On Lower and Upper Bounds of Matrices

ON LOWER AND UPPER BOUNDS OF MATRICES

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ABSTRACT. Using an approach of Bergh, we give an alternate proof of Bennett's result on lower bounds for non-negative matrices acting on non-increasing non-negative sequences in l^p when $p \geq 1$ and its dual version, the upper bounds when $0 < p \leq 1$. We also determine such bounds explicitly for some families of matrices.

1. Introduction

Let $p > 0$ and l^p be the space of all complex sequences $\mathbf{a} = (a_n)_{n\geq 1}$ satisfying:

$$||\mathbf{a}||_p = \Big(\sum_{i=1}^{\infty}|a_i|^p\Big)^{1/p} < \infty.$$

When $p > 1$, the celebrated Hardy's inequality [15, Theorem 326] asserts that for any $\mathbf{a} \in l^p$,

$$\sum_{n=1}^{\infty}\Big|\frac{1}{n}\sum_{k=1}^{n}a_k\Big|^p \leq \Big(\frac{p}{p-1}\Big)^p\sum_{k=1}^{\infty}|a_k|^p. \tag{1.1}$$

Hardy's inequality can be interpreted as the l^p operator norm of the Cesàro matrix C, given by $c_{j,k} = 1/j, k \leq j$, and 0 otherwise, is bounded on l^p and has norm $\leq p/(p-1)$ (The norm is in fact $p/(p-1)$). It is known that the Cesàro operator is not bounded below, or the converse of inequality (1.1) does not hold for any positive constant. However, if one assumes C acting only on non-increasing non-negative sequences in l^p, then such a lower bound does exist, and this is first obtained by Lyons in [18] for the case of l^2 with the best possible constant. For the general case concerning the lower bounds for an arbitrary non-negative matrix acting on non-increasing non-negative sequences in l^p when $p \geq 1$, Bennett [3] determined the best possible constant. When $0 < p \leq 1$, one can also consider a dual question and this has been studied in [4], [8] and [6]. Let $A = (a_{j,k}), 1 \leq j \leq m, 1 \leq k \leq n$ with $a_{j,k} \geq 0$, we can summarize the main results in this area in the following

Theorem 1.1 ([3, Theorem 2], [6, Theorem 4])**.** *Let* $\mathbf{x} = (x_1, \ldots, x_n), x_1 \geq \ldots \geq x_n \geq 0$, $p \geq 1$, $0 < q \leq p$, *then*

$$||A\mathbf{x}||_q \geq \lambda||\mathbf{x}||_p, \tag{1.2}$$

where

$$||A\mathbf{x}||_q^q = \sum_{j=1}^{m}(\sum_{k=1}^{n}a_{j,k}x_k)^q$$

and

$$\lambda^q = \min_{1\leq r\leq n} r^{-q/p}\sum_{j=1}^{m}(\sum_{k=1}^{r}a_{j,k})^q. \tag{1.3}$$

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Inequality (1.2) *is reversed when* $0 < p \leq 1$ *and* $q \geq p$ *with* min *replaced by* max *in* (1.3). *Moreover, there is equality in* (1.2) *if* $\mathbf{x}$ *has the form* $x_k = x_1, 1 \leq k \leq s$ *and* $x_k = 0, k > s$ *where* s *is any value of* r *where the minimum or maximum in* (1.3) *occurs.*

One may also consider the integral analogues of Theorem 1.1 and there is a rich literature on this area and we shall refer the reader to the articles [8], [19], [11], [21], [16], [9], [10], [2], [12] and the reference therein for the related studies. We point out here one may deduce Theorem 1.1 from its integral analogues by considering suitable integrals on suitable measure spaces (see for example, [2] and [12]).

A special case of Theorem 1.1 appeared in [4], where Bennett established the following inequality for $0 < p < 1, x_1 \geq x_2 \geq \ldots \geq 0$,

$$\sum_{n=1}^{\infty}\Big(\frac{1}{n}\sum_{k=n}^{\infty}x_k\Big)^p \leq \frac{\pi p}{\sin \pi p}\sum_{n=1}^{\infty}x_n^p. \tag{1.4}$$

The constant $\pi p/\sin(\pi p)$ is best possible. An integral analogue of the above inequality was established by Bergh in [8] and he then used it to deduce a slightly weaker result than inequality (1.4).

Our interest in Theorem 1.1 starts from the following inequality ($0 < p < 1$) for any non-negative $\mathbf{x}$:

$$\sum_{n=1}^{\infty}\Big(\frac{1}{n}\sum_{k=n}^{\infty}x_k\Big)^p \geq c_p\sum_{n=1}^{\infty}x_n^p. \tag{1.5}$$

It is shown in [15, Theorem 345] that the above inequality holds with $c_p = p^p$ for $0 < p < 1$ and it is also noted there that the constant p^p may not be best possible and the best possible constant was in fact later obtained by Levin and Stečkin [17, Theorem 61] to be $(p/(1-p))^p$ for $0 < p \leq 1/3$. Recently, the author [14] has extended the result of Levin and Stečkin to hold for $0 < p \leq 0.346$. Inequalities of type (1.5) with more general weights are also studied in [14], among which the following one for $0 < p < 1, \alpha \geq 1$:

$$\sum_{n=1}^{\infty}\Big(\frac{1}{n^{\alpha}}\sum_{k=n}^{\infty}((k+1)^{\alpha}-k^{\alpha})x_k\Big)^p \geq c_{p,\alpha}\sum_{n=1}^{\infty}x_n^p.$$

Here $c_{p,\alpha}$ is a constant and note that the above inequality gives back (1.5) when $\alpha = 1$. In view of (1.4), it's then natural to consider the reversed inequality if we assume further that $x_1 \geq x_2 \geq \ldots \geq 0$.

It is our goal in this paper to first give an alternate proof of Theorem 1.1 in Section 2 using the approach of Bergh in [8] and then using Theorem 1.1 to prove the following result in Section 3:

Theorem 1.2. *Let* $0 < p < 1, \alpha \geq 1, \alpha p < 1, 0 \leq t \leq 1, x_1 \geq x_2 \geq \ldots \geq 0$. *We have*

$$\sum_{n=1}^{\infty}\Big(\frac{1}{n^{\alpha}}\sum_{k=n}^{\infty}((k+t)^{\alpha}-(k+t-1)^{\alpha})x_k\Big)^p \leq \frac{1}{\alpha}B(\frac{1}{\alpha}-p, p+1)\sum_{n=1}^{\infty}x_n^p, \tag{1.6}$$

where $B(x,y), x > 0, y > 0$ *is the beta function*

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt.$$

Inequality (1.6) *also holds for* $t = 1, 0 < \alpha < 1$, $(1-\alpha)/(1+\alpha^2) \leq p \leq 1$. *Moreover, the constant is best possible. Inequality* (1.6) *reverses when* $t = 1, \alpha > 0, p \geq 1$ *with the best possible constant* $(2^{\alpha}-1)^p$.

We note that the case $\alpha = 1, t = 1, 0 < p < 1$ in the above theorem gives back inequality (1.4) and the integral analogue of Theorem 1.2 has been studied in [16]. Some consequences of Theorem 1.2 are deduced in Section 4 and other applications of Theorem 1.1 are given in Sections 5 and 6.

2. Proof of Theorem 1.1

We need a lemma first:

Lemma 2.1. *Let $p \geq 1, 0 < q \leq p$, then for any positive sequences $(a_j)_{1\leq j\leq m}$ and any non-negative sequence $(b_j)_{1\leq j\leq m}$, we have*

$$\Big(\sum_{j=1}^m (a_j+b_j)^q\Big)^{p/q-1}\Big(\sum_{j=1}^m (a_j+b_j)^{q-1}a_j\Big) \geq \Big(\sum_{j=1}^m a_j^q\Big)^{p/q}. \tag{2.1}$$

The above inequality reverses when $0 < p \leq 1, q \geq p$.

Proof. We shall only consider the case $p \geq 1, 0 < q \leq p$ here, the other case is being analogue. We recast inequality (2.1) as

$$\Big(\sum_{j=1}^m (a_j+b_j)^q\Big)^{1-q/p}\Big(\sum_{j=1}^m (a_j+b_j)^{q-1}a_j\Big)^{q/p} \geq \sum_{j=1}^m a_j^q.$$

Applying Hölder's inequality to the left-hand side expression above, we obtain

$$\Big(\sum_{j=1}^m (a_j+b_j)^q\Big)^{1-q/p}\Big(\sum_{j=1}^m (a_j+b_j)^{q-1}a_j\Big)^{q/p} \geq \sum_{j=1}^m (a_j+b_j)^{q(1-1/p)}a_j^{q/p} \geq \sum_{j=1}^m a_j^q.$$

This completes the proof. □

We now prove Theorem 1.1. As the proofs are similar for both cases, we shall focus only on establishing (1.2) for $p \geq 1, 0 < q \leq p$ and we shall also leave the discussion on the cases of equality to the reader. We may also assume $a_{j,k} > 0$ for all j, k and the general case follows from a limiting process. By homogeneity, we see that one can make inequality (1.2) valid by taking λ to be $\lambda_0 = \min\{||A\mathbf{x}||_q : ||x||_p = 1, x_1 \geq \ldots \geq x_n \geq 0\}$. By compactness, λ_0 is attained at some $\mathbf{x}_0 \neq 0$. We may assume the right-hand side expression of (1.3) is > 0 for otherwise inequality (1.2) holds trivially. This readily implies that $\lambda_0 \neq 0$. Certainly λ_0 is no more than the right-hand expression of (1.3) and suppose now that λ_0 is strictly less than the right-hand expression of (1.3) and it's attained at a vector $\mathbf{x}_0$ satisfying: $(\mathbf{x}_0)_k = x, 1 \leq k \leq i$ for some k with $1 \leq i \leq n-1$ and $(\mathbf{x}_0)_{i+1} = 1 < x$ (by homogeneity). We now regard x as a variable and consider the following function:

$$f(x) = \frac{||A\mathbf{x}_0||_q^p}{||\mathbf{x}_0||_p^p}.$$

We then have at $\mathbf{x}_0$,

$$f'(x) = \frac{p}{||\mathbf{x}_0||_p^p}\Big(x^{-1}||A\mathbf{x}_0||_q^{p-q}\sum_{j=1}^m(\sum_{k=1}^n a_{j,k}(\mathbf{x}_0)_k)^{q-1}\sum_{k=1}^i a_{j,k}(\mathbf{x}_0)_k - ix^{p-1}\frac{||A\mathbf{x}_0||_q^p}{||\mathbf{x}_0||_p^p}\Big).$$

