\mathbf{a}|| := (\sum_{n=1}^{\infty}|a_n|^p)^{1/p} < \infty.$$

The celebrated Hardy's inequality ([17, Theorem 326]) asserts that for $p>1$,

$$\sum_{n=1}^{\infty}\Big|\frac{1}{n}\sum_{k=1}^{n}a_k\Big|^p \leq \Big(\frac{p}{p-1}\Big)^p\sum_{n=1}^{\infty}|a_n|^p. \tag{1.1}$$

Hardy's inequality can be regarded as a special case of the following inequality:

$$\sum_{n=1}^{\infty}\Big|\sum_{k=1}^{\infty}c_{n,k}a_k\Big|^p \leq U\sum_{n=1}^{\infty}|a_n|^p,$$

in which $C=(c_{n,k})$ and the parameter $p$ are assumed fixed ($p>1$), and the estimate is to hold for all complex sequences $\mathbf{a}$. The $l^p$ operator norm of $C$ is then defined as the $p$-th root of the smallest value of the constant $U$:

$$||C||_{p,p}=U^{\frac{1}{p}}.$$

Hardy's inequality thus asserts that the Cesáro matrix operator $C$, given by $c_{n,k}=1/n, k\leq n$ and 0 otherwise, is bounded on $l^p$ and has norm $\leq p/(p-1)$. (The norm is in fact $p/(p-1)$.)

We say a matrix $A=(a_{n,k})$ is a lower triangular matrix if $a_{n,k}=0$ for $n<k$ and a lower triangular matrix $A$ is a summability matrix if $a_{n,k}\geq 0$ and $\sum_{k=1}^{n}a_{n,k}=1$. We say a summability matrix $A$ is a weighted mean matrix if its entries satisfy:

$$a_{n,k}=\lambda_k/\Lambda_n,\ 1\leq k\leq n;\quad \Lambda_n=\sum_{i=1}^{n}\lambda_i, \lambda_i\geq 0, \lambda_1>0. \tag{1.2}$$

Hardy's inequality (1.1) now motivates one to determine the $l^p$ operator norm of an arbitrary summability matrix $A$. In an unpublished dissertation [9], Cartlidge studied weighted mean matrices as operators on $l^p$ and obtained the following result (see also [1, p. 416, Theorem C]).

**Theorem 1.1.** *Let $1<p<\infty$ be fixed. Let $A$ be a weighted mean matrix given by* (1.2). *If*

$$L=\sup_n\Big(\frac{\Lambda_{n+1}}{\lambda_{n+1}}-\frac{\Lambda_n}{\lambda_n}\Big)<p\ , \tag{1.3}$$

*then $||A||_{p,p}\leq p/(p-L)$.*

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There are several published proofs of Cartlidge's result. Borwein [5] proved a far more general result than Theorem 1.1 on the $l^p$ norms of generalized Hausdorff matrices. Rhoades [21, Theorem 1] obtained a slightly general result than Theorem 1.1, using a modification of the proof of Cartlidge. Recently, the author [13] also gave a simple proof of Theorem 1.1.

It is our goal in this paper to extend the result of Theorem 1.1. This is motivated by Carleman's proof [8] of the well-known Carleman's inequality, which asserts that for convergent infinite series $\sum a_n$ with non-negative terms, one has

$$\sum_{n=1}^{\infty}(\prod_{k=1}^{n} a_k)^{\frac{1}{n}} \leq e\sum_{n=1}^{\infty} a_n,$$

with the constant $e$ best possible.

In [16], the author studied the following weighted version of Carleman's inequality via Carleman's approach:

$$\sum_{n=1}^{\infty}\Big(\prod_{k=1}^{n} a_k^{\lambda_k/\Lambda_n}\Big) \leq E\sum_{n=1}^{\infty} a_n, \tag{1.4}$$

where the notations are as in (1.2). The task there is to determine the best constant $E$ so that inequality (1.4) holds for any convergent infinite series $\sum a_n$ with non-negative terms. The main result in [16] is the following:

**Theorem 1.2.** [16, Theorem 1.2] *Suppose that*

$$M = \sup_n \frac{\Lambda_n}{\lambda_n}\log\Big(\frac{\Lambda_{n+1}/\lambda_{n+1}}{\Lambda_n/\lambda_n}\Big) < +\infty,$$

*then inequality* (1.4) *holds with* $E = e^M$.

