A best approximation for the difference of expressions related to the power means

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Abstract

Let $n$ be a positive integer, let $p > q$, and let $0 < a < b$.

It is proved that the maximum of

$$\frac{a_1^p + \cdots + a_n^p}{n} - \left(\frac{a_1^q + \cdots + a_n^q}{n}\right)^{\frac{p}{q}}$$

when $a_1, \ldots, a_n \in [a, b]$ is attained if and only if $k$ of the variables $a_1, \ldots, a_n$ are equal to $a$ and $n-k$ are equal to $b$, where $k$ is either

$$\left[\frac{b^q - D_{p,q}^q(a, b)}{b^q - a^q}\cdot n\right]$$

or

$$\left[\frac{b^q - D_{p,q}^q(a, b)}{b^q - a^q}\cdot n\right] + 1,$$

and $D_{p,q}(a, b)$ denotes the Stolarsky mean of $a$ and $b$. Moreover, if $n, p$ and $q$ are fixed, then

$$\lim_{b \searrow a} \frac{k}{n} = \frac{1}{2}.$$
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1 Introduction and main results

Given the positive real numbers $a$ and $b$ and the real numbers $p$ and $q$, the difference or Stolarsky mean $D_{p,q}(a,b)$ of $a$ and $b$ is defined by (see, for instance, [6] or [3]).

$$D_{p,q}(a,b) := \begin{cases} 
\frac{q(a^p - b^p)}{p(a^q - b^q)} \frac{1}{p-q} & \text{if } pq(p-q)(b-a) \neq 0, \\
\left(\frac{a^p - b^p}{p(ln a - ln b)}\right)^{\frac{1}{p}} & \text{if } p(a-b) \neq 0, q = 0, \\
\frac{q(ln a - ln b)}{(a^q - b^q)}^{-\frac{1}{q}} & \text{if } q(a-b) \neq 0, p = 0, \\
exp\left(-\frac{1}{p} + \frac{a^p ln a - b^p ln b}{a^p - b^p}\right) & \text{if } q(a-b) \neq 0, p = q, \\
(ab)^{\frac{1}{2}} & \text{if } a - b \neq 0, p = q = 0, \\
a & \text{if } a - b = 0.
\end{cases}$$

Note that $D_{2p,p}(a,b)$ is the power mean of order $p$ of $a$ and $b$:

$$D_{2p,p}(a,b) = M_p(a,b) := \begin{cases} 
\left(\frac{a^p + b^p}{2}\right)^{\frac{1}{p}} & \text{if } p \neq 0 \\
(ab)^{\frac{1}{2}} & \text{if } p = 0.
\end{cases}$$

The power mean can be defined not only for two numbers, but for any finite set of nonnegative real numbers. Given $a_1, \ldots, a_n \in [0, \infty[$, and
$p \in \mathbb{R}$, the power mean $M_p(a_1, \ldots, a_n)$ of $a_1, \ldots, a_n$ is defined by

$$M_p(a_1, \ldots, a_n) = \begin{cases} 
\left(\frac{a_1^p + \cdots + a_n^p}{n}\right)^{\frac{1}{p}} & \text{if } p \neq 0 \\
(a_1 \cdots a_n)^{\frac{1}{n}} & \text{if } p = 0 
\end{cases}$$

It is well known (see for instance [1], [5], [4] or [2]), that for fixed $a_1, \ldots, a_n$, the function $p \in \mathbb{R} \mapsto M_p(a_1, \ldots, a_n) \in \mathbb{R}$ is nondecreasing. Moreover, if $q < p$, then $M_q(a_1, \ldots, a_n) < M_p(a_1, \ldots, a_n)$, unless $a_1 = \cdots = a_n$. This result implies that for every $p > q$ one has

$$\frac{a_1^p + \cdots + a_n^p}{n} - \left(\frac{a_1^q + \cdots + a_n^q}{n}\right)^{\frac{p}{q}} \geq 0,$$

with equality if and only if $a_1 = \cdots = a_n$. Therefore, for fixed $p$ and $q$ such that $p > q$, the function $f : [0, \infty)^n \to \mathbb{R}$ defined by (1.1) $f(a_1, \ldots, a_n) =

\frac{a_1^p + \cdots + a_n^p}{n} - \left(\frac{a_1^q + \cdots + a_n^q}{n}\right)^{\frac{p}{q}},$

satisfies $f(a_1, \ldots, a_n) \geq 0$ for all $a_1, \ldots, a_n \in [0, \infty)$.

