A Best Approximation for the Difference of Expressions Related to the Power Means

A best approximation for the difference of expressions related to the power means

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Abstract

Let n be a positive integer, let $p > q$, and let $0 < a < b$.

It is proved that the maximum of

$$\frac{a_1^p + \cdots + a_n^p}{n} - \left(\frac{a_1^q + \cdots + a_n^q}{n}\right)^{\frac{p}{q}}$$

when $a_1, \ldots, a_n \in [a, b]$ is attained if and only if k of the variables $a_1, \ldots, a_n$ are equal to a and $n - k$ are equal to b, where k is either

$$\left[\frac{b^q - D_{p,q}^q(a,b)}{b^q - a^q} \cdot n\right]$$

or

$$\left[\frac{b^q - D_{p,q}^q(a,b)}{b^q - a^q} \cdot n\right] + 1,$$

and $D_{p,q}(a, b)$ denotes the Stolarsky mean of a and b. Moreover, if n, p and q are fixed, then

$$\lim_{b \searrow a} \frac{k}{n} = \frac{1}{2}.$$

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1 Introduction and main results

Given the positive real numbers a and b and the real numbers p and q, the difference or Stolarsky mean $D_{p,q}(a,b)$ of a and b is defined by (see, for instance, [6] or [3]).

$$D_{p,q}(a,b) := \begin{cases} \left(\dfrac{q(a^p-b^p)}{p(a^q-b^q)}\right)^{\frac{1}{p-q}} & \text{if } pq(p-q)(b-a)\neq 0, \\ \left(\dfrac{a^p-b^p}{p(\ln a-\ln b)}\right)^{\frac{1}{p}} & \text{if } p(a-b)\neq 0, q=0, \\ \left(\dfrac{q(\ln a-\ln b)}{(a^q-b^q)}\right)^{-\frac{1}{q}} & \text{if } q(a-b)\neq 0, p=0, \\ exp\left(-\frac{1}{p}+\frac{a^p\ln a-b^p\ln b}{a^p-b^p}\right) & \text{if } q(a-b)\neq 0, p=q, \\ (ab)^{\frac{1}{2}} & \text{if } a-b\neq 0, p=q=0, \\ a & \text{if } a-b=0. \end{cases}$$

Note that $D_{2p,p}(a,b)$ is the power mean of order p of a and b:

$$D_{2p,p}(a,b) = M_p(a,b) := \begin{cases} \left(\dfrac{a^p+b^p}{2}\right)^{\frac{1}{p}} & \text{if } p\neq 0 \\ (ab)^{\frac{1}{2}} & \text{if } p=0. \end{cases}$$

The power mean can be defined not only for two numbers, but for any finite set of nonnegative real numbers. Given $a_1,\ldots,a_n \in [0,\infty[$, and

$p \in \mathbb{R}$, the power mean $M_p(a_1, \ldots, a_n)$ of $a_1, \ldots, a_n$ is defined by

$$M_p(a_1,\ldots,a_n) = \begin{cases} \left(\dfrac{a_1^p + \cdots + a_n^p}{n}\right)^{\frac{1}{p}} & \text{if } p \neq 0 \\ \\ (a_1 \cdots a_n)^{\frac{1}{n}} & \text{if } p = 0 \end{cases}$$

It is well known (see for instance [1],[5], [4] or [2]), that for fixed $a_1, \ldots, a_n$, the function $p \in \mathbb{R} \mapsto M_p(a_1, \ldots, a_n) \in \mathbb{R}$ is nondecreasing. Moreover, if $q < p$, then $M_q(a_1, \ldots, a_n) < M_p(a_1, \ldots, a_n)$, unless $a_1 = \cdots = a_n$. This result implies that for every $p > q$ one has

$$\frac{a_1^p + \cdots + a_n^p}{n} - \left(\frac{a_1^q + \cdots + a_n^q}{n}\right)^{\frac{p}{q}} \geq 0,$$

with equality if and only if $a_1 = \cdots = a_n$. Therefore, for fixed p and q such that $p > q$, the function $f : [0, \infty]^n \to \mathbb{R}$ defined by (1.1)$f(a_1, \ldots, a_n)$=

$$\frac{a_1^p + \cdots + a_n^p}{n} - \left(\frac{a_1^q + \cdots + a_n^q}{n}\right)^{\frac{p}{q}},$$

satisfies $f(a_1, \ldots, a_n) \geq 0$ for all $a_1, \ldots, a_n \in [0, \infty)$.