We note here one can obtain Carleman's inequality from inequality (1.1) by a change of variables $a_k \to a_k^{1/p}$ in (1.1) and on letting $p \to +\infty$. Similarly, Cartlidge's result (Theorem 1.1) implies that when (1.3) is satisfied, then for any $\mathbf{a} \in l^p$, one has

$$\sum_{n=1}^{\infty}\Big|\sum_{k=1}^{n}\frac{\lambda_k a_k}{\Lambda_n}\Big|^p \leq \Big(\frac{p}{p-L}\Big)^p\sum_{n=1}^{\infty}|a_n|^p. \tag{1.5}$$

By a change of variables $a_k \to a_k^{1/p}$ in (1.5) and on letting $p \to +\infty$, one obtains inequality (1.4) with $E = e^L$ as long as (1.3) is satisfied with $p$ replaced by $+\infty$ there. It was shown in [16] that $M \leq L$ and hence it is natural for us to expect to obtain a stronger result than that of Cartlidge's concerning $l^p$ operator norms of weighted mean matrices using Carleman's approach. We point out here Carleman's approach is essentially a use of Lagrange multipliers, which we shall explain in details in the next section. This approach is more technically involved so we are looking for other methods that be used to achieve our goal in this paper while technically simpler compared with Carleman's approach.

There is a rich literature on many different proofs of Hardy's inequality (1.1) as well as its generalizations and extensions. Notably, there are Knopp's approach [18] and Redheffer's "recurrent inequalities" [20]. It is shown in [14] that these two methods above are essentially the same and we shall further show that Knopp's approach can be regarded as an approximation to Carleman's approach in this paper in Section 2. Hence instead of Carleman's approach, there is not much lost using Knopp's approach when studying Hardy-type inequalities, yet technically it is much easier to handle.

In this paper, we shall use Knopp's approach to prove the following extension of Theorem 1.1 (we note here the case $n = 1$ of (1.3) implies $L > 0$) in Section 3:

**Theorem 1.3.** *Let $1 < p < \infty$ be fixed. Let $A$ be a weighted mean matrix given by* (1.2)*. If for any integer $n \geq 1$, there exists a positive constant $0 < L < p$ such that*

$$\frac{\Lambda_{n+1}}{\lambda_{n+1}} \leq \frac{\Lambda_n}{\lambda_n}\Big(1-\frac{L\lambda_n}{p\Lambda_n}\Big)^{1-p}+\frac{L}{p}\ , \tag{1.6}$$

*then $||A||_{p,p} \leq p/(p-L)$.*

We point out there Theorem 1.1 can be regarded as the case $p \rightarrow 1^+$ of the above theorem while Theorem 1.2 can be regarded as the case $p \rightarrow +\infty$ of the above theorem

Note that for $p > 1$,

$$\Big(1-\frac{L\lambda_n}{p\Lambda_n}\Big)^{1-p} \geq 1+(1-\frac{1}{p})\frac{L\lambda_n}{\Lambda_n}+(1-\frac{1}{p})\frac{\lambda_n^2}{\Lambda_n^2}\frac{L^2}{2}.$$

It follows from this and (1.6) that we have the following

**Corollary 1.1.** *Let $1 < p < \infty$ be fixed. Let $A$ be a weighted mean matrix given by* (1.2)*. If for any integer $n \geq 1$, there exists a positive constant $0 < L < p$ such that*

$$\frac{\Lambda_{n+1}}{\lambda_{n+1}}-\frac{\Lambda_n}{\lambda_n} \leq L+\Big(\frac{\lambda_n}{2\Lambda_n}\Big)\Big(1-\frac{1}{p}\Big)L^2\ , \tag{1.7}$$

*then $||A||_{p,p} \leq p/(p-L)$.*

An application of Theorem 1.3 will be given in Section 4.