We set $a_j = \sum_{k=1}^i a_{j,k}(\mathbf{x}_0)_k$ and $b_j = \sum_{k=i+1}^n a_{j,k}(\mathbf{x}_0)_k$ (note that $a_j > 0$) in Lemma 2.1 to see that

$$||A\mathbf{x}_0||_q^{p-q}\sum_{j=1}^m(\sum_{k=1}^n a_{j,k}(\mathbf{x}_0)_k)^{q-1}\sum_{k=1}^i a_{j,k}(\mathbf{x}_0)_k \geq \Big(\sum_{j=1}^m(\sum_{k=1}^i a_{j,k}x)^q\Big)^{p/q}.$$

It follows that

$$f'(x) \geq \frac{p}{||\mathbf{x}_0||_p^p}\Big(x^{-1}\Big(\sum_{j=1}^m(\sum_{k=1}^i a_{j,k}x)^q\Big)^{p/q} - ix^{p-1}\lambda_0^p\Big) > 0.$$

This leads to a contradiction and Theorem 1.1 is thus proved.

3. Proof of Theorem 1.2

We first consider the case when $0 < p < 1$. We may certainly focus on establishing our assertion for inequality (1.6) with the infinite sums there replaced by any finite sums, say from 1 to N. We now consider the case $\alpha \geq 1, t = 1$. Theorem 1.1 readily implies that in this case, the best constant is given by $\max_{1 \leq r \leq N} s_r$, where

$$s_r = r^{-1} \sum_{k=1}^{r} \Big(\frac{(r+1)^\alpha - k^\alpha}{k^\alpha} \Big)^p.$$

Suppose we can show that the sequence (s_r) is non-decreasing, then the maximum occurs when $r = N$, and as $N \to \infty$, one obtains the constant in (1.6) easily and this also shows that the constant there is best possible. It rests thus to show the sequence (s_r) is non-decreasing. To show this, we use the trick of Bennett in [4] (see also [7, Proposition 7]) on considering the following function:

$$f_{\alpha,p}(x) = \Big(\frac{1 - x^\alpha}{x^\alpha} \Big)^p + \Big(\frac{1 - (1-x)^\alpha}{(1-x)^\alpha} \Big)^p.$$

For $n \geq 1$ and any given function f defined on $(0, 1)$, we define

$$A_n(f) = \frac{1}{n} \sum_{r=1}^{n} f\Big(\frac{r}{n+1} \Big).$$

Note that we then have $s_n = 2A_n(f_{\alpha,p})$. It then suffices to show that $A_n(f_{\alpha,p})$ increases with n. A result of Bennett and Jameson [7, Theorem 1] asserts that if f is a convex function on $(0, 1)$, then $A_n(f)$ increases with n. Thus, it suffices to show that $f_{\alpha,p}$ is convex on $(0, 1)$ and direct calculation shows that

$$\begin{aligned} f''_{\alpha,p}(x) &= \alpha p x^{-\alpha p - 2}(1 - x^\alpha)^{p-2}(\alpha p + 1 - (1+\alpha)x^\alpha) \\ &\quad + \alpha p (1-x)^{-\alpha p - 2}(1 - (1-x)^\alpha)^{p-2}(\alpha p + 1 - (1+\alpha)(1-x)^\alpha). \end{aligned}$$

As $f''_{\alpha,p}(x) = f''_{\alpha,p}(1-x)$, it suffices to show $f''_{\alpha,p}(x) \geq 0$ for $0 < x \leq 1/2$. Assuming $0 < x \leq 1/2$, we recast $f''_{\alpha,p}(x)$ as

$$\begin{aligned} f''_{\alpha,p}(x) &= \alpha p (1-x)^{-\alpha p - 2}(1 - (1-x)^\alpha)^{p-2} \\ &\quad \cdot \Big(g_{\alpha,p}(x)(\alpha p + 1 - (1+\alpha)x^\alpha) + (\alpha p + 1 - (1+\alpha)(1-x)^\alpha) \Big), \end{aligned}$$

where

$$g_{\alpha,p}(x) = \Big(\frac{1 - x^\alpha}{x^\alpha} \cdot \frac{(1-x)^\alpha}{1 - (1-x)^\alpha} \Big)^p \cdot \Big(\frac{1-x}{1-x^\alpha} \cdot \frac{1 - (1-x)^\alpha}{1 - (1-x)} \Big)^2.$$

It is easy to show that both factors of $g_{\alpha,p}(x)$ are ≥ 1 and that $\alpha p + 1 - (1+\alpha)x^\alpha \geq 0$ when $0 < x \leq 1/2$. We bound $g_{\alpha,p}(x)$ by

$$g_{\alpha,p}(x) \geq \frac{1-x}{1-x^\alpha} \cdot \frac{1 - (1-x)^\alpha}{1 - (1-x)}.$$

It then follows that $f''_{\alpha,p}(x) \geq 0$ as long as

$$\frac{1-x}{1-x^\alpha} \cdot \frac{1 - (1-x)^\alpha}{1 - (1-x)} (\alpha p + 1 - (1+\alpha)x^\alpha) + (\alpha p + 1 - (1+\alpha)(1-x)^\alpha) \geq 0.$$

It suffices to establish the above inequality for $p = 0$ and in this case we recast it as $h_\alpha(x) + h_\alpha(1 - x) \geq 0$ where

$$h_\alpha(x) = \frac{1-x}{1-x^\alpha}(1 - (1+\alpha)x^\alpha) = (1+\alpha)(1-x) - \frac{\alpha(1-x)}{1-x^\alpha}.$$

It is easy to show that $h_\alpha(x)$ is concave on $(0,1)$ and it follows from the theory of majorization (see, for example, Section 6 of [7]) that for any $0<x<1$, we have

$$h_\alpha(x)+h_\alpha(1-x)\geq \lim_{x\to 0^+}(h_\alpha(x)+h_\alpha(1-x))=0.$$

This now completes the proof of Theorem 1.2 when $0<p<1, \alpha\geq 1, t=1$. Before we move to the proof of other cases, we point out here an alternative proof of $h_\alpha(x)+h_\alpha(1-x)\geq 0$ is that one can show easily that $h_\alpha(x)$ is an increasing function of $\alpha\geq 1$ for fixed x so that $h_\alpha(x)+h_\alpha(1-x)\geq \lim_{\alpha\to 1^+}(h_\alpha(x)+h_\alpha(1-x))=0$. This will be our approach for the case $0<p\leq 1, 0<\alpha<1$ in what follows.

Now the general case $0<p<1, \alpha\geq 1, 0\leq t\leq 1$, we note that the left-hand side expression of (1.6) is termwise no larger than the corresponding term when $t=1$. Therefore, inequality (1.6) follows from the case $t=1$. To show the constant is best possible, we use Theorem 1.1 again to see that the best constant is given by $\max_{1\leq r\leq N} s(t)_r$, where

$$s(t)_r=r^{-1}\sum_{k=1}^{r}\Big(\frac{(r+t)^\alpha-(k+t-1)^\alpha}{k^\alpha}\Big)^p\geq r^{-1}\sum_{k=1}^{r}\Big(\frac{r^\alpha-k^\alpha}{k^\alpha}\Big)^p.$$

It follows that $\lim_{N\to\infty} s(t)_N\geq \frac{1}{\alpha}B(\frac{1}{\alpha}-p, p+1)$, this combining with our discussions above completes the proof of Theorem 1.2 when $0<p<1, \alpha\geq 1$.

For the case $0<p\leq 1, 0<\alpha<1$, we can use the same approach as above except this time we bound $g_{\alpha,p}(x)$ by

$$g_{\alpha,p}(x)\geq \Big(\frac{1-x}{1-x^\alpha}\cdot\frac{1-(1-x)^\alpha}{1-(1-x)}\Big)^2.$$

We define for $0<x<1$,

$$u_{\alpha,p}(x)=\Big(\frac{1-x}{1-x^\alpha}\Big)^2(\alpha p+1-(1+\alpha)x^\alpha).$$

It remains to show $u_{\alpha,p}(x)+u_{\alpha,p}(1-x)\geq 0$ (note that this also implies that that $u_{\alpha,p}(1/2)\geq 0$). On considering the limit as $x\to 0^+$, we see that it is necessary to have $(1-\alpha)/(1+\alpha^2)\leq p\leq 1$. We now assume this condition for p and note that it suffices to establish $u_{\alpha,p}(x)+u_{\alpha,p}(1-x)\geq 0$ for $p=(1-\alpha)/(1+\alpha^2)$. We write $u_{\alpha,(1-\alpha)/(1+\alpha^2)}(x)=(1+\alpha)/(1+\alpha^2)v(\alpha,x)$ with

$$v(\alpha,x)=(1-x)^2\frac{1-(1+\alpha^2)x^\alpha}{(1-x^\alpha)^2}.$$

It remains thus to show $v(\alpha,x)+v(\alpha,1-x)\geq 0$. Calculation shows

$$\frac{\partial v}{\partial\alpha}=\frac{x^\alpha(1-x)^2}{\alpha(1-x^\alpha)^3}w_\alpha(x^\alpha),$$

with $w_\alpha(t)=-2\alpha^2+(1-\alpha^2)\ln t-(1+\alpha^2)t\ln t+2\alpha^2 t, 0<t\leq 1$. As $w_\alpha(1)=w'_\alpha(1)=0$ and $w''_\alpha(t)<0$, one sees easily that $w_\alpha(t)\leq 0$ for $0<t\leq 1$ and it follows that when $0<x<1$,

$$v(\alpha,x)+v(\alpha,1-x)\geq \lim_{\alpha\to 1^-}(v(\alpha,x)+v(\alpha,1-x))=0.$$

This completes the proof of Theorem 1.2 when $0<p\leq 1, 0<\alpha<1$.