Having in mind that the minimum of $f$ over $[0, \infty)^n$ is 0 and it is attained when $a_1 = \cdots = a_n$, it is natural to ask when is attained the maximum of $f$. Since

$$\sup_{a_1, \ldots, a_n \in [0, \infty]} f(a_1, \ldots, a_n) = \infty,$$

this question is relevant only when all the variables $a_1, \ldots, a_n$ of $f$ are restricted to a compact interval $[a, b] \subseteq [0, \infty]$. The answer is given in the next theorem:
Theorem 1. Given the positive integer $n$, the real numbers $p > q > 0$ and $0 < a < b$, and the function $f : [a, b]^n \to \mathbb{R}$, defined by (1.1), the following assertions are true:

1°. The function $f$ attains its maximum if and only if

$$a_1 = \cdots = a_k = a \text{ and } a_{k+1} = \cdots = a_n = b,$$

where $k$ is either

$$\left[ \frac{b^q - D_{p,q}^q(a, b)}{b^q - a^q} \right] \cdot n$$

or

$$\left[ \frac{b^q - D_{p,q}^q(a, b)}{b^q - a^q} \right] + 1.$$

2°. If $n, p$ and $q$ are held fixed, then it holds that

$$\lim_{b \searrow a} \frac{k}{n} = \frac{1}{2}.$$

As an application of Theorem 1, we solve the following problem, (see [1], p.70-72): given the positive integer $n$, determine the smallest value of $\alpha$ such that

$$a_1^2 + \cdots + a_n^2 - \left( \frac{a_1 + \cdots + a_n}{n} \right)^2 \leq \alpha \max_{1 \leq i \leq j \leq n} (a_i - a_j)^2$$

holds true for all positive real numbers $a_1, \ldots, a_n$.

Theorem 2. Given the positive integer $n$, the smallest value of $\alpha$ such that (1.2) holds true for all positive real numbers $a_1, \ldots, a_n$ is

$$\alpha = \left[ \frac{n}{2} \right] \left[ \frac{n + 1}{2} \right].$$
2 Proofs

Proof of Theorem 1

1° Since $f$ is continuous on the compact interval $[a, b]^n$, there is a point $(\overline{a}_1, \ldots, \overline{a}_n) \in [a, b]^n$ at which $f$ attains its maximum. If $(\overline{a}_1, \ldots, \overline{a}_n)$ is an interior point of $[a, b]^n$, then

$$\frac{\partial f}{\partial a_i}(\overline{a}_1, \ldots, \overline{a}_n) = 0$$

for all $i = 1, \ldots, n$.

Therefore

$$\frac{p}{n} \cdot \overline{a}_i^{p-1} - \frac{q}{n} \cdot \frac{q \overline{a}_i^{q-1}}{n} \left( \frac{\overline{a}_1^q + \cdots + \overline{a}_n^q}{n} \right)^{\frac{p}{q} - 1} = 0,$$

whence

$$\overline{a}_i = \left( \frac{\overline{a}_1^q + \cdots + \overline{a}_n^q}{n} \right)^{\frac{1}{q}}$$

for all $i = 1, \ldots, n$.