## 2. Carleman's approach versus Knopp's approach

Our goal in general is to find conditions on $\lambda_k$'s so that the following inequality holds for some constant $U$ and for any $\mathbf{a} \in l^p$:

$$\sum_{n=1}^{\infty}\Big|\sum_{k=1}^{n}\frac{\lambda_k a_k}{\Lambda_n}\Big|^p \leq U\sum_{n=1}^{\infty}|a_n|^p.$$

It suffices to consider the cases with the infinite summations above replaced by any finite summations, say from 1 to $N \geq 1$ here. We may also assume $a_k \geq 0$ from now on and we shall define

$$A_n = \sum_{k=1}^{n}\frac{\lambda_k a_k}{\Lambda_n}.$$

Carleman's approach is to determine the maximum value $\mu_N$ of $\sum_{n=1}^N A_n^p$ subject to the constraint $\sum_{n=1}^N a_n^p = 1$ using Lagrange multipliers. We first show that we may further assume that $a_n > 0$ for all $1 \leq n \leq N$ when the maximum is reached. For otherwise, we may assume without loss of generality that $a_i = 0, a_{i+1} > 0$ for some $1 \leq i \leq N-1$ when the maximum is reached. We can now assume $a_n$'s are fixed for $n \neq i, i+1$. Note that by our assumption that $\sum_{n=1}^N a_n^p = 1$, this implies that the value of $a_i^p + a_{i+1}^p$ is constant as well and hence defines $a_{i+1}$ explicitly as a function of $a_i$. We now regard $\sum_{n=1}^N A_n^p$ as a function of $a_i$ and it is then easy to check that it is an increasing function of $a_i$ near $a_i = 0$, by which it means that on increasing the value of $a_i$ from 0 to a small positive number while decreasing the value of $a_{i+1}$ and keeping other variables fixed, we will increase the value of $\sum_{n=1}^N A_n^p$, a contradiction.

We now define

$$F(\mathbf{a};\mu) = \sum_{n=1}^{N} A_n^p - \mu(\sum_{n=1}^{N} a_n^p - 1), \tag{2.1}$$

where $\mathbf{a} = (a_n)_{1 \leq n \leq N}$. By the Lagrange method and our discussions above, we have to solve $\nabla F = 0$, or the following system of equations:

$$\mu a_k^p = \sum_{n=k}^{N} \frac{\lambda_k A_n^{p-1}}{\Lambda_n} a_k, \quad 1 \leq k \leq N; \quad \sum_{n=1}^{N} a_n^p = 1. \tag{2.2}$$

We note that on summing over $1 \leq k \leq N$ of the first $N$ equations above, we get

$$\sum_{n=1}^{N} A_n^p = \mu.$$

Hence we have $\mu = \mu_N$ in this case, which allows us to recast the equations (2.2) as:

$$\mu_N \frac{a_k^{p-1}}{\lambda_k} = \sum_{n=k}^{N} \frac{A_n^{p-1}}{\Lambda_n}, \quad 1 \leq k \leq N; \quad \sum_{n=1}^{N} a_n^p = 1.$$

On subtracting consecutive equations, we can rewrite the above system of equations as:

$$\mu_N \left(\frac{a_k^{p-1}}{\lambda_k} - \frac{a_{k+1}^{p-1}}{\lambda_{k+1}}\right) = \frac{A_k^{p-1}}{\Lambda_k}, \quad 1 \leq k \leq N-1; \quad \mu_N \frac{a_N^{p-1}}{\lambda_N} = \frac{A_N^{p-1}}{\Lambda_N}; \quad \sum_{n=1}^{N} a_n^p = 1. \tag{2.3}$$

Now we define for $1 \leq k \leq N-1$,

$$\omega_k = \frac{\Lambda_k}{\lambda_k} - \frac{\Lambda_k a_{k+1}^{p-1}}{\lambda_{k+1} a_k^{p-1}},$$

so that we can further rewrite our system of equations as:

$$\mu_N a_k^{p-1} \omega_k = A_k^{p-1}, \quad 1 \leq k \leq N-1; \quad \mu_N \frac{a_N^{p-1}}{\lambda_N} = \frac{A_N^{p-1}}{\Lambda_N}; \quad \sum_{n=1}^{N} a_n^p = 1.$$

It is easy to check that for $1 \leq k \leq N-2$,

$$\omega_{k+1}^{\frac{1}{p-1}} = \frac{\Lambda_k}{\Lambda_{k+1}} \Big(\frac{\omega_k}{\frac{\lambda_{k+1}}{\Lambda_k}(\Lambda_k/\lambda_k - \omega_k)}\Big)^{\frac{1}{p-1}} + \frac{\lambda_{k+1}}{\Lambda_{k+1}} \Big(\frac{1}{\mu_N}\Big)^{\frac{1}{p-1}}.$$