Lastly, when $p\geq 1$, the assertion of the theorem follows as long as we can show the sequence (s_r) is increasing, where (s_r) is defined as above. In this case, it's easy to see that the function $x\mapsto(1-x^\alpha)^p x^{-\alpha p}$ is convex on $(0,1)$ when $\alpha>0, p\geq 1$ so that our discussions above can be applied here and this completes the proof.

4. Some consequences of Theorem 1.2

In this section we deduce some consequences from Theorem 1.2. We note that $(k+1)^\alpha - k^\alpha \geq \alpha k^{\alpha-1}$ when $\alpha \geq 1$ and it is also easy to show by induction that

$$\alpha \sum_{n=k}^{r} n^{\alpha-1} \geq r^\alpha - k^\alpha.$$

A similar argument to the proof of Theorem 1.2 then allows us to establish

Corollary 4.1. *Let $0 < p < 1, \alpha \geq 1, \alpha p < 1, x_1 \geq x_2 \geq \ldots \geq 0$. We have*

$$\sum_{n=1}^{\infty} \Big(\frac{1}{n^\alpha} \sum_{k=n}^{\infty} \alpha k^{\alpha-1} x_k \Big)^p \leq \frac{1}{\alpha} B(\frac{1}{\alpha} - p, p+1) \sum_{n=1}^{\infty} x_n^p. \tag{4.1}$$

The constant is best possible.

We recall here the function $L_r(a,b)$ for $a > 0, b > 0, a \neq b$ and $r \neq 0, 1$ (the only case we shall concern here) is defined as $L_r^{r-1}(a,b) = (a^r - b^r)/(r(a-b))$. We also write $L_\infty(a,b)$ as $\lim_{r\to\infty} L_r(a,b)$ and note that $L_\infty(a,b) = \max(a,b)$. Using this notation, the matrix $(a_{j,k})$ associated to inequality (4.1) is thus given by $a_{j,k} = L_\infty^{\alpha-1}(k,k-1)/\sum_{i=1}^{j} L_\alpha^{\alpha-1}(i,i-1)$ when $k \geq j$ and $a_{j,k} = 0$ otherwise.

It is known [1, Lemma 2.1] that the function $r \mapsto L_r(a,b)$ is strictly increasing on $\mathbb{R}$, this combining with Corollary 4.1 allows us to establish the first assertion of the following

Corollary 4.2. *Let $0 < p < 1, \beta \geq \alpha > 1, \alpha p < 1, x_1 \geq x_2 \geq \ldots \geq 0$. We have*

$$\sum_{n=1}^{\infty} \Big(\frac{1}{\sum_{i=1}^{n} L_\beta^{\alpha-1}(i,i-1)} \sum_{k=n}^{\infty} L_\beta^{\alpha-1}(k,k-1) a_k \Big)^p \leq \frac{1}{\alpha} B(\frac{1}{\alpha} - p, p+1) \sum_{n=1}^{\infty} x_n^p.$$

The constant is best possible when $\alpha \geq 2$.

To show the constant is best possible when $\alpha \geq 2$, we first show that for $n \geq 1, \beta \geq \alpha \geq 2$,

$$\frac{\sum_{i=1}^{n+1} L_\beta^{\alpha-1}(i,i-1)}{\sum_{i=1}^{n} L_\beta^{\alpha-1}(i,i-1)} \geq \frac{(n+2)^\alpha}{(n+1)^\alpha}. \tag{4.2}$$

By [13, Lemma 3.1], it suffices to show for $n \geq 1$,

$$\frac{L_\beta^{\alpha-1}(n+1,n)}{L_\beta^{\alpha-1}(n,n-1)} \geq \frac{(n+2)^\alpha - (n+1)^\alpha}{(n+1)^\alpha - n^\alpha}.$$

The above inequality follows from the following inequalities:

$$\frac{L_\beta^{\alpha-1}(n+1,n)}{L_\beta^{\alpha-1}(n,n-1)} \geq \frac{(n+1)^{\alpha-1}}{n^{\alpha-1}} \geq \frac{(n+2)^\alpha - (n+1)^\alpha}{(n+1)^\alpha - n^\alpha}. \tag{4.3}$$

As $\beta \geq 2$, we have by convexity,

$$\frac{1}{n} + \frac{n-1}{n}(\frac{n-1}{n})^{\beta-1} \geq (\frac{1}{n} + \frac{n-1}{n} \cdot \frac{n-1}{n})^{\beta-1} \geq (\frac{n}{n+1})^{\beta-1}.$$

One checks easily that this implies the first inequality in (4.3) and the second inequality of (4.3) can be shown similarly.

Now, to see the constant is best possible, we note that Theorem 1.1 implies that the constant is no smaller than

$$\begin{aligned} & \lim_{N\to\infty}\frac{1}{N}\sum_{j=1}^{N}\Big(\frac{\sum_{k=j}^{N}L_{\beta}^{\alpha-1}(k,k-1)}{\sum_{i=1}^{j}L_{\beta}^{\alpha-1}(i,i-1)}\Big)^{p} \\ \geq & \lim_{N\to\infty}\frac{1}{N}\sum_{j=1}^{N}\Big(\frac{\sum_{k=j+1}^{N}L_{\beta}^{\alpha-1}(k,k-1)}{\sum_{i=1}^{j}L_{\beta}^{\alpha-1}(i,i-1)}\Big)^{p} \geq \lim_{N\to\infty}\frac{1}{N}\sum_{j=1}^{N}\Big(\frac{(N+1)^{\alpha}}{(j+1)^{\alpha}}-1\Big)^{p}, \end{aligned}$$

where the last inequality follows from (4.2) and the last limit also gives the constant in Corollary 4.2.

Note the particular case $\beta=\infty$ of Corollary 4.2 gives

$$\sum_{n=1}^{\infty}\Big(\frac{1}{\sum_{i=1}^{n}i^{\alpha-1}}\sum_{k=n}^{\infty}k^{\alpha-1}a_k\Big)^{p} \leq \frac{1}{\alpha}B(\frac{1}{\alpha}-p,p+1)\sum_{n=1}^{\infty}x_n^p.$$

5. Applications of Theorem 1.1 to weighted mean matrices

In this section we give more applications of Theorem 1.1. We remark first that the problem of finding lower bounds of non-negative weighted mean matrices acting on non-increasing non-negative sequences in l^p when $p\geq 1$ has been studied in [5] and [20]. Here we recall that a weighted mean matrix $(a_{j,k})$ is given by $a_{j,k}=\lambda_k/\Lambda_j$ for $1\leq k\leq j$ and $a_{j,k}=0$ otherwise, where $\Lambda_n=\sum_{i=1}^{n}\lambda_i, \lambda_1>0$. We also recall that a Nörlund matrix $(a_{j,k})$ is given by $a_{j,k}=\lambda_{j-k+1}/\Lambda_j$ for $1\leq k\leq j$ and $a_{j,k}=0$ otherwise. In what follows, we shall say a weighted mean (or a Nörlund) matrix A is generated by (λ_n) if its entries are given as above. In the weighted mean matrix case, it is shown in [5, Theorem 4] that when $\lambda_i=i^{\alpha}, \alpha\geq 1$ or $-1<\alpha\leq 0, (1+\alpha)p>0$, the corresponding minimum in (1.3) is reached at $r=1$. The case $\alpha\geq 1$ is also shown in [20, Corollary 9]. We now give an alternative proof of the case $-1<\alpha\leq 0$ based on the idea used in the proof of Theorem 4 in [3]. We also give a companion result concerning the upper bound when $0<\alpha\leq 1$ and $0<p\leq 1$. We have

Corollary 5.1. *Let* $\mathbf{x}$ *be a non-negative non-increasing sequence,* $p>1$, $-1<\alpha\leq 0, (\alpha+1)p>1$, *then*

$$\sum_{j=1}^{\infty}\Big(\sum_{k=1}^{j}\frac{k^{\alpha}}{\sum_{i=1}^{j}i^{\alpha}}x_k\Big)^{p} \geq \sum_{j=1}^{\infty}\Big(\frac{1}{\sum_{i=1}^{j}i^{\alpha}}\Big)^{p}||\mathbf{x}||_p^p. \tag{5.1}$$

The above inequality reverses when $0<p\leq 1$, $0<\alpha\leq 1, (1+\alpha)p>1$. *The constant is best possible in either case.*

Proof. Note that the condition $(\alpha+1)p>1$ ensures that the constant in (5.1) is finite. We consider the case $p>1$ first. For any weighted mean matrix A generated by (λ_n) with $\lambda_1>0$, Theorem 1.1 implies that for any non-increasing sequence $\mathbf{x}$, $||A\mathbf{x}||_p\geq\lambda||\mathbf{x}||_p$ with

$$\begin{aligned} \lambda^p &= \inf_r r^{-1}\sum_{j=1}^{\infty}(\sum_{k=1}^{\min(r,j)}\frac{\lambda_k}{\Lambda_j})^p = 1+\inf_r r^{-1}\sum_{j=r+1}^{\infty}\Big(\frac{\Lambda_r}{\Lambda_j}\Big)^p \\ &= 1+\inf_r\sum_{k=1}^{\infty}r^{-1}\sum_{i=1}^{r}\Big(\frac{\Lambda_r}{\Lambda_{kr+i}}\Big)^p. \end{aligned}$$

To show the infimum is achieved at $r=1$, it suffices to show $\Lambda_r/\Lambda_{(k+1)r}\geq\Lambda_1/\Lambda_{k+1}$ for any $k\geq 1$. When $\lambda_n=n^{\alpha}$ with $-1<\alpha\leq 0$, this is easily shown by induction and this completes the proof for the first assertion of the corollary.