But, if $\overline{a}_1 = \cdots = \overline{a}_n$, then $f(\overline{a}_1, \ldots, \overline{a}_n) = 0$ and $f$ cannot attain its maximum at $(\overline{a}_1, \ldots, \overline{a}_n)$. Consequently, $(\overline{a}_1, \ldots, \overline{a}_n)$ lies on the boundary of $[a, b]^n$. Taking into account that $f$ is symmetric in its variables, and that

$$f(\underbrace{a, \ldots, a}_n) = f(\underbrace{b, \ldots, b}_n) = 0,$$

it follows that there exist $k \in \{1, \ldots, n-1\}$ and $l \in \{k + 1, \ldots, n\}$ such that

$$\overline{a}_1 = \cdots = \overline{a}_k = a \quad \text{and} \quad \overline{a}_{k+1} = \cdots = \overline{a}_l = b.$$

If $l < n$ then $\overline{a}_{l+1}, \ldots, \overline{a}_n \in (a, b)$. 
We consider the function $g_l : (a, b)^{n-l} \to \mathbb{R}$, defined by

$$g_l(a_{l+1}, \ldots, a_n) = f(a, \ldots, a, b, \ldots, b, a_{l+1}, \ldots, a_n).$$

Note that $g_l$ attains its maximum at $(\overline{a}_{l+1}, \ldots, \overline{a}_n)$, which is an interior point of $[a, b]^{n-l}$.

By virtue of the Fermat theorem, we deduce that for all $i \in l + 1, \ldots, n$ one has

$$\frac{\partial g_l}{\partial a_i}(\overline{a}_{l+1}, \ldots, \overline{a}_n) = 0$$

hence

$$\frac{p}{n} \cdot \frac{(n-l)^{q-1}}{q} \left(\frac{\overline{a}_1^q + \cdots + \overline{a}_n^q}{n}\right)^{\frac{p}{q} - 1} = 0,$$

where $c$ satisfies

$$c^q = \frac{ka^q + (l-k)b^q + (n-l)c^q}{n}.$$

A simple computation shows that

$$c^q = \frac{ka^q + (l-k)b^q}{l}.$$

We have

$$g_l(c, \ldots, c) = \frac{ka^p + (l-k)b^p + (n-l)c^p}{n} - c^p$$

$$= \frac{k(a^p - b^p) + l \left[b^p - \left(b^q - \frac{k}{l}(b^q - a^q)^{\frac{p}{q}}\right)^{\frac{q}{p}}\right]}{n} = M_k.$$
Consider now the function $h : [k + 1, n] \to \mathbb{R}$, defined by

$$h(x) = x \left[ b^p - \left( b^q - \frac{k}{x} (b^q - a^q) \right)^{\frac{p}{q}} \right].$$

We claim that $h$ is increasing. Indeed, one has

$$h'(x) = \left[ b^p - \left( b^q - \frac{k}{x} (b^q - a^q) \right)^{\frac{p}{q}} \right]$$

$$= b^p - b^q - \frac{k}{x} (b^q - a^q) \left( b^q - \frac{k}{x} (b^q - a^q) \right)^{\frac{p}{q} - 1} \cdot \frac{k}{x} \cdot \frac{1}{x^2} (b^q - a^q).$$

Let $\alpha = b^q - a^q$, $\eta = \frac{k}{x} < 1$, and let

$$\varphi(\eta) := b^p - (b^q - \alpha \eta)^{\frac{p}{q}} - \frac{p}{q} \cdot \frac{k}{x} (b^q - a^q) \left( b^q - \frac{k}{x} (b^q - a^q) \right)^{\frac{p}{q} - 1}.$$

Since

$$a^q < b^q - \alpha \eta = b^q - \frac{k}{x} (b^q - a^q) < b^q,$$

it follows that $h'(x) > 0$. Therefore $h$ is increasing as claimed. Finally, we get

$$\max g_l = \frac{k (a^p - b^p) + h(l)}{n} \leq \frac{k (a^p - b^p) + h(n)}{n}$$

$$= \frac{k a^p + (n - k) b^p}{n} - \left[ \frac{ka^q + (n - k) b^q}{n} \right]^{\frac{p}{q}} = M_k.$$

Our problem is now reduced to the one of finding the $k \in [0, \ldots, n]$ for which $M_k$ attains its maximum, where

$$M_k = \frac{a^p - b^p}{n} k + b^p - \left( \frac{a^q - b^q}{n} k + b^q \right)^{\frac{p}{q}}.$$
To do this, we consider the function $g : [0, n] \to \mathbb{R}$, defined by

$$g(x) = \frac{a^p - b^p}{n} x + b^p - \left( \frac{a^q - b^q}{n} x + b^q \right)^{\frac{p}{q}}.$$

It is clear that our function satisfies

$$g(k) := M_k, \text{ for } k \in [0, \ldots, n]$$

We find first the extremal points of $g$ which lie in the interior of the interval $[0, n]$.