We now define a sequence of real functions $\Omega_k(\mu)$ inductively by setting $\Omega_1(\mu) = 1/\mu$ and

$$\Omega_{k+1}^{\frac{1}{p-1}}(\mu) = \frac{\Lambda_k}{\Lambda_{k+1}} \Big(\frac{\Omega_k(\mu)}{\frac{\lambda_{k+1}}{\Lambda_k}(\Lambda_k/\lambda_k - \Omega_k(\mu))}\Big)^{\frac{1}{p-1}} + \frac{\lambda_{k+1}}{\Lambda_{k+1}} \Big(\frac{1}{\mu}\Big)^{\frac{1}{p-1}}.$$

We note that $\Omega_k(\mu_N) = \omega_k$ for $1 \leq k \leq N-1$ and

$$\begin{aligned}
\Omega_N^{\frac{1}{p-1}}(\mu_N) &= \frac{\Lambda_{N-1}}{\Lambda_N} \Big(\frac{\omega_{N-1}}{\frac{\lambda_N}{\Lambda_{N-1}}(\Lambda_{N-1}/\lambda_{N-1} - \omega_{N-1})}\Big)^{\frac{1}{p-1}} + \frac{\lambda_N}{\Lambda_N} \Big(\frac{1}{\mu_N}\Big)^{\frac{1}{p-1}} \\
&= \frac{\Lambda_{N-1}}{\Lambda_N} \Big(\frac{A_{N-1}^{p-1}}{\mu_N a_N^{p-1}}\Big)^{\frac{1}{p-1}} + \frac{\lambda_N}{\Lambda_N} \Big(\frac{1}{\mu_N}\Big)^{\frac{1}{p-1}} = \Big(\frac{1}{\mu_N}\Big)^{\frac{1}{p-1}} \frac{A_N}{a_N} \\
&= \Big(\frac{\Lambda_N}{\lambda_N}\Big)^{\frac{1}{p-1}}.
\end{aligned}$$

We now define another sequence of real functions $\eta_k(\mu)$ by setting

$$\eta_k(\mu) = \Big(\frac{\Lambda_k}{\lambda_k}\Big)^{p-1} \Omega_k(\mu)$$

so that it satisfies the following relation:

$$\eta_{k+1}^{\frac{1}{p-1}}(\mu) = \frac{\Lambda_k}{\lambda_{k+1}}\Big(\frac{\Lambda_k\eta_k(\mu)/\lambda_{k+1}}{(\Lambda_k/\lambda_k)^p-\eta_k(\mu)}\Big)^{\frac{1}{p-1}} + \Big(\frac{1}{\mu}\Big)^{\frac{1}{p-1}}. \tag{2.4}$$

Note that we have seen above that $\eta_N(\mu_N) = (\Lambda_N/\lambda_N)^p$ and Carleman's idea is to show that the above relation (2.4) leads to a contradiction if $\mu$ is large and this forces $\mu_N$ to be small. For example, one can show by induction that if (1.7) is satisfied and $\mu > (1-L/p)^{-p}$, then for $k \geq 1$,

$$\eta_k^{\frac{1}{p-1}}(\mu) < (b+c)\Big(\frac{\Lambda_k}{\lambda_k}\Big) - c, \quad b = (1-L/p)^{p/(p-1)}, \quad c = \frac{L}{p}(1-L/p)^{1/(p-1)}.$$

It follows that if the above assertion is established, then for $1 \leq n \leq N$,

$$0 < \eta_n^{\frac{1}{p-1}}(\mu) < (b+c)\Big(\frac{\Lambda_n}{\lambda_n}\Big) - c < (b+c)\frac{\Lambda_n}{\lambda_n} < \frac{\Lambda_n}{\lambda_n} \leq \Big(\frac{\Lambda_n}{\lambda_n}\Big)^{\frac{p}{p-1}} = \eta_N^{\frac{1}{p-1}}(\mu).$$

this forces $\mu_N \leq (1-L/p)^{-p}$ and the assertion for Corollary 1.1 will follow. We further note here that one can compared the above example to the case considered in [15], where the $l^2$ norms of weighted mean matrices were treated using linear algebra techniques. It is easy to see that the method used there can be regarded essentially as the special case $p=2$ in the proof of Corollary 1.1. We shall leave the details for the reader to verify.