Now consider the case $0 < p \leq 1$. Let $\Lambda_i = \sum_{j=1}^{i} j^{\alpha}$. Theorem 1.1 and our discussions above imply that the best constant for the reversed inequality of (5.1) is given by

$$1 + \sup_r r^{-1} \sum_{j=r+1}^{\infty} \Big(\frac{\Lambda_r}{\Lambda_j}\Big)^p = 1 + \sup_r a_r.$$

The assertion of the corollary follows if we can show the sequence (a_r) is decreasing and by Lemma 7 of [5] (see the remark after that) with $x_n = \Lambda_n^{-p}$ there, it suffices to show $1 + n(\Lambda_{n+1}/\Lambda_n)^p \leq (n+1)(\Lambda_{n+2}/\Lambda_{n+1})^p$ for $n \geq 1$ and one can see easily that it suffices to establish this for $p = 1$ but in this case, this is given by Lemma 8 of [5] and this completes the proof. □

Our next result concerns with the bounds for the weighted mean matrix generated by $\lambda_i = i^{\alpha} - (i-1)^{\alpha}, \alpha p > 1$:

Corollary 5.2. *Let* $\mathbf{x}$ *be a non-negative non-increasing sequence,* $p \geq 1$*,* $\alpha > 1/p$*, then*

$$\sum_{j=1}^{\infty} \Big(\sum_{k=1}^{j} \frac{k^{\alpha} - (k-1)^{\alpha}}{j^{\alpha}} x_k\Big)^p \geq \zeta(\alpha p) ||\mathbf{x}||_p^p, \tag{5.2}$$

where $\zeta(x)$ *denotes the Riemann zeta function and the constant is best possible. The above inequality reverses when* $0 < p \leq 1, \alpha p > 1$ *with the best constant* $\alpha p/(\alpha p - 1)$*.*

Proof. The proof for the $p \geq 1$ case can be easily obtained by applying similar ideas to that used in the proof of Theorem 4 in [3] so we shall leave it to the reader. When $0 < p \leq 1$, we note by Theorem 1.1 and the proof of Corollary 5.1, the best constant for the reversed inequality of (5.2) is given by

$$1 + \sup_r \sum_{k=1}^{\infty} r^{-1} \sum_{i=1}^{r} \Big(\frac{\Lambda_r}{\Lambda_{kr+i}}\Big)^p = 1 + \sup_r \sum_{k=1}^{\infty} \sum_{i=1}^{r} \frac{(k + i/r)^{-\alpha p}}{r}.$$

It follows from Theorem 3A of [7] that the term inside the first sum of the last expression above is increasing with r, and it is easy to see that as $r \to +\infty$, it approaches the value $(k^{1-\alpha p} - (k+1)^{1-\alpha p})/(\alpha p - 1)$ and this completes the proof. □

It is an open problem to determine the lower bounds of the weighted mean matrices generated by $\lambda_n = n^{\alpha}, 0 < \alpha < 1$ acting on non-increasing non-negative sequences in l^p when $p \geq 1$. In connection to this, Bennett [5, p. 65] asked to determine the monotonicity of the following sequence for $p > 1, (1+\alpha)p > 1$, when $\Lambda_n = \sum_{i=1}^{n} i^{\alpha}$:

$$\frac{\Lambda_n^p}{n} \sum_{k>n} \Lambda_k^{-p}. \tag{5.3}$$

The following condition is sufficient for the above sequence to be increasing, given by [5, Theorem 3] (see also [20, Theorem 8]):

$$1 + n\Big(\frac{\Lambda_{n+1}}{\Lambda_n}\Big)^p - (n+1)\Big(\frac{\Lambda_{n+2}}{\Lambda_{n+1}}\Big)^p \geq 0. \tag{5.4}$$

Suppose the above condition is satisfied, then we deduce from it that

$$n\Big(\frac{\Lambda_{n+1}}{\Lambda_n}\Big)^p \geq (n+1)\Big(\frac{\Lambda_{n+2}}{\Lambda_{n+1}}\Big)^p - 1 \geq (n+2)\Big(\frac{\Lambda_{n+3}}{\Lambda_{n+2}}\Big)^p - 2 \geq \ldots \geq (n+k)\Big(\frac{\Lambda_{n+k+1}}{\Lambda_{n+k}}\Big)^p - k,$$

for any $n, k \geq 1$. When $\Lambda_n = \sum_{i=1}^{n} i^{\alpha}$, we note by the Euler-Maclaurin formula, one easily finds that for $\alpha > 0$,

$$\sum_{i=1}^{n} i^{\alpha} = \frac{n^{\alpha+1}}{1+\alpha} + \frac{n^{\alpha}}{2} + O(1 + n^{\alpha-1}). \tag{5.5}$$

We then deduce from this that when $\Lambda_n=\sum_{i=1}^n i^{\alpha}, \alpha>0$,

$$\lim_{k\to+\infty}(n+k)\Big(\frac{\Lambda_{n+k+1}}{\Lambda_{n+k}}\Big)^p-k=n+(1+\alpha)p.$$

It follows from this that we have

$$\frac{\Lambda_{n+1}}{\Lambda_n}\geq (1+\frac{(1+\alpha)p}{n})^{1/p}.$$

Using Taylor expansion and (5.5) again, we find that in order for the above inequality to hold, it is necessary to have $p\geq 2/(1+\alpha)$. It is therefore interesting to ask whether the above inequality holds or not for $p=2/(1+\alpha)$ and this in fact is known, as we have the following

Lemma 5.1. *For $0\leq\alpha\leq 1$ or $\alpha\geq 3$, we have for $k\geq n\geq 1$,*

$$\frac{\sum_{i=1}^n i^{\alpha}}{\sum_{i=1}^k i^{\alpha}}\leq\Big(\frac{n(n+1)}{k(k+1)}\Big)^{\frac{\alpha+1}{2}}. \tag{5.6}$$

The above inequality reverses when $1\leq\alpha\leq 3$. In particular, we have for $0\leq\alpha\leq 1$ or $\alpha\geq 3$,

$$\sum_{i=1}^n i^{\alpha}\leq\frac{(n(n+1))^{\frac{\alpha+1}{2}}}{\alpha+1}. \tag{5.7}$$

The above inequality reverses when $1\leq\alpha\leq 3$.

Proof. This lemma is a restatement of Corollary 3.1 of [13] (note that in the statement of [13, Corollary 3.1], one needs to interchange the place of the words "increasing" and "decreasing"). In what follows, we shall give a simper proof. We first note that inequality (5.7) follows from the corresponding cases of (5.6) on letting $k\to+\infty$ in (5.6) so that it suffices to establish (5.6). We shall only prove the case for $\alpha\geq 3$, the proof for the other cases are similar. We may assume $k=n+1$ here and by Lemma 3.1 of [13], it suffices to establish (5.6) for $n=1$ as well as the following inequality for all $n\geq 1$:

$$\frac{(n+1)^{\alpha}}{(n+2)^{\alpha}}\leq\frac{\Big((n+2)(n+1)\Big)^{(1+\alpha)/2}-\Big(n(n+1)\Big)^{(1+\alpha)/2}}{\Big((n+3)(n+2)\Big)^{(1+\alpha)/2}-\Big((n+1)(n+2)\Big)^{(1+\alpha)/2}}. \tag{5.8}$$

The above inequality is easily seen to be equivalent to $f(n+2)\leq f(n+1)$ where

$$f(x)=\frac{(x+1)^{(1+\alpha)/2}-(x-1)^{(1+\alpha)/2}}{x^{(\alpha-1)/2}}=\frac{1+\alpha}{2}\int_0^1\Big((1+t/x)^{(\alpha-1)/2}+(1-t/x)^{(\alpha-1)/2}\Big)dt.$$

One shows easily the last expression above is a decreasing function of $x\geq 1$ when $\alpha\geq 3$ so that (5.8) holds. Moreover, the case $n=1, k=2$ of (5.6) is just $f(2)\leq f(1)$ and this completes the proof. □

The above lemma implies that (in combining the arguments given in [5, Theorem 3] or [20, Theorem 8]) the sequence defined in (5.3) for $\Lambda_n=\sum_{i=1}^n i^{\alpha}$ is increasing for n large enough when $0\leq\alpha\leq 1$ or $\alpha\geq 3$, as long as $p\geq 2/(1+\alpha)$ and it is decreasing for n large enough when $1\leq\alpha\leq 3$, as long as $1/(1+\alpha)<p\leq 2/(1+\alpha)$. It's also shown in [5] that the sequence is increasing for $\alpha\geq 1, p\geq 1$ and decreasing for $0<\alpha\leq 1, 1/(1+\alpha)<p\leq 1$. In what follows, we shall give an extension of this result. But we first need a few lemmas:

Lemma 5.2. *For $1\leq\alpha\leq 3$, we have*

$$\sum_{i=1}^n i^{\alpha}\geq\frac{1}{1+\alpha}\frac{4n^2(n+1)^{\alpha}}{4n+1+\alpha}. \tag{5.9}$$

The above inequality reverses when $\alpha\geq 3$.