Having in mind that the minimum of f over $[0, \infty)^n$ is 0 and it is attained when $a_1 = \cdots = a_n$, it is natural to ask when is attained the maximum of f. Since

$$\sup_{a_1,\ldots,a_n \in [0,\infty[} f(a_1, \ldots, a_n) = \infty,$$

this question is relevant only when all the variables $a_1, \ldots, a_n$ of f are restricted to a compact interval $[a, b] \subseteq [0, \infty[$. The answer is given in the next theorem:

Theorem 1. *Given the positive integer n, the real numbers $p > q > 0$ and $0 < a < b$, and the function $f : [a,b]^n \to \mathbb{R}$, defined by (1.1), the following assertions are true:*

1°. *The function f attains its maximum if and only if*

$a_1 = \cdots = a_k = a$ *and* $a_{k+1} = \cdots = a_n = b$, *where k is either*

$$\left[\frac{b^q - D_{p,q}^q(a,b)}{b^q - a^q} \cdot n\right]$$

or

$$\left[\frac{b^q - D_{p,q}^q(a,b)}{b^q - a^q} \cdot n\right] + 1.$$

2°. *If n,p and q are held fixed, then it holds that*

$$\lim_{b \searrow a} \frac{k}{n} = \frac{1}{2}.$$

As an application of Theorem 1, we solve the following problem, (see [1],p.70-72): given the positive integer n, determine the smallest value of α such that

(1.2)

$$\frac{a_1^2 + \cdots + a_n^2}{n} - \left(\frac{a_1 + \cdots + a_n}{n}\right)^2 \le \alpha \max_{1 \le i \le j \le n} (a_i - a_j)^2$$

holds true for all positive real numbers $a_1, \ldots, a_n$.

Theorem 2. *Given the positive integer n, the smallest value of α such that (1.2) holds true for all positive real numbers $a_1, \ldots, a_n$ is*

$$\alpha = \left[\frac{n}{2}\right]\left[\frac{n+1}{2}\right].$$

2 Proofs

Proof of Theorem 1

1° Since f is continuous on the compact interval $[a,b]^n$, there is a point $(\overline{a}_1, \ldots, \overline{a}_n) \in [a,b]^n$ at which f attains its maximum. If $(\overline{a}_1, \ldots, \overline{a}_n)$ is an interior point of $[a,b]^n$, then

$\frac{\partial f}{\partial a_i}(\overline{a}_1, \ldots, \overline{a}_n) = 0$ for all $i = 1, \ldots, n$.

Therefore

$$p \cdot \frac{\overline{a}_i{}^{p-1}}{n} - \frac{p}{q} \cdot \frac{q\overline{a}_i{}^{q-1}}{n} \left(\frac{\overline{a}_1{}^q + \cdots + \overline{a}_n{}^q}{n} \right)^{\frac{p}{q}-1} = 0,$$

whence

$$\overline{a}_i = \left(\frac{\overline{a}_1{}^q + \cdots + \overline{a}_n{}^q}{n} \right)^{\frac{1}{q}}$$

for all $i = 1, \ldots, n$.

But, if $\overline{a}_1 = \cdots = \overline{a}_n$, then $f(\overline{a}_1, \ldots, \overline{a}_n) = 0$ and f cannot attain its maximum at $(\overline{a}_1, \ldots, \overline{a}_n)$. Consequently, $(\overline{a}_1, \ldots, \overline{a}_n)$ lies on the boundary of $[a,b]^n$. Taking into account that f is symmetric in its variables, and that

$f(\underbrace{a, \ldots, a}_{n}) = f(\underbrace{b, \ldots, b}_{n}) = 0$, it follows that there exist $k \in \{1, \ldots, n-1\}$ and $l \in \{k+1, \ldots, n\}$ such that

$$\overline{a}_1 = \cdots = \overline{a}_k = a \qquad \text{and} \qquad \overline{a}_{k+1} = \cdots = \overline{a}_l = b.$$

If $l < n$ then $\overline{a}_{l+1}, \ldots, \overline{a}_n \in (a,b)$.