We now give a short account on Knopp's approach [18] on proving Hardy's inequality (1.1). In fact, we explain this more generally for the case involving weighted mean matrices. Using the notations in Section 1 and once again restricting our attention to any finite summation, say from 1 to $N \geq 1$ here, we are looking for a positive constant $U$ such that

$$\sum_{n=1}^N \Big|\frac{1}{\Lambda_n}\sum_{k=1}^n \lambda_k a_k\Big|^p \leq U\sum_{n=1}^N |a_n|^p \tag{2.5}$$

holds for all complex sequences $\mathbf{a}$ with $p > 1$ being fixed. To motivate the approach, we may assume $a_n \geq 0$ and we are using Carleman's approach to find the maximum value $\mu_N$ of $\sum_{n=1}^N A_n^p$ subject to the constraint $\sum_{n=1}^N a_n^p = 1$. Suppose this is done and we find that the maximum value is reached at a sequence $\mathbf{w} = \{w_n\}_{n=1}^N$. Hence (2.2) is satisfied with $a_n$'s there replaced by $w_n$'s and $\mu = \mu_N$. This motivates us to consider, for an arbitrary sequence $\mathbf{a} = \{a_n\}_{n=1}^N$, the following expression

$$\mu_N\sum_{n=1}^N |a_n|^p = \sum_{k=1}^N w_k^{-(p-1)}\Big(\sum_{n=k}^N \frac{\lambda_k}{\Lambda_n}\Big(\sum_{j=1}^n \frac{\lambda_j}{\Lambda_n}w_j\Big)^{p-1}\Big)|a_k|^p.$$

Thus inequality (2.5) will follows from this with $U = \mu_N$ if one can show the right-hand side expression above is no less than the left-hand side expression of (2.5).

Knopp's idea is to reverse the process discussed above by finding an auxiliary sequence $\mathbf{w} = \{w_n\}_{n=1}^N$ of positive terms such that by Hölder's inequality,

$$\begin{aligned}\Big(\sum_{k=1}^n \lambda_k|a_k|\Big)^p &= \Big(\sum_{k=1}^n \lambda_k|a_k|w_k^{-\frac{1}{q}}\cdot w_k^{\frac{1}{q}}\Big)^p \\ &\leq \Big(\sum_{k=1}^n \lambda_k^p|a_k|^p w_k^{-(p-1)}\Big)\Big(\sum_{j=1}^n w_j\Big)^{p-1}\end{aligned}$$

so that

$$\sum_{n=1}^{N}\Big|\frac{1}{\Lambda_n}\sum_{k=1}^{n}\lambda_k a_k\Big|^p \leq \sum_{n=1}^{N}\frac{1}{\Lambda_n^p}\Big(\sum_{k=1}^{n}\lambda_k^p|a_k|^p w_k^{-(p-1)}\Big)\Big(\sum_{j=1}^{n}w_j\Big)^{p-1}$$
$$= \sum_{k=1}^{N}w_k^{-(p-1)}\lambda_k^p\Big(\sum_{n=k}^{N}\frac{1}{\Lambda_n^p}\Big(\sum_{j=1}^{n}w_j\Big)^{p-1}\Big)|a_k|^p.$$

Suppose now one can find for each $p>1$ a positive constant $U$, a sequence $\mathbf{w}$ of positive terms, such that for any integer $1 \leq n \leq N$,

$$\Big(\sum_{i=1}^{n}w_i\Big)^{p-1} \leq U\Lambda_n^p\Big(\frac{w_n^{p-1}}{\lambda_n^p}-\frac{w_{n+1}^{p-1}}{\lambda_{n+1}^p}\Big), \tag{2.6}$$

where we define $w_{N+1}=0$. Then it is easy to see that inequality (2.5) follows from this. When $\lambda_n=1$ for all $n$, Knopp's choice for $\mathbf{w}$ is given inductively by setting $w_1=1$ and

$$\sum_{i=1}^{n}w_i=\frac{n-1/p}{1-1/p}w_n.$$

and one can show that (2.6) holds in this case with $U=q^p$ and Hardy's inequality (1.1) follows from this.