Proof. We only give the proof for the case $1 \leq \alpha \leq 3$ and the proof for the other case is similar. It follows from (5.6) with $k = n+1$ that we have for $1 \leq \alpha \leq 3$,

$$\frac{\sum_{i=1}^{n} i^{\alpha}}{\sum_{i=1}^{n+1} i^{\alpha}} \geq \left(\frac{n}{n+2}\right)^{\frac{\alpha+1}{2}}. \tag{5.10}$$

We deduce from this that for $1 \leq \alpha \leq 3$,

$$\sum_{i=1}^{n} i^{\alpha} \geq \frac{n^{(1+\alpha)/2}(n+1)^{\alpha}}{(n+2)^{(1+\alpha)/2} - n^{(1+\alpha)/2}}. \tag{5.11}$$

It suffices to show the right-hand side expression above is no less than the right-hand side expression of (5.9). One easily sees that this follows from the following inequality for $1 \leq \alpha \leq 3, 0 \leq x \leq 1$:

$$1 + (1+\alpha)x + \frac{(1+\alpha)^2x^2}{4} - (1+2x)^{(1+\alpha)/2} \geq 0.$$

The above inequality can be shown easily and this completes the proof. □

Lemma 5.3. *Let $0 \leq x \leq 1$, the following inequality holds when $1 \leq \alpha \leq 3$:*

$$\Big((1+x)^{2-\alpha}(1+2x)^{(\alpha-1)/2} - 1\Big)\Big((1+2x)^{(1+\alpha)/2} - 1\Big) - (1+\alpha)x^2 \geq 0. \tag{5.12}$$

The above inequality reverses when $\alpha \geq 3$.

Proof. We regard the left-hand side expression of (5.12) as a function of α and note that its second derivative with respect to α equals $(1+2x)^{(\alpha-1)/2}h(\alpha; x)$, where

$$\begin{aligned} h(\alpha; x) \quad &= \quad \ln^2\Big(\frac{1+2x}{1+x}\Big)(1+x)^{2-\alpha}(1+2x)^{(\alpha+1)/2} - \Big(\frac{\ln(1+2x)}{2}\Big)^2(1+2x) \\ &\quad - \ln^2\Big(\frac{(1+2x)^{1/2}}{1+x}\Big)(1+x)^{2-\alpha}. \end{aligned}$$

We again regard $h(\alpha; x)$ as a function of α and note that

$$h'(\alpha; x) = (1+x)^{2-\alpha}\ln\Big(\frac{(1+2x)^{1/2}}{1+x}\Big)\Big(\ln^2\Big(\frac{1+2x}{1+x}\Big)(1+2x)^{(\alpha+1)/2} + \ln\Big(\frac{(1+2x)^{1/2}}{1+x}\Big)\ln(1+x)\Big).$$

We want to show the last factor of the right-hand side expression above is non-negative when $\alpha \geq 1$ and it suffices to show this for $\alpha = 1$ and in this case, this expression becomes

$$\begin{aligned} & \ln^2\Big(\frac{1+2x}{1+x}\Big)(1+2x) + \ln\Big(\frac{(1+2x)^{1/2}}{1+x}\Big)\ln(1+x) \\ = \quad & (1+2x)\ln^2(1+2x) - (2(1+2x) - 1/2)\ln(1+2x)\ln(1+x) + 2x\ln^2(1+x) \\ \geq \quad & (1+2x)\ln^2(1+2x) - (2(1+2x) - 1/2)\ln(1+2x)\ln(1+x) + x\ln(1+x)\ln(1+2x) \\ = \quad & (1+2x)\ln(1+2x)\Big(\ln(1+2x) - 3\ln(1+x)/2\Big) \geq 0. \end{aligned}$$

It follows that $h'(\alpha; x) \leq 0$. As the left-hand side expression of (5.12) takes value 0 when $\alpha = 1$ and 3, the assertion of the lemma follows if we can show the derivative with respect to α of the left-hand side expression of (5.12) is ≥ 0 (> 0 for $x \neq 0$) at $\alpha = 1$. Calculation shows this is

$$\begin{aligned} & \ln\Big(\frac{1+2x}{1+x}\Big)(1+x)(1+2x) - \frac{\ln(1+2x)}{2}(1+2x) - \ln\Big(\frac{(1+2x)^{1/2}}{1+x}\Big)(1+x) - x^2 \\ = \quad & x\Big((3/2+2x)\ln(1+2x) - 2(1+x)\ln(1+x) - x\Big). \end{aligned}$$

It is easy to show the second factor in the last expression above is ≥ 0 for $0 \leq x \leq 1$ (> 0 for $x \neq 0$) and this completes the proof. □

Now we are ready to prove the following

Theorem 5.1. *For* $1 < \alpha \leq 3$ *and* $1/(1+\alpha) < p \leq 1/2$*, the sequence defined in* (5.3) *for* $\Lambda_n = \sum_{i=1}^n i^\alpha$ *is decreasing. For* $\alpha \geq 3$ *and* $p \geq 1/2$*, the sequence defined in* (5.3) *for* $\Lambda_n = \sum_{i=1}^n i^\alpha$ *is increasing.*

Proof. We only prove the case for $1 \leq \alpha \leq 3$ here and the proof for the case $\alpha \geq 3$ is similar. We only point out that in the $\alpha \geq 3$ case, one needs to use the fact (which is easy to show) that the right-hand side expression of (5.10) is no greater than $n(n+1)^\alpha/(1+\alpha)$ (and hence $\leq n(n+1)^\alpha/\sqrt{1+\alpha}$). Now we return to the proof of our assertion for $1 \leq \alpha \leq 3$ and by the remark after Lemma 7 of [5] (with $x_n = \Lambda_n^{-p}$ there), it suffices to prove the reversed inequality of (5.4) for $p = 1/2$, which is equivalent to

$$2n\Big(\frac{\Lambda_{n+1}}{\Lambda_n}\Big)^{1/2} \leq \frac{(n+2)^\alpha(n+1)^2}{\Lambda_{n+1}} - \frac{(n+1)^\alpha n^2}{\Lambda_n} + 2n. \tag{5.13}$$

We now show for $1 \leq \alpha \leq 3$, we have

$$\frac{(n+2)^\alpha(n+1)^2}{\Lambda_{n+1}} - \frac{(n+1)^\alpha n^2}{\Lambda_n} \geq 1+\alpha. \tag{5.14}$$

We recast this as

$$(n+2)^\alpha(n+1)^2 - (n+1)^\alpha n^2 \geq (1+\alpha)(n+1)^\alpha + (1+\alpha)\Lambda_n + \frac{(n+1)^{2\alpha}n^2}{\Lambda_n}. \tag{5.15}$$

We now regard Λ_n as a variable on the right-hand side expression above and it is easy to see this is a convex function with the unique critical point being $n(n+1)^\alpha/\sqrt{1+\alpha}$. Note that we have $\sum_{i=1}^n i^\alpha \leq n(n+1)^\alpha/(1+\alpha)$ for $\alpha \geq 1$ (this follows from [5, Lemma 8]). It follows that it suffices to establish (5.15) with Λ_n replaced by the lower bound given in (5.11). Equivalently, we can then multiply both sides of (5.14) by Λ_n and in the resulting expression replace the values of Λ_n/Λ_{n+1} and Λ_n by the values given by the right-hand side expressions of (5.10) and (5.11) respectively. Then after some simplifications and on setting $x = 1/n$, we see that inequality (5.14) is a consequence of inequality (5.12) for $0 \leq x \leq 1$. Substituting (5.14) in (5.13) and squaring both sides, we find that it suffices to show (5.9) and Lemma 5.2 now leads to the assertion of the theorem. □

We now apply our results above to prove the following

Theorem 5.2. *Let* $\mathbf{x}$ *be a non-negative non-increasing sequence,* $0 < p \leq 1$, $\alpha \geq 3, (\alpha+1)p > 2$, *then*

$$\sum_{j=1}^\infty \Big(\sum_{k=1}^j \frac{k^\alpha}{\sum_{i=1}^j i^\alpha} x_k\Big)^p \leq \frac{(1+\alpha)p}{(1+\alpha)p-1}||\mathbf{x}||_p^p. \tag{5.16}$$

The constant is best possible. The above inequality also holds when $1 < \alpha \leq 3, 1/(1+\alpha) < p \leq 1/2$ *with the best possible constant* $\sum_{j=1}^\infty \left(\sum_{i=1}^j i^\alpha\right)^{-p}$.

Proof. The second assertion of the theorem is a direct consequence of Theorems 1.1 and 5.1. To prove the first assertion of the theorem, we let $\Lambda_{n,\alpha} = \sum_{i=1}^n i^\alpha$ and Theorem 1.1 implies that the best constant in (5.16) is given by

$$1 + \sup_r r^{-1} \sum_{j=r+1}^\infty \Big(\frac{\Lambda_{r,\alpha}}{\Lambda_{j,\alpha}}\Big)^p \leq 1 + \sup_r r^{-1} \sum_{j=r+1}^\infty \Big(\frac{\Lambda_{r,1}}{\Lambda_{j,1}}\Big)^{\frac{(1+\alpha)p}{2}} = 1 + \sup_r b_r,$$

by Lemma 5.1. We want to show (b_r) is increasing and by Lemma 7 of [5] with $x_n = \Lambda_{n,1}^{-(1+\alpha)p/2}$ there, it suffices to show $1 + n(\Lambda_{n+1,1}/\Lambda_{n,1})^{(1+\alpha)p/2} \geq (n+1)(\Lambda_{n+2,1}/\Lambda_{n+1,1})^{(1+\alpha)p/2}$ for $n \geq 1$

and one sees easily that it suffices to establish this for $(1+\alpha)p = 2$, in which case the inequality becomes an identity. It follows that $\sup_r b_r = \lim_{r\to+\infty} b_r$ and we note that

$$b_r = \sum_{k=1}^{\infty}\frac{1}{r}\sum_{i=1}^{r}\Big(\frac{r(r+1)}{(kr+i)(kr+i+1)}\Big)^{\frac{(1+\alpha)p}{2}} = \sum_{k=1}^{\infty}\frac{1}{r}\sum_{i=1}^{r}\Big((k+i/r)^{-(1+\alpha)p}+O(1/r)\Big).$$

It follows that as $r \to +\infty$, the inner sum of the last expression above approaches the value $(k^{1-(\alpha+1)p}-(k+1)^{1-(\alpha+1)p})/((1+\alpha)p-1)$ so that $\lim_{r\to+\infty} b_r = 1/((1+\alpha)p-1)$. We then deduce that the best constant in (5.16) is $\leq (1+\alpha)p/((1+\alpha)p-1)$. On the other hand, the first inequality of [13, (1.3)] implies that $\Lambda_{r,\alpha}/\Lambda_{j,\alpha} \geq (r/j)^{1+\alpha}$ when $j \geq r$ so that Corollary 5.2 implies that the best constant in (5.16) is $\geq (1+\alpha)p/((1+\alpha)p-1)$. This now completes the proof. □

We now return to the question of determining the monotonicity of the sequence given in (5.4) for $\Lambda_n = \sum_{i=1}^n i^\alpha$ and note that the most interesting case here is $0 < \alpha < 1 < p$ (see [5, p. 65]), in view of the connection to the open problem of determining the lower bounds of the weighted mean matrices generated by $\lambda_n = n^\alpha, 0 < \alpha < 1$ acting on non-increasing non-negative sequences in l^p when $p \geq 1$. In what follows, we shall give a partial solution to this and we point out here that we have not tried to optimize the choice of the auxiliary function appearing in the proof of Theorem 5.3 and one may be able to obtain better lower bounds for α appearing in Theorem 5.3 as well as Corollary 5.3.