We consider the function $g_l : (a,b)^{n-l} \to \mathbb{R}$, defined by

$$g_l(a_{l+1}, \dots, a_n) = f(\underbrace{a, \dots, a}_{k}, \underbrace{b, \dots, b}_{l-k}, a_{l+1}, \dots, a_n).$$

Note that g_l attains its maximum at $(\overline{a}_{l+1}, \dots, \overline{a}_n)$, which is an interior point of $[a,b]^{n-l}$.

By virtue of the Fermat theorem, we deduce that for all $i \in l+1, \dots, n$ one has

$\frac{\partial g_l}{\partial a_i}(\overline{a}_{l+1}, \dots, \overline{a}_n) = 0$ for all $i = l+1, \dots, n$, that is

$$p \cdot \frac{\overline{a}_i{}^{p-1}}{n} - \frac{p}{q} \cdot \frac{q\overline{a}_i{}^{q-1}}{n} \left(\frac{\overline{a}_1{}^q + \cdots + \overline{a}_n{}^q}{n} \right)^{\frac{p}{q}-1} = 0,$$

hence

$$\overline{a}_i = \left(\frac{\overline{a}_1{}^q + \cdots + \overline{a}_n{}^q}{n} \right)^{\frac{1}{q}} = c,$$

where c satisfies

$$c^q = \frac{ka^q + (l-k)b^q + (n-l)c^q}{n}.$$

A simple computation shows that

$$c^q = \frac{ka^q + (l-k)b^q}{l}.$$

We have

$$g_l(\underbrace{c, \dots, c}_{n-l}) = \frac{ka^p + (l-k)b^p + (n-l)c^p}{n} - c^p$$

$$= \frac{k(a^p - b^p) + l\left[b^p - \left(b^q - \frac{k}{l}(b^q - a^q) \right)^{\frac{p}{q}} \right]}{n} = M_k.$$

Consider now the function $h : [k+1, n] \to \mathbb{R}$, defined by

$$h(x) = x\left[b^p - \left(b^q - \frac{k}{x}(b^q - a^q)\right)^{\frac{p}{q}}\right].$$

We claim that h is increasing. Indeed, one has

$$\begin{aligned} h'(x) = &\left[b^p - \left(b^q - \frac{k}{x}(b^q - a^q)\right)^{\frac{p}{q}}\right] \\ &- x \cdot \frac{p}{q}\left(b^q - \frac{k}{x}(b^q - a^q)\right)^{\frac{p}{q}-1} \frac{k}{x^2}(b^q - a^q) \\ = b^p - &\left[b^q - \frac{k}{x}(b^q - a^q)\right]^{\frac{p}{q}} - \frac{p}{q} \cdot \frac{k}{x}(b^q - a^q)\left[b^q - \frac{k}{x}(b^q - a^q)\right]^{\frac{p}{q}-1}. \end{aligned}$$

Let $\alpha = b^q - a^q, \quad \eta = \dfrac{k}{x} < 1,$ and let

$$\varphi(\eta) := b^p - (b^q - \alpha\eta)^{\frac{p}{q}} - \frac{p}{q}\alpha\eta(b^q - \alpha\eta)^{\frac{p}{q}-1}$$

Since

$$a^q < b^q - \alpha\eta = b^q - \frac{k}{x}(b^q - a^q) < b^q,$$

it follows that $h'(x) > 0$. Therefore h is increasing as claimed. Finally, we get

$$\begin{aligned} \max g_l = &\frac{k(a^p - b^p) + h(l)}{n} \le \frac{k(a^p - b^p) + h(n)}{n} \\ = &\frac{ka^p + (n-k)b^p}{n} - \left[\frac{ka^q + (n-k)b^q}{n}\right]^{\frac{p}{q}} = M_k. \end{aligned}$$

Our problem is now reduced to the one of finding the $k \in [0, \ldots, n]$ for which M_k attains its maximum, where

$$M_k = \frac{a^p - b^p}{n}k + b^p - \left(\frac{a^q - b^q}{n}k + b^q\right)^{\frac{p}{q}}.$$

To do this, we consider the function $g : [0, n] \to \mathbb{R}$, defined by

$$g(x) = \frac{a^p - b^p}{n} x + b^p - \left(\frac{a^q - b^q}{n} x + b^q \right)^{\frac{p}{q}}.$$

It is clear that our function satisfies

$$g(k) := M_k, fork \in [0, \dots, n]$$

We find first the extremal points of g which lie in the interior of the interval $[0, n]$.