We note here by a change of variables $w_k \rightarrow \lambda_k a_k$, we can recast (2.6) as

$$U\Big(\frac{a_k^{p-1}}{\lambda_k}-\frac{a_{k+1}^{p-1}}{\lambda_{k+1}}\Big) \geq \frac{A_k^{p-1}}{\Lambda_k}, \quad 1 \leq k \leq N-1; \quad U\frac{a_N^{p-1}}{\lambda_N} \geq \frac{A_N^{p-1}}{\Lambda_N}, \tag{2.7}$$

where $A_n$'s are defined as above. Compare the above with (2.3), we see that the above inequalities are certainly implies by (2.3). On the other hand, if the above inequalities hold with positive $a_n$'s, then we may assume the $a_n$'s are properly normalized so that the last equation in (2.3) is satisfied. It is then easy to see that this leads to

$$U\frac{a_k^{p-1}}{\lambda_k} \geq \sum_{n=k}^{N}\frac{A_n^{p-1}}{\Lambda_n}, \quad 1 \leq k \leq N; \quad \sum_{n=1}^{N}a_n^p=1.$$

Thus we have $\nabla F(\mathbf{a};U) \leq 0$, for $F$ defined by (2.1), which leads to

$$\sum_{n=1}^{N}A_n^p \leq U.$$

Moreover, $\nabla F(\mathbf{a};U) \leq 0$ suggests that, heuristically one may expect to obtain the value $\mu_N$ at a point $\mathbf{a}'$ which is pointwise smaller than $\mathbf{a}$ and this implies that $\mu_N \leq U$. Hence Knopp's approach can be regarded as an approximation to Carleman's approach on using Lagrange multipliers.

## 3. Proof of Theorem 1.3

We now apply Knopp's method to give a proof of Theorem 1.3. It suffices to find a sequence $\mathbf{a}=\{a_n\}_{n=1}^N$ of positive terms so that inequalities (2.7) are satisfied with $U=(p/(p-L))^p$. We now define our sequence inductively by setting $a_1=1$ and for $n \geq 1$,

$$A_n=\frac{1}{\Lambda_n}\sum_{i=1}^{n}\lambda_i a_i=(1+\beta-\frac{\beta\lambda_n}{\Lambda_n})a_n,$$

where $\beta = L/(p-L)$. Equivalently, this is amount to taking $\sum_{i=1}^n w_i = ((1+\beta)\Lambda_n/\lambda_n - \beta)w_n$ for those $w_i$'s satisfying (2.6). One checks easily that the above relation leads to the following relation between $a_n$ and $a_{n+1}$:

$$a_{n+1} = \frac{1}{1+\beta}(1+\beta - \frac{\beta\lambda_n}{\Lambda_n})a_n.$$

It is then easy to see that inequalities (2.7) follow from the following inequality for $n \geq 1$:

$$U\Big(\frac{\Lambda_n}{\lambda_n}(1+\beta-\frac{\beta\lambda_n}{\Lambda_n})^{1-p} - (1+\beta)^{1-p}(\frac{\Lambda_{n+1}}{\lambda_{n+1}}-1)\Big) \geq 1.$$

We now set $x = \Lambda_n/\lambda_n$, $y = \Lambda_{n+1}/\lambda_{n+1}$ to rewrite the above inequality as:

$$U(1+\beta)^{1-p}\Big(x\Big(1-\frac{\beta}{(1+\beta)x}\Big)^{1-p} - (y-1)\Big) \geq 1.$$

The above inequality now follows from (1.6) and this completes the proof of Theorem 1.3.

## 4. An Application of Theorem 1.3

The following inequality were claimed to hold by Bennett ( [2, p. 40-41]; see also [3, p. 407]):

$$\sum_{n=1}^{\infty}\Big|\frac{1}{\sum_{i=1}^n i^{\alpha-1}}\sum_{i=1}^n i^{\alpha-1}a_i\Big|^p \leq \Big(\frac{\alpha p}{\alpha p-1}\Big)^p\sum_{n=1}^{\infty}|a_n|^p, \tag{4.1}$$

whenever $\alpha > 0, p > 1, \alpha p > 1$. This was proved for the cases $p > 1, \alpha \geq 2$ or $0 < \alpha \leq 1, \alpha p > 1$ by the author [12] and Bennett himself [4] independently. Recently, the author [14] has shown that the above inequality holds for $p \geq 2, 1 \leq \alpha \leq 1+1/p$ or $1 < p \leq 4/3, 1+1/p \leq \alpha \leq 2$.