In these points, due to the Theorem of Fermat we have that

$$g'(x) = \frac{a^p - b^p}{n} - \frac{p}{q} \cdot \frac{a^q - b^q}{n} \left( \frac{a^q - b^q}{n} x + b^q \right)^{\frac{p}{q} - 1} = 0,$$

that is

$$\frac{q}{p} \frac{(a^p - b^p)}{(a^q - b^q)} = \left( \frac{a^q - b^q}{n} x + b^q \right)^{\frac{p}{q} - 1}$$

hence, as we have seen in the definition of the Stolarski mean that we are using in our case,

$$D_{p,q}^{p-q}(a, b) = \left[ \frac{a^q - b^q}{n} x + b^q \right]^{\frac{p-q}{q}}$$

and from here,

$$x^* = \frac{b^q - D_{p,q}^{p-q}(a, b)}{b^q - a^q} \cdot n,$$

is the only extremal point contained in the interior of $[0, n]$.

Taking into account that the second derivative of $g$ is :

$$g''(x) = -\frac{p}{q} \cdot \left( \frac{p}{q} - 1 \right) \cdot \left( \frac{a^q - b^q}{n} x + b^q \right)^{\frac{p}{q} - 2} < 0,$$
we get that the extremal point $x^*$ we have just found, is a point of maximum for $g$.

This relation also tells us that the function $g'$ is decreasing on the interval $(0,n)$. Because $g'(x^*) = 0$, we get then that $g'(y) > 0$ for $y \in (0,x^*)$, and also that $g'(y) < 0$ for $y \in (x^*,n)$.

Finally this means that $g$ is increasing on $(0,x^*)$ and decreasing on $(x^*,n)$.

We conclude that:

$$g(1) < g(2) < \cdots < g([x^*])$$

and

$$g(n) < g(n-1) < \cdots < g([x^*] + 1).$$

From here we get that in order to obtain the maximum for $M_k$, $k$ has to take one of the values $[x^*]$ and $[x^*] + 1$, where

$$x^* = \frac{b^q - D_{p,q}^q(a,b)}{b^q - a^q} \cdot n.$$

**Remark.** Because in our case

$$pq(p - q)(b - a) \neq 0,$$

the Stolarsky mean has the property that $a < D_{p,q}^q(a,b) < b$, so we clearly have that $0 < x < n$.

2° Let

$$\ell = \lim_{b \searrow a} \frac{k}{n} = \lim_{b \searrow a} \frac{b^q - \left[ \frac{q(b^p - a^p)}{p(b^q - a^q)} \right]^\frac{q}{p-q}}{b^q - a^q}.$$

Using l'Hospital’s rule we get
\[
\ell = \lim_{b \downarrow a} \frac{qb^{q-1} - \frac{q}{p-q} \left( \frac{q(b^p - a^p)}{p(b^q - a^q)} \right)^{\frac{p}{p-q} - 1} \cdot \frac{q}{p}}{qb^{q-1}}
\]

But
\[
\lim_{b \downarrow a} \frac{b^p - a^p}{b^q - a^q} = \frac{p}{q} \cdot a^{p-q},
\]
so,
\[
\ell = \lim_{b \downarrow a} \left\{ 1 - \frac{q}{(p-q)p} a^{2q-p} \cdot \ell \right\}
\]