We now prove a few lemmas:

Lemma 5.4. *Let* $\Lambda_n = \sum_{i=1}^n i^\alpha$. *For* $0.14 \leq \alpha \leq 1$, $n \geq 1$, *we have*

$$\frac{n(n+1)^{2\alpha}}{\Lambda_n^2} - \frac{(n+1)(n+2)^{2\alpha}}{\Lambda_{n+1}^2} - \frac{0.94(1+\alpha)}{(n+1)^2} \geq 0. \tag{5.17}$$

Proof. We first prove inequality (5.17) holds when $n = 1$ for all $0 \leq \alpha \leq 1$. In fact we shall prove the following stronger inequality:

$$2^{2\alpha} - \frac{2\cdot 3^{2\alpha}}{(1+2^\alpha)^2} - \frac{1+\alpha}{2} \geq 0.$$

Now using the bound $1+2^\alpha \geq 2^{1+\alpha/2}$, we see that the above inequality is a consequence of the following inequality:

$$2^{2\alpha+1} - (9/2)^\alpha - 1 - \alpha \geq 0.$$

It is easy to show that the left-hand side expression above, as a function of α, $0 \leq \alpha \leq 1$, is convex and increasing and as it takes the value 0 at $\alpha = 0$, this completes the proof for the case $n = 1$ of (5.17).

Now we assume $n \geq 2$ and note that the reversed inequalities (5.10) and (5.11) are still valid when $0 \leq \alpha \leq 1$ and it is easy to see, using the reversed inequality of (5.11) for $0 \leq \alpha \leq 1$, that the left-hand side expression of (5.17) is a decreasing function of Λ_n and hence it suffices to establish (5.17) on multiplying both sides of (5.21) by Λ_n^2 and in the resulting expression replacing the values of Λ_n/Λ_{n+1} and Λ_n by the values given by the right-hand side expressions of (5.10) and (5.11) respectively. We are now led to show the following inequality:

$$n(n+1)^{2\alpha} \geq (n+1)(n+2)^{2\alpha}\Big(\frac{n}{n+2}\Big)^{1+\alpha} + \frac{0.94(1+\alpha)}{(n+1)^2}n^{1+\alpha}(n+1)^{2\alpha}\Big((n+2)^{(\alpha+1)/2} - n^{(\alpha+1)/2}\Big)^{-2}.$$

After some simplifications and on setting $x = 1/n$, we can recast the above inequality as

$$1 \geq (1+x)^{1-2\alpha}(1+2x)^{\alpha-1} + \frac{0.94(1+\alpha)x^3}{(1+x)^2}\Big((1+2x)^{(\alpha+1)/2} - 1\Big)^{-2}. \tag{5.18}$$

By Hadamard's inequality, which asserts for a continuous convex function $h(u)$ on $[a, b]$,

$$\frac{1}{b-a}\int_a^b h(u)du \geq h(\frac{a+b}{2}),$$

we see that

$$(1+2x)^{(\alpha+1)/2}-1=\frac{1+\alpha}{2}\int_0^{2x}(1+t)^{(\alpha-1)/2}dt \geq (1+\alpha)x(1+x)^{(\alpha-1)/2}.$$

It suffices to prove (5.18) with $(1+2x)^{(\alpha+1)/2}-1$ replaced by this lower bound above which leads to the following inequality (with $x = 1/n$) for $0 \leq x \leq 1/2$:

$$1 \geq \frac{1+x}{1+2x}\Big(\frac{1+2x}{(1+x)^2}\Big)^{\alpha}+\frac{0.94x(1+x)^{-1-\alpha}}{(1+\alpha)}. \tag{5.19}$$

Note that we have

$$\begin{aligned} & \frac{1+x}{1+2x}\Big(\frac{1+2x}{(1+x)^2}\Big)^{\alpha}+\frac{0.94x(1+x)^{-1-\alpha}}{(1+\alpha)} \\ = & \frac{1+x}{1+2x}\Big(\frac{1+2x}{(1+x)^2}\Big)^{\alpha}+\frac{x}{1+2x}\Big(\Big(\frac{0.94(1+2x)}{(1+\alpha)(1+x)^{1+\alpha}}\Big)^{1/\alpha}\Big)^{\alpha} \\ \leq & \Big(\frac{1+x}{1+2x}\cdot\frac{1+2x}{(1+x)^2}+\frac{x}{1+2x}\cdot\Big(\frac{0.94(1+2x)}{(1+\alpha)(1+x)^{1+\alpha}}\Big)^{1/\alpha}\Big)^{\alpha}. \end{aligned}$$

Hence it suffices to show the last expression above is ≤ 1, which is equivalent to showing for $\alpha \geq 0.14, 0 \leq x \leq 1/2$,

$$(1+\alpha)(1+x)-0.94(1+2x)^{1-a} \geq 0. \tag{5.20}$$

To see this, observe that the left-hand side expression above is an increasing function of α, hence it suffices to check the above inequality for $\alpha = 0.14$, in which case we also observe that the left-hand side expression above is a convex function of x and its derivative at $x = 1/2$ is negative. It follows that one only needs to check the case when $x = 1/2$ and one checks easily that (5.20) holds in this case. This now establishes inequality (5.19) and hence completes the proof. □

Lemma 5.5. *Let* $\Lambda_n = \sum_{i=1}^n i^{\alpha}$. *For* $0.14 \leq \alpha \leq 1$, $n \geq 1$, *we have*

$$\frac{2n(n+1)^{\alpha}}{\Lambda_n}-\frac{2(n+1)(n+2)^{\alpha}}{\Lambda_{n+1}}+\frac{0.94(1+\alpha)}{(n+1)^2} \geq 0. \tag{5.21}$$

Proof. We first prove inequality (5.21) holds when $n = 1$ for all $0 \leq \alpha \leq 1$, in which case the inequality becomes

$$2^{\alpha+1}-\frac{4\cdot 3^{\alpha}}{1+2^{\alpha}}+\frac{0.94(1+\alpha)}{4} \geq 0.$$

Now using the bound $1 + 2^{\alpha} \geq 2^{1+\alpha/2}$, we see that the above inequality is a consequence of the following inequality:

$$2^{\alpha+1}-2\cdot(3/\sqrt{2})^{\alpha}+\frac{0.94(1+\alpha)}{4} \geq 0.$$

It is easy to show that the left-hand side expression above, as a function of α, $0 \leq \alpha \leq 1$, is concave so that it suffices to check its values at $\alpha = 0$ and $\alpha = 1$, in both cases the above inequality can be verified easily and this completes the proof for the case $n = 1$ of (5.21).

Now assume $n \geq 2$ and we recast inequality (5.21) as

$$\frac{2n(n+1)^{2\alpha}}{\Lambda_n}+\frac{0.94(1+\alpha)}{(n+1)^2}\Lambda_n+2n(n+1)^{\alpha}-2(n+1)(n+2)^{\alpha}+\frac{0.94(1+\alpha)}{(n+1)^2}(n+1)^{\alpha} \geq 0.$$

We now regard Λ_n as a variable on the left-hand side expression above and it is easy to see this is a convex function with the unique critical point being $\sqrt{(2n)}(n+1)^{\alpha+1}/\sqrt{0.94(1+\alpha)}$.

Note that the reversed inequalities (5.10) and (5.11) are still valid when $0 \leq \alpha \leq 1$ and we want to show first that the upper bound given in the reversed inequality in (5.11) for Λ_n is no greater than $\sqrt{(2n)}(n+1)^{\alpha+1}/\sqrt{0.94(1+\alpha)}$. In fact it suffices to show it is no greater than $\sqrt{2}n(n+1)^{\alpha+1/2}/\sqrt{1+\alpha}$, which is equivalent to showing the following inequality

$$n^{(\alpha-1)/2} \leq \frac{\sqrt{2}}{\sqrt{1+\alpha}}(n+1)^{1/2}\Big((n+2)^{(\alpha+1)/2} - n^{(\alpha+1)/2}\Big). \tag{5.22}$$

Note that it follows from the mean value theorem, we have $(n+2)^{(\alpha+1)/2} - n^{(\alpha+1)/2} \geq (1+\alpha)(n+2)^{(\alpha-1)/2}$. Using this in (5.22), we see that it remains to show

$$(1+2/n)^{(1-\alpha)/2} \leq \sqrt{2(1+\alpha)}(n+1)^{1/2}.$$

But we have $(1+2/n)^{(1-\alpha)/2} \leq (1+2/n)^{1/2} \leq \sqrt{3}$ and on the other hand, we have $\sqrt{2(1+\alpha)}(n+1)^{1/2} \geq \sqrt{2}(1+1)^{1/2} = 2$ so (5.22) holds. This being given, it follows from our discussions above that in order for (5.21) to hold, it suffices to multiply both sides of (5.21) by Λ_n and in the resulting expression replace the values of Λ_n/Λ_{n+1} and Λ_n by the values given by the right-hand side expressions of (5.10) and (5.11) respectively. Then after some simplifications and on setting $x = 1/n$, we see that it suffices to show for $0 \leq x \leq 1/2$,

$$2 - 2(1+x)^{1-\alpha}(1+2x)^{(\alpha-1)/2} + \frac{0.94(1+\alpha)x^3}{(1+x)^2}\Big((1+2x)^{(1+\alpha)/2} - 1\Big)^{-1} \geq 0.$$