In these points, due to the Theorem of Fermat we have that

$$g'(x) = \frac{a^p - b^p}{n} - \frac{p}{q} \cdot \frac{a^q - b^q}{n} \left(\frac{a^q - b^q}{n} x + b^q \right)^{\frac{p}{q}-1} = 0,$$

that is

$$\frac{q\,(a^p - b^p)}{p\,(a^q - b^q)} = \left(\frac{a^q - b^q}{n} x + b^q \right)^{\frac{p}{q}-1}$$

hence, as we have seen in the definition of the Stolarski mean that we are using in our case,

$$D_{p,q}^{p-q}(a, b) = \left[\frac{a^q - b^q}{n} x + b^q \right]^{\frac{p-q}{q}}$$

and from here,

$$x^* = \frac{b^q - D_{p,q}^q(a, b)}{b^q - a^q} \cdot n,$$

is the only extremal point contained in the interior of $[0, n]$.

Taking into account that the second derivative of g is :

$$g''(x) = -\frac{p}{q} \cdot (\frac{p}{q} - 1) \cdot \left(\frac{a^q - b^q}{n} \right)^2 \cdot \left(\frac{a^q - b^q}{n} x + b^q \right)^{\frac{p}{q}-2} < 0,$$

we get that the extremal point x^* we have just found, is a point of maximum for g.

This relation also tells us that the function g' is decreasing on the interval (0,n). Because $g'(x^*) = 0$, we get then that $g'(y) > 0$ for $y \in (0, x^*)$, and also that $g'(y) < 0$ for $y \in (x^*, n)$.

Finally this means that g is increasing on $(0, x^*)$ and decreasing on (x^*, n).

We conclude that:

$$g(1) < g(2) < \cdots < g([x^*])$$

and

$$g(n) < g(n-1) < \cdots < g([x^*]+1).$$

¿From here we get that in order to obtain the maximum for M_k, k has to take one of the values $[x^*]$ and $[x^*]+1$, where

$$x^* = \frac{b^q - D^q_{p,q}(a,b)}{b^q - a^q} \cdot n.$$

Remark. Because in our case

$$pq(p-q)(b-a) \neq 0,$$

the Stolarsky mean has the property that $a < D^q_{p,q}(a,b) < b$, so we clearly have that $0 < x < n$.

2° Let

$$\ell = \lim_{b \searrow a} \frac{k}{n} = \lim_{b \searrow a} \frac{b^q - \left[\dfrac{q(b^p - a^p)}{p(b^q - a^q)}\right]^{\frac{q}{p-q}}}{b^q - a^q}.$$

Using l'Hospital's rule we get

$$\ell = \lim_{b \searrow a} \frac{qb^{q-1} - \dfrac{q}{p-q}\left[\dfrac{q(b^p - a^p)}{p(b^q - a^q)}\right]^{\frac{q}{p-q}-1} \cdot \dfrac{q}{p} \cdot \overline{\ell}}{qb^{q-1}}$$

But

$$\lim_{b \searrow a} \frac{b^p - a^p}{b^q - a^q} = \frac{p}{q} \cdot a^{p-q},$$

so,

$$\ell = \lim_{b \searrow a} \left\{ 1 - \frac{q}{(p-q)p} a^{2q-p} \cdot \overline{\ell} \right\}$$

where

$$\begin{aligned}\overline{\ell} &= \lim_{b \searrow a} \frac{pb^{p-1}(b^q - a^q) - qb^{q-1}(b^p - a^p)}{b^{q-1}(b^q - a^q)^2} \\ &= \lim_{b \searrow a} \frac{(p-q)b^p - pb^{p-q}a^q + qa^p}{(b^q - a^q)^2}.\end{aligned}$$