where
\[
\overline{\ell} = \lim_{b \downarrow a} \frac{p(p-q)b^{p-1} - p(p-q)b^{p-q-1}a^q}{b^{p-1}(b^q - a^q)^2}
\]
\[
= \lim_{b \downarrow a} \frac{p(p-q)b^{p-q} - p(p-q)b^{p-2q}a^q}{2q(b^q - a^q)^2}
\]
\[
= \lim_{b \downarrow a} \frac{p(p-q)(p-q)b^{p-q} - p(p-q)(p-2q)b^{p-2q-1}a^q}{2q^2b^{p-1}}
\]
\[
= \lim_{b \downarrow a} \frac{p(p-q)(p-q)b^{p-q} - p(p-q)(p-2q)b^{p-2q-1}a^q}{2q^2b^{p-1}}
\]
\[
= \lim_{b \downarrow a} \frac{p(p-q)(p-q)b^{p-q} - p(p-q)(p-2q)b^{p-2q-1}a^q}{2q^2b^{p-1}}
\]
\[
= \frac{p}{2q^2}(p-q)qa^{p-2q} = \frac{1}{2}(p-q)\frac{p}{q}.
\]

Finally,
\[
\ell = 1 - \frac{q}{(p-q)p} \cdot \frac{1}{2}(p-q)\frac{p}{q} = \frac{1}{2}.
\]

In conclusion, \(\lim_{b \downarrow a} \frac{k}{\eta} = \frac{1}{2}\), for any \(p > q\).

**Remark.** The proofs are the same in the dcase when \(0 > q > p\). In this case we have the next theorem.
Theorem 3. Given the positive integer \( n \), the real numbers \( |p| > |q| > 0 \) and \( 0 < a < b \), and the function \( f : [a, b]^n \to \mathbb{R} \), defined by (1.1), the following assertions are true:

1°. The function \( f \) attains its maximum if and only if

\[
a_1 = \cdots = a_k = a \quad \text{and} \quad a_{k+1} = \cdots = a_n = b, \quad \text{where} \ k \ \text{is either} \ \left[ \frac{b^q - D_{p,q}^q(a, b)}{b^q - a^q} \cdot n \right] + 1.
\]

or

\[
\left[ \frac{b^q - D_{p,q}^q(a, b)}{b^q - a^q} \cdot n \right] + 1.
\]

2°. If \( n, p \) and \( q \) are held fixed, then it holds that

\[
\lim_{b \searrow a} \frac{k}{n} = \frac{1}{2}.
\]

Remark. From the monotonicity of function \( p \mapsto M_p(a_1, \ldots, a_n) \), we could see that for \( p > q \):

\[
\left( \frac{a_1^p + \cdots + a_n^p}{n} \right)^{\frac{1}{p}} \geq \left( \frac{a_1^q + \cdots + a_n^q}{n} \right)^{\frac{1}{q}},
\]

with equality if and only of \( a_1 = \cdots = a_n \). It follows clearly that the inequality mentioned before, is equivalent to:

\[
\frac{a_1^p + \cdots + a_n^p}{n} - \left( \frac{a_1^q + \cdots + a_n^q}{n} \right)^{\frac{p}{q}} \geq 0.
\]

Proof of Theorem 2

Considering \( p = 2, \ q = 1 \) in Theorem 1, we can see that:

\[
D_{2,1}(a, b) = \frac{1}{2} \cdot \frac{b^2 - a^2}{b - a} = \frac{1}{2} (b + a)
\]
and it follows that
\[
\frac{k}{n} = \frac{b - \frac{1}{2}(b + a)}{b - a} = \frac{1}{2}.
\]

From here, we get immediately the best constant \(\alpha\) for which:
\[
\frac{a_1^2 + \cdots + a_n^2}{n} - \left(\frac{a_1 + \cdots + a_n}{n}\right)^2 \leq \alpha \max_{1 \leq i \leq j \leq n} (a_i - a_j)^2.
\]

Following the steps mentioned before, the function gets the maximum for \(a_1 = \cdots = a_k = a,\)
\[a_{k+1} = \cdots = a_n = b,\]
where \(k = \left[\frac{n}{2}\right],\) or \(k = \left[\frac{n+1}{2}\right].\)

We have that
\[
\frac{a_1^2 + \cdots + a_n^2}{n} - \left(\frac{a_1 + \cdots + a_n}{n}\right)^2 \leq \frac{(b - a)^2}{n^2} (nk - k^2).
\]

So the best constant \(\alpha\) will be
\[
\alpha = \left[\frac{n}{2}\right] \left[\frac{n+1}{2}\right].
\]

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