By the mean value theorem again, we see that $(1+2x)^{(1+\alpha)/2} - 1 \leq (1+\alpha)x$. Replacing this in the above inequality, we see that it suffices to show $h_\alpha(x^2/(1+x)^2) \geq 0$, where

$$h_\alpha(t) = 2 - 2(1-t)^{(\alpha-1)/2} + 0.94t.$$

As $h_\alpha(t)$ is a concave function of t, and note that $x^2/(1+x)^2 \leq 1/9$, in order for $h_\alpha(x^2/(1+x)^2) \geq 0$, it suffices to check $h_\alpha(0) \geq 0$ and $h_\alpha(1/9) \geq 0$. This leads to the condition $\alpha \geq 1 - 2\ln(1 + 0.94/18)/\ln(9/8) < 0.14$. This now completes the proof. □

Now we are ready to prove the following

Theorem 5.3. *For $0.14 \leq \alpha \leq 1$ and $p \geq 2$, the sequence defined in (5.3) for $\Lambda_n = \sum_{i=1}^n i^\alpha$ is increasing.*

Proof. By Lemma 7 of [5] (with $x_n = \Lambda_n^{-p}$ there), it suffices to prove inequality of (5.4) for $p = 2$, which is

$$1 + n\Big(1 + \frac{(n+1)^\alpha}{\Lambda_n}\Big)^2 - (n+1)\Big(1 + \frac{(n+2)^\alpha}{\Lambda_{n+1}}\Big)^2 \geq 0.$$

Expanding the squares, we can recast the above inequality as

$$\frac{2n(n+1)^\alpha}{\Lambda_n} - \frac{2(n+1)(n+2)^\alpha}{\Lambda_{n+1}} + \frac{n(n+1)^{2\alpha}}{\Lambda_n^2} - \frac{(n+1)(n+2)^{2\alpha}}{\Lambda_{n+1}^2} \geq 0.$$

The assertion of the theorem now follows by combining Lemma 5.4 and Lemma 5.5. □

We will now show the sequence defined in (5.3) for $\Lambda_n = \sum_{i=1}^n i^\alpha$ is increasing for all $0 \leq \alpha \leq 1$, provided p is large enough. We first need two lemmas:

Lemma 5.6. *For $n \geq 1, 0 \leq \alpha \leq 1$ and $p \geq 1$, the function*

$$f_n(x) = 1 + n\Big(1 + \frac{(n+1)^\alpha}{x}\Big)^p - (n+1)\Big(1 + \frac{(n+2)^\alpha}{(n+1)^\alpha + x}\Big)^p$$

is a decreasing function for $x \leq \frac{n^{(1+\alpha)/2}(n+1)^\alpha}{(n+2)^{(1+\alpha)/2} - n^{(1+\alpha)/2}}$.

Proof. We have

$$f_n'(x)=p(n+1)\Big(1+\frac{(n+2)^\alpha}{(n+1)^\alpha+x}\Big)^{p-1}\frac{(n+2)^\alpha}{((n+1)^\alpha+x)^2}-pn\Big(1+\frac{(n+1)^\alpha}{x}\Big)^{p-1}\frac{(n+1)^\alpha}{x^2}.$$

To show $f_n'(x)\leq 0$, it suffices to show the following inequalities:

$$\begin{aligned}1+\frac{(n+2)^\alpha}{(n+1)^\alpha+x} &\leq 1+\frac{(n+1)^\alpha}{x},\\ \frac{(n+1)(n+2)^\alpha}{((n+1)^\alpha+x)^2} &\leq \frac{n(n+1)^\alpha}{x^2}.\end{aligned}$$

It's also easy to see that one only needs to show the above inequalities for $x=\frac{n^{(1+\alpha)/2}(n+1)^\alpha}{(n+2)^{(1+\alpha)/2}-n^{(1+\alpha)/2}}$, in which case both inequalities are easy to prove and this completes the proof. □

Lemma 5.7. *For $n\geq 1, 0\leq\alpha\leq 1$, we have*

$$(n+1)^\alpha+\frac{n^{(1+\alpha)/2}(n+1)^\alpha}{(n+2)^{(1+\alpha)/2}-n^{(1+\alpha)/2}}\geq\frac{(n+1)^{(1+\alpha)/2}(n+2)^\alpha}{(n+3+1/n^2)^{(1+\alpha)/2}-(n+1)^{(1+\alpha)/2}}.$$

Proof. Let $x=1/n$, it is easy to see that we can recast the above inequality as $f(\alpha;x)\geq 0$ for $x=1/n$, where

$$f(\alpha;x)=(1+3x+x^3)^{(1+\alpha)/2}(1+x)^{(\alpha-1)/2}(1+2x)^{(1-\alpha)/2}-(1+x)^\alpha(1+2x)^{(1-\alpha)/2}-(1+2x)^{(1+\alpha)/2}+1.$$

We regard $f(\alpha;x)$ as a function of α and note that

$$\begin{aligned}&(1+2x)^{-(1+\alpha)/2}f'(\alpha;x)\\ =\ &\frac{1}{2}\ln\Big(\frac{(1+3x+x^3)(1+x)}{(1+2x)}\Big)\cdot\Big(\frac{1+3x+x^3}{1+x}\Big)^{1/2}\cdot\Big(\frac{(1+3x+x^3)(1+x)}{(1+2x)^2}\Big)^{\alpha/2}\\ &-\ln\Big(\frac{(1+x)}{(1+2x)^{1/2}}\Big)\cdot\Big(\frac{1+x}{1+2x}\Big)^\alpha-\ln(1+2x)^{1/2}\\ \geq\ &\frac{1}{2}\ln\Big(\frac{(1+3x+x^3)(1+x)}{(1+2x)}\Big)\cdot\Big(\frac{1+3x+x^3}{1+x}\Big)^{1/2}\cdot\Big(\frac{(1+3x+x^3)(1+x)}{(1+2x)^2}\Big)^{\alpha/2}\\ &-\ln\Big(\frac{(1+x)}{(1+2x)^{1/2}}\Big)-\ln(1+2x)^{1/2}.\end{aligned}$$

It's easy to see that when $0\leq x\leq 1/2$, we have $(1+3x+x^3)(1+x)\leq(1+2x)^2$ and one verifies directly that when $x=1$, the last expression above is ≥ 0 for either $\alpha=0,1$. Therefore, in order to show $f'(\alpha;x)\geq 0$ for $x=1/n$, it suffices to assume $0\leq x\leq 1/2$ and assume $\alpha=1$ in the last expression above. Therefore, it rests to show $h(x)\geq 0$ for $0\leq x\leq 1/2$, where

$$h(x)=\frac{1}{2}\ln\Big(\frac{(1+3x+x^3)(1+x)}{(1+2x)}\Big)\cdot\frac{1+3x+x^3}{1+2x}-\ln(1+x).$$

Direction calculation shows that

$$\frac{2(1+2x)^2}{1+3x^2+4x^3}h'(x)=\frac{x(-2+x+8x^2+6x^3)}{(1+x)(1+3x^2+4x^3)}+\ln\Big(\frac{(1+3x+x^3)(1+x)}{(1+2x)}\Big),$$

and the derivative of the last expression above equals $x^2h_1(x)/((1+x)(1+2x)(1+3x+x^3)(1+3x^2+4x^3)^2)$, where

$$h_1(x)=96x^8+292x^7+436x^6+592x^5+610x^4+603x^3+511x^2+258x+56\geq 0.$$

As it is easy to check $h'(0)=h(0)=0$, this now implies $h(x)\geq 0$ for $0\leq x\leq 1/2$ and it follows that $f(\alpha;x)$ is an increasing function of α for $x=1/n$. In order to completes the proof, it remains to show $f(0;x)\geq 0$ and we recast this as

$$(1+3x+x^3)^{1/2}(1+x)^{-1/2}+(1+2x)^{-1/2}\geq 2.$$

The above inequality can be verified by taking squares and this completes the proof. □

Now we are ready to prove the following

Theorem 5.4. *For $0\leq\alpha\leq 1$ and $p\geq 8/(1+\alpha)$, the sequence defined in (5.3) for $\Lambda_n=\sum_{i=1}^n i^\alpha$ is increasing.*

Proof. Let $\Lambda_n=\sum_{i=1}^n i^\alpha$ and it suffices to show inequality (5.4) for $p\geq 8/(1+\alpha)$. Note that in our case we can recast inequality (5.4) as $f_n(\Lambda_n)\geq 0$ where $f_n(x)$ is defined as in Lemma 5.6. It follows from the reversed inequality of (5.11) (note that it holds when $0\leq\alpha\leq 1$) and Lemma 5.6 that it suffices to show $f_n(\frac{n^{(1+\alpha)/2}(n+1)^\alpha}{(n+2)^{(1+\alpha)/2}-n^{(1+\alpha)/2}})\geq 0$. Equivalently, this is

$$1+n(\frac{n+2}{n})^{p(1+\alpha)/2}-(n+1)\Big(1+\frac{(n+2)^\alpha}{(n+1)^\alpha+\frac{n^{(1+\alpha)/2}(n+1)^\alpha}{(n+2)^{(1+\alpha)/2}-n^{(1+\alpha)/2}}}\Big)^p\geq 0.$$

We now apply Lemma 5.7 to see that it suffices to show

$$1+n(\frac{n+2}{n})^{p(1+\alpha)/2}-(n+1)(\frac{n+3+1/n^2}{n+1})^{p(1+\alpha)/2}\geq 0.$$

As $p\geq 8/(1+\alpha)$, it suffices to prove the above inequality with $p(1+\alpha)/2$ replaced by 4. In this case, on setting $x=1/n$, we can recast the above inequality as

$$(1+x)^3(x+(1+2x)^4)-(1+3x+x^3)^4=x^3(20+76x+60x^2-34x^3-20x^4-54x^5-4x^6-12x^7-x^9)\geq 0.$$

This now completes the proof. □

It follows readily from Theorem 1.1, Theorem 5.3 and Theorem 5.4 that we have the following

Corollary 5.3. *Let $\mathbf{x}$ be a non-negative non-increasing sequence, then for $p\geq 2$, $0.14\leq\alpha\leq 1$, or for $0\leq\alpha\leq 1$, $p\geq 8/(1+\alpha)$, we have*

$$\sum_{j=1}^\infty\Big(\sum_{k=1}^j\frac{k^\alpha}{\sum_{i=1}^j i^\alpha}x_k\Big)^p\geq\sum_{j=1}^\infty\Big(\frac{1}{\sum_{i=1}^j i^\alpha}\Big)^p||\mathbf{x}||_p^p.$$

The constant is best possible.