As an application of Theorem 1.3 or rather, Corollary 1.1, we now prove the following

**Theorem 4.1.** *For fixed $p \geq 3$, let $3/2 \leq \alpha < 2$ be a number that satisfies $(\alpha-1)2^{\alpha-1} \geq 1-1/p$, then inequality (4.1) holds for such an $\alpha$.*

*Proof.* For simplicity, we make a change of variable $\alpha - 1 \mapsto \alpha$ so that by Corollary 1.1, we look for conditions on $0 < \alpha < 1$ such that the following inequality holds for any integer $n \geq 1$:

$$\frac{\sum_{k=1}^{n+1}k^\alpha}{(n+1)^\alpha} - \frac{\sum_{k=1}^n k^\alpha}{n^\alpha} \leq \frac{1}{\alpha+1} + \frac{n^\alpha}{2\sum_{k=1}^n k^\alpha}\Big(1-\frac{1}{p}\Big)\frac{1}{(\alpha+1)^2}.$$

It is easy to see on letting $x = 1/\sum_{k=1}^n k^\alpha$ that the above inequality is equivalent to $f_n(x) \geq 0$ for $0 \leq x \leq 1$, where

$$f_n(x) = 1 + n^\alpha x\Big(\frac{1}{\alpha+1} + \frac{n^\alpha}{2}\Big(1-\frac{1}{p}\Big)\frac{x}{(\alpha+1)^2}\Big) - \frac{n^\alpha}{(n+1)^\alpha}\Big(1+(n+1)^\alpha x\Big).$$

Now we need two lemmas:

**Lemma 4.1.** *For $p \geq 2$ and $1/2 \leq \alpha < 1$, we have*

$$\frac{1}{2^\alpha} \leq \frac{1}{\alpha+1} + \frac{1-1/p}{2(\alpha+1)^2}.$$

*Proof.* As $p \geq 2$, we have $1-1/p \geq 1/2$. Hence our assertion is a consequence of the following inequality:

$$\frac{1}{2^\alpha} \leq \frac{1}{\alpha+1} + \frac{1}{4(\alpha+1)^2}.$$

It is easy to see that the above inequality is equivalent to $h(\alpha) \geq 0$ for $1/2 \leq \alpha < 1$, where

$$h(\alpha) = 2^\alpha(5+4\alpha) - 4(1+\alpha)^2.$$

Note that for $1/2 \leq \alpha < 1$,

$$h''(\alpha) = (\ln 2)^2 \cdot 2^\alpha(5+4\alpha) + 8(\ln 2)2^\alpha - 8 \geq h''(1/2) > 0.$$

This combined with the observation that $h(1/2) > 0, h'(1/2) > 0$ now implies that $h(\alpha) \geq 0$ for $1/2 \leq \alpha < 1$ and this completes the proof. $\square$

The above lemma implies that $f_1(x) \geq 0$ so now we may assume $n \geq 2$ and we now need the following

**Lemma 4.2.** [19, Lemma 1, p.18] *For an integer $n \geq 1$,*

$$\sum_{i=1}^{n} i^{\alpha} \geq \frac{1}{\alpha+1} n(n+1)^{\alpha}, \quad 0 \leq \alpha \leq 1.$$

The above lemma implies that $x \leq (\alpha+1)/(n(n+1)^{\alpha})$ for our consideration here. It follows from this that when $\alpha 2^{\alpha} \geq 1 - 1/p$, we have

$$\begin{aligned} f_n'(x) &= n^{\alpha}\Big(\frac{1}{\alpha+1} + n^{\alpha}\Big(1-\frac{1}{p}\Big)\frac{x}{(\alpha+1)^2} - 1\Big) \leq n^{\alpha}\Big(\frac{1}{\alpha+1} + \frac{n^{\alpha-1}}{(n+1)^{\alpha}}\frac{(1-\frac{1}{p})}{(\alpha+1)} - 1\Big) \\ &\leq n^{\alpha}\Big(\frac{1}{\alpha+1} + \frac{1}{2^{\alpha}}\frac{(1-\frac{1}{p})}{(\alpha+1)} - 1\Big) \leq 0. \end{aligned}$$

We then deduce that for $x \leq (\alpha+1)/(n(n+1)^{\alpha})$,

$$f_n(x) \geq f_n\Big(\frac{\alpha+1}{n(n+1)^{\alpha}}\Big) = \frac{n^{\alpha}}{(n+1)^{\alpha}} g(\frac{1}{n}),$$

where

$$g(y) = (1+y)^{\alpha} + y\Big(1 + \frac{1-1/p}{2} y(1+y)^{-\alpha}\Big) - 1 - (\alpha+1)y.$$

Note that when $0 < \alpha < 1$,

$$(1+y)^{\alpha} \geq 1 + \alpha y + \alpha(\alpha-1)y^2/2; \quad (1+y)^{-\alpha} \geq 1 - \alpha y.$$