Using l'Hospital's rule we get

$$\begin{aligned}\overline{\ell} &= \lim_{b \searrow a} \frac{p(p-q)b^{p-1} - p(p-q)b^{p-q-1}a^q}{2qb^{q-1}(b^q - a^q)} \\ &= \lim_{b \searrow a} \frac{p(p-q)b^{p-q} - p(p-q)b^{p-2q}a^q}{2q(b^q - a^q)} \\ &= \lim_{b \searrow a} \frac{p(p-q)(p-q)b^{p-q-1} - p(p-q)(p-2q)b^{p-2q-1}a^q}{2q^2b^{q-1}} \\ &= \frac{p}{2q^2}(p-q)qa^{p-2q} = \frac{1}{2}(p-q)\frac{p}{q}.\end{aligned}$$

Finally,

$$\ell = 1 - \frac{q}{(p-q)p} \cdot \frac{1}{2}(p-q)\frac{p}{q} = \frac{1}{2}.$$

In conclusion, $\lim\limits_{b \searrow a} \dfrac{k}{\eta} = \dfrac{1}{2}$, for any $p > q$.

Remark. The proofs are the same in the dcase when $0 > q > p$. In this case we have the next theorem.

Theorem 3. *Given the positive integer n, the real numbers $|p| > |q| > 0$ and $0 < a < b$, and the function $f : [a,b]^n \to \mathbb{R}$, defined by (1.1), the following assertions are true:*

1°. *The function f attains its maximum if and only if*

$a_1 = \cdots = a_k = a$ *and* $a_{k+1} = \cdots = a_n = b$, *where k is either*

$$\left[\frac{b^q - D_{p,q}^q(a,b)}{b^q - a^q} \cdot n\right]$$

or

$$\left[\frac{b^q - D_{p,q}^q(a,b)}{b^q - a^q} \cdot n\right] + 1.$$

2°. *If n,p and q are held fixed, then it holds that*

$$\lim_{b \searrow a} \frac{k}{n} = \frac{1}{2}.$$

Remark. From the monotonicity of function $p \mapsto M_p(a_1, \ldots, a_n)$, we could see that for $p > q$:

$$\left(\frac{a_1^p + \cdots + a_n^p}{n}\right)^{\frac{1}{p}} \geq \left(\frac{a_1^q + \cdots + a_n^q}{n}\right)^{\frac{1}{q}},$$

with equality if and only of $a_1 = \cdots = a_n$. It follows clearly that the inequality mentioned before, is equivalent to:

$$\frac{a_1^p + \cdots + a_n^p}{n} - \left(\frac{a_1^q + \cdots + a_n^q}{n}\right)^{\frac{p}{q}} \geq 0.$$

Proof of Theorem 2

Considering $p = 2$, $q = 1$ in Theorem 1, we can see that:

$$D_{2,1}(a,b) = \frac{1}{2} \cdot \frac{b^2 - a^2}{b - a} = \frac{1}{2}(b + a)$$

and it follows that

$$\frac{k}{n} = \frac{b - \frac{1}{2}(b+a)}{b-a} = \frac{1}{2}.$$

From here, we get immediately the best constant α for which:

$$\frac{a_1^2 + \cdots + a_n^2}{n} - \left(\frac{a_1 + \cdots + a_n}{n}\right)^2 \leq \alpha \max_{1 \leq i \leq j \leq n} (a_i - a_j)^2.$$

Following the steps mentioned before, the function gets the maximum for $a_1 = \cdots = a_k = a$,

$a_{k+1} = \cdots = a_n = b$,
where $k = \left[\frac{n}{2}\right]$, or $k = \left[\frac{n+1}{2}\right]$.
We have that

$$\frac{a_1^2 + \cdots + a_n^2}{n} - \left(\frac{a_1 + \cdots + a_n}{n}\right)^2 \leq \frac{(b-a)^2}{n^2}(nk - k^2).$$

So the best constant α will be

$$\alpha = \left[\frac{n}{2}\right]\left[\frac{n+1}{2}\right].$$

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