6. Applications of Theorem 1.1 to Nörlund matrices

It is asked in [20] to determine the lower bounds for Nörlund matrices and motivated by this, we apply a similar idea to that used in the proof of Theorem 4 in [3] to prove the following

Lemma 6.1. *Let $\mathbf{x}$ be a non-negative non-increasing sequence. Let $p\geq 1$ and let A be an infinite Nörlund matrix generated by (λ_j) with $\lambda_1>0$. Suppose that Λ_j/Λ_{j+1} is increasing for $j\geq 1$ and for any integer $k\geq 1, r\geq 1$, $\Lambda_k/\Lambda_{k+1}\geq\Lambda_{kr}/\Lambda_{(k+1)r}$. Then $||A\mathbf{x}||_p\geq\lambda||\mathbf{x}||_p$ with the best possible constant (provided that the infinite sum converges)*

$$\lambda^p=1+\sum_{j=2}^\infty(1-\frac{\Lambda_{j-1}}{\Lambda_j})^p.$$

Proof. Theorem 1.1 implies that $||A\mathbf{x}||_p \geq \lambda ||\mathbf{x}||_p$ with

$$\begin{aligned}\lambda^p &= \inf_r r^{-1}\sum_{j=1}^{\infty}(\sum_{k=1}^{\min(r,j)}\frac{\lambda_{j-k+1}}{\Lambda_j})^p = 1+\inf_r r^{-1}\sum_{j=r+1}^{\infty}(\sum_{k=1}^{r}\frac{\lambda_{j-k+1}}{\Lambda_j})^p \\ &= 1+\inf_r r^{-1}\sum_{j=r+1}^{\infty}(1-\frac{\Lambda_{j-r}}{\Lambda_j})^p = 1+\inf_r \sum_{k=1}^{\infty}a_k(r),\end{aligned}$$

where

$$a_k(r) = r^{-1}\sum_{j=kr+1}^{(k+1)r}(1-\frac{\Lambda_{j-r}}{\Lambda_j})^p.$$

It therefore remains to show that $a_k(r) \geq a_k(1)$. To show this, it suffices to show that for $k \geq 1, r \geq 1, kr+1 \leq j \leq (k+1)r$, we have

$$1-\frac{\Lambda_{j-r}}{\Lambda_j} \geq 1-\frac{\Lambda_k}{\Lambda_{k+1}}.$$

The assumption Λ_j/Λ_{j+1} is increasing for $j \geq 1$ implies that

$$1-\frac{\Lambda_{j-r}}{\Lambda_j} \geq 1-\frac{\Lambda_{(k+1)r-r}}{\Lambda_{(k+1)r}}.$$

This combines with the other assumption implies the assertion of the lemma. □

If we take $\Lambda_j = j^{\alpha}, \alpha > 0$ in Lemma 6.1, then the assumptions there are easily verified and we thus have

Corollary 6.1. *Let* $\mathbf{x}$ *be a non-negative non-increasing sequence,* $p > 1$*,* $\alpha > 0$*, then*

$$\sum_{j=1}^{\infty}\Big(\sum_{k=1}^{j}\frac{(j-k+1)^{\alpha}-(j-k)^{\alpha}}{j^{\alpha}}x_k\Big)^p \geq \sum_{j=1}^{\infty}\Big(\frac{j^{\alpha}-(j-1)^{\alpha}}{j^{\alpha}}\Big)^p||\mathbf{x}||_p^p.$$

The constant is best possible.

We remark here that when the assumptions of Lemma 6.1 are satisfied by some sequence (Λ_n), then the same assumptions are also satisfied by the sequence $(\sum_{i=1}^n \Lambda_i)$. To see this, we let $\Lambda'_n = \sum_{i=1}^n \Lambda_n$ and note that the fact $\Lambda'_n/\Lambda'_{n+1}$ is increasing follows from [13, Lemma 3.1]. To show $\Lambda'_k/\Lambda'_{k+1} \geq \Lambda'_{kr}/\Lambda'_{(k+1)r}$, we apply [13, Lemma 3.1] again to see that it suffices to show for $r \geq 1, n \geq 0$,

$$\frac{\Lambda_{n+1}}{\Lambda_{n+2}} \geq \frac{\sum_{i=rn+1}^{r(n+1)}\Lambda_i}{\sum_{i=r(n+1)+1}^{r(n+2)}\Lambda_i}.$$

The above inequality holds since by our assumptions for (Λ_n), we have for $1 \leq i \leq r$, $\Lambda_{rn+i}/\Lambda_{r(n+1)+i} \leq \Lambda_{rn+r}/\Lambda_{r(n+1)+r} \leq \Lambda_{n+1}/\Lambda_{n+2}$.

We now take $\Lambda'_j = \sum_{i=1}^j i^{\alpha}, \alpha \geq 0$ so that by our remark above and Corollary 6.1, we have

Corollary 6.2. *Let* $\mathbf{x}$ *be a non-negative non-increasing sequence,* $p > 1$*,* $\alpha \geq 0$*, then*

$$\sum_{j=1}^{\infty}\Big(\sum_{k=1}^{j}\frac{(j-k+1)^{\alpha}}{\sum_{i=1}^j i^{\alpha}}x_k\Big)^p \geq \sum_{j=1}^{\infty}\Big(\frac{j^{\alpha}}{\sum_{i=1}^j i^{\alpha}}\Big)^p||\mathbf{x}||_p^p.$$

The constant is best possible.

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References

[1] H. Alzer, Sharp bounds for the ratio of q-gamma functions, *Math. Nachr.*, **222** (2001), 5–14.
[2] S. Barza, L.-E. Persson and J. Soria, Sharp weighted multidimensional integral inequalities for monotone functions, *Math. Nachr.*, **210** (2000), 43–58.
[3] G. Bennett, Lower bounds for matrices, *Linear Algebra Appl.*, **82** (1986), 81–98.
[4] G. Bennett, Some elementary inequalities. II., *Quart. J. Math. Oxford Ser. (2)*, **39** (1988), 385–400.
[5] G. Bennett, Lower bounds for matrices. II., *Canad. J. Math.*, **44** (1992), 54–74.
[6] G. Bennett and K.-G. Grosse-Erdmann, Weighted Hardy inequalities for decreasing sequences and functions, *Math. Ann.*, **334** (2006), 489–531.
[7] G. Bennett and G. Jameson, Monotonic averages of convex functions, *J. Math. Anal. Appl.*, **252** (2000), 410–430.
[8] J. Bergh, Hardy's inequality—a complement, *Math. Z.*, **202** (1989), 147–149.
[9] J. Bergh, V. I. Burenkov and L.-E. Persson, Best constants in reversed Hardy's inequalities for quasimonotone functions, *Acta Sci. Math. (Szeged)*, **59** (1994), 221–239.
[10] J. Bergh, V. I. Burenkov and L.-E. Persson, On some sharp reversed Hölder and Hardy type inequalities, *Math. Nachr.*, **169** (1994), 19–29.
[11] V. I. Burenkov, On the exact constant in the Hardy inequality with $0 < p < 1$ for monotone functions, *Proc. Steklov Inst. Math.*, **194** (1993), 59–63.
[12] M. J. Carro, J. A. Raposo and J. Soria, Recent developments in the theory of Lorentz spaces and weighted inequalities, *Mem. Amer. Math. Soc.*, **187** (2007), 128pp.
[13] P. Gao, Sums of powers and majorization, *J. Math. Anal. Appl.*, **340** (2008), 1241-1248.
[14] P. Gao, On a result of Levin and Stečkin, arXiv:0902.4070.
[15] G. H. Hardy, J. E. Littlewood and G. Pólya, *Inequalities*, Cambridge Univ. Press, 1952.
[16] S.-Z. Lai, Weighted norm inequalities for general operators on monotone functions, *Trans. Amer. Math. Soc.*, **340** (1993), 811-836.
[17] V. I. Levin and S. B. Stečkin, Inequalities, *Amer. Math. Soc. Transl. (2)*, **14** (1960), 1–29.
[18] R. Lyons, A lower bound on the Cesàro operator, *Proc. Amer. Math. Soc.*, **86** (1982), 694.
[19] C. J. Neugebauer, Weighted norm inequalities for averaging operators of monotone functions, *Publ. Mat.*, **35** (1991), 429-447.
[20] B. E. Rhoades and P. Sen, Lower bounds for some factorable matrices, *Int. J. Math. Math. Sci.*, **2006** (2006), Art. ID 76135, 13 pp.
[21] V. Stepanov, The weighted Hardy's inequality for nonincreasing functions, *Trans. Amer. Math. Soc.*, **338** (1993), 173-186.

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