We conclude from the above estimations that when $0 \leq y \leq 1/2$, $p \geq 3$, $1/2 \leq \alpha < 1$,

$$\begin{aligned} g(y) &\geq 1 + \alpha y + \alpha(\alpha-1)y^2/2 + y\Big(1 + \frac{1-1/p}{2} y(1-\alpha y)\Big) - 1 - (\alpha+1)y \\ &= \frac{y^2}{2}\Big(\alpha(\alpha-1) + (1-1/p)(1-\alpha y)\Big) \geq \frac{y^2}{2}\Big(\alpha(\alpha-1) + (1-1/p)(1-\alpha/2)\Big) \\ &\geq \frac{y^2}{2}\Big(\alpha(\alpha-1) + 2/3(1-\alpha/2)\Big) \geq 0. \end{aligned}$$

This now implies that $f_n(x) \geq 0$ so that the assertion of Theorem 4.1 follows. $\square$

We note here that when $\alpha \geq 1.65, (\alpha-1)2^{\alpha-1} \geq 1 > 1 - 1/p$ for any $p > 1$. Hence Theorem 4.1 implies the following

**Corollary 4.1.** *Inequality (4.1) holds for $p \geq 3$, $1.65 \leq \alpha < 2$.*

## 5. Further Discussions

We end this paper using Knopp's approach to give an upper bound for $l^p$ norms of lower triangular matrices with non-negative entries in general. We now prove the following

**Theorem 5.1.** *Let $A = (a_{n,k})$ be a lower triangular matrix with non-negative entries. If*

$$S = \sup_{n,k}(n a_{n,k}) < +\infty,$$

*then $||A||_{p,p} \leq Sq$.*

*Proof.* Similar to our treatment earlier in Section 2, we see that for a sequence $\mathbf{w}$ of positive terms,

$$\sum_{n=1}^{\infty}\Big|\sum_{k=1}^{n}a_{n,k}a_k\Big|^p \leq \sum_{k=1}^{\infty}w_k^{-(p-1)}\Big(\sum_{n=k}^{\infty}a_{n,k}^p\Big(\sum_{j=1}^{n}w_j\Big)^{p-1}\Big)|a_k|^p.$$

We now choose $\mathbf{w}$ inductively by setting $w_1 = 1$ and

$$\sum_{i=1}^{n}w_i = \frac{n-\beta}{1-\beta}w_n,$$

for some $0 < \beta < 1$ to be determined later. Note that the above relation implies that

$$w_{n+1} = \frac{n-\beta}{n}w_n, \quad w_n = O(\frac{1}{n^{\beta}}).$$

Moreover, it is easy to see that

$$1-\Big(1-\frac{\beta}{n}\Big)^{p-1} \geq \frac{(p-1)\beta}{n}\Big(1-\frac{\beta}{n}\Big)^{p-1}.$$

It follows from this that

$$\Big(\frac{\sum_{j=1}^{n}w_j}{n}\Big)^{p-1} \leq \frac{n}{(p-1)\beta(1-\beta)^{p-1}}\Big(w_n^{p-1}-w_{n+1}^{p-1}\Big).$$

We then deduce that

$$\sum_{n=k}^{\infty}a_{n,k}^p\Big(\sum_{j=1}^{n}w_j\Big)^{p-1} \leq \frac{S^p}{(p-1)\beta(1-\beta)^{p-1}}w_k^{p-1}.$$

Note that $\max_{0<\beta<1}(p-1)\beta(1-\beta)^{p-1} = q^{-p}$ at $\beta = 1/p$. On taking $\beta = 1/p$, this proves our assertion for Theorem 5.1. □

As an application of Theorem (5.1), we note that a lower triangular matrix $A$ is said to be a Nörlund matrix if its entries satisfy:

$$a_{n,k} = \lambda_{n-k+1}/\Lambda_n, \ 1 \leq k \leq n; \quad \Lambda_n = \sum_{i=1}^{n}\lambda_i, \lambda_i \geq 0, \lambda_1 > 0.$$

Comparing with the case of weighted mean matrices, results on the $l^p$ norms of Nörlund matrices are less satisfactory. If we further assume that $\lambda_k$'s are increasing, then Theorem 5.1 gives that

$$||A||_{p,p} \leq \sup_n(\frac{n\lambda_n}{\Lambda_n})q.$$

This is a result in [11]. We refer the reader to [7], [10] and [6] and for related results in this area.

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