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A RE-EXAMINATION OF A FOLDOVER DESIGN

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(9 EQRM 5)

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TECHNICAL REPORT

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Efficiency, Quality and Reliability
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The Foldover of the 12 run Plackett-Burman design is shown to be resolution V in every set of 5 variables. The identification, estimation and criticism of a model involving a small number of main effects and the corresponding two factor interactions is discussed and illustrated with an example.

Keywords: Two-level designs; foldover designs; effect sparsity; search designs.

1. INTRODUCTION

The two level Plackett-Burman designs given by Plackett and Burman (1946) are a very useful class of designs when the number of two factor interactions is zero. They provide orthogonal and 100% efficient estimates of the main effects. When interactions are possible they are probably less useful. For example Draper and Stoneman (1966) and Margolin (1968), showed that in the Plackett-Burman design involving 11 factors in 12 runs ($2^{11}//12$) every main effect is biased by plus or minus one third of every interaction not involving the associated factor.

The important projective properties of regular fractional factorial designs were first discussed by Box and Hunter (1961). Bisgaard in Box, Bisgaard and Fung (1989, Section 3.4) examined the projective properties of the Plackett-Burman designs. He showed that projected onto 3-space the 12 run design provides a full 2^3 design plus a 2^{3-1} design. These results indicate therefore that even where interactions are possible the Plackett-Burman designs may be used in screening situations under the assumption of effect sparsity.

The $2^{11}//12$ is of resolution III. We can use the foldover theorem of Box and Wilson (1951) to generate a $2^{12}//24$ Plackett-Burman foldover design and therefore estimate the main effects with full efficiency unbiased by the two factor interactions. We can also obtain estimates of the strings of two factor interactions. Despite this the use of these designs to determine the two factor interactions has generated little interest. Daniel (1976, p 225), in analysing these designs, has said: "The prospect of disentangling a number of long strings of 2fi's, although ameliorated somewhat by the known patterns, is not enticing."

The purpose of this paper is to examine some properties of the $2^{12}//24$ Plackett-Burman foldover and to determine if interactions can be identified and estimated efficiently in the case of effect sparsity.

2. SOME PROPERTIES OF THE $2^{12}//24$ PLACKETT-BURMAN FOLDOVER DESIGN

In the context of designing small run composite designs, Draper (1985) showed that there are only two essentially different choices of five columns from a 12 run Plackett-Burman design. In other words of the 792 projections of the 11 columns of the design onto 5 - space, there are only two types which are given in Table 1. We see that, as Draper points out, design (a) has a pair of repeat runs while design (b) has a minor-image pair of runs.

a					b				
-	-	-	-	-	-	-	-	-	-
-	-	-	-	-	+	+	+	+	+
-	-	+	+	+	-	-	+	+	+
-	+	-	+	+	-	+	+	-	+
-	+	+	-	+	+	-	-	+	+
-	+	+	+	-	+	+	-	+	-
+	-	-	+	+	+	+	+	-	-
+	-	+	-	+	+	-	+	-	-
+	-	+	+	-	+	-	-	-	+
+	+	-	-	+	-	+	-	+	-
+	+	-	+	-	-	+	-	-	+
+	+	+	-	-	-	-	+	+	-

Table 1 : The two essentially different choices of five columns from a 12 run Plackett-Burman design given by Draper (1985).

We can use Draper's result to determine the projective properties of the Plackett-Burman foldover. The two essentially different 12 run designs are given in Figure 1 together with the foldover runs.

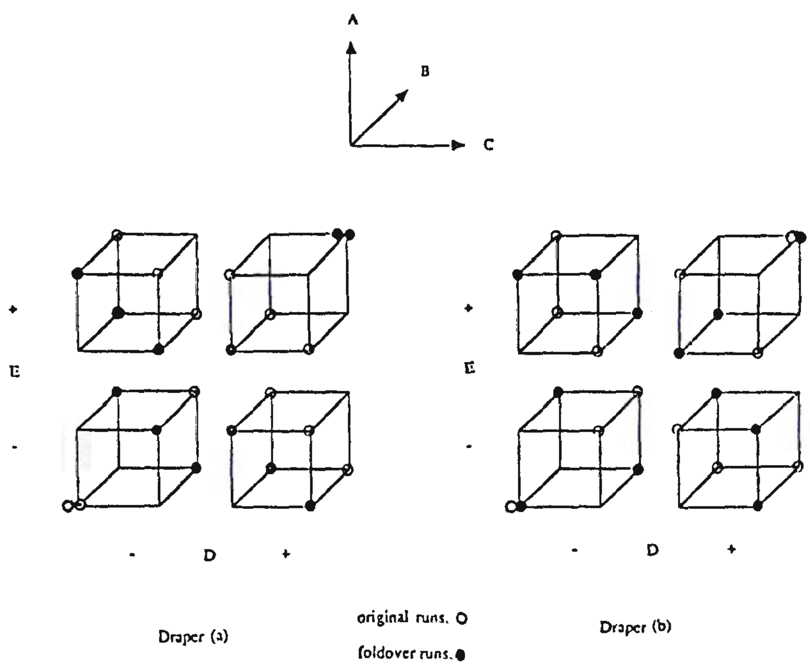

Figure 1 : The two essentially different choices of five columns from the folded over 12 run Plackett-Burman design.

The point to notice is that although the two 12 run designs are different, the two 24 run designs are the same. This covers the case when the projection is onto 5 of the original 11 variables. The other possibility is when the projection is onto 4 of the original 11 variables plus the 12th variable which we can add when we do the foldover. By projecting the two designs given by Draper onto 4 - space and then folding over and adding the extra variable we in fact get the same result as before. Hence for the Plackett-Burman foldover $2^{12}//24$ design, the 792 projections onto 5 - space are of only one type. The design consists of the runs (1) and abcde replicated twice plus all two letter runs and all three letter runs and therefore the design is balanced with respect to all the factors.

The design can be rewritten as { (1), ad, cde, abe, cd, ac, bce, abcde, de, ae, bd, ab, cde, ace, bcde, abc, (1), ade, be, abd, ce, acd, bc, abcde}. The order has been chosen so that we can easily see that the design found is in fact the union of 3 fractional replicates. The first eight runs are the fractional replicate with alias subgroup I = BE = ABCD = ACDE, the second eight runs are the fractional replicate with alias subgroup I = -BE = -AD = ABDE, and the third eight runs are the fractional replicate with alias subgroup I = AD = ABCE = BCDE. This representation is not unique and suffers from the fact that it is not symmetric in all the factors, since the factor C does not appear in any of the two letter words. However it is useful since it enables the result for projections onto 4 - space and 3 - space to be easily derived.

To see what the projective properties for 4 - space are, we can use the projection onto 5 - space and collapse one of the factors, A, B, C, D or E. For example, collapsing A, we drop A from the alias subgroups and obtain the alias subgroups I = BE, I = -BE and I = BCDE. The first two combine to form a full 2^4 design in B, C, D and E, while the third gives a 2^{4-1} in B, C, D and E. Collapsing B, we get I = ACDE, I = -AD and I = AD. Collapsing D, we get I = BE, I = -BE and I = ABCE. Collapsing E, we get I = ABCD, I = -AD and I = AD. Finally for C we get I = BE, I = -BE = -AD = ABDE (replicated twice) and I = AD. I = BE is the union of the fractional replicates I =BE=AD=ABDE and I = BE = -AD = -ABDE while I = AD is the union of the fractional replicates I = BE = AD = ABDE and I = -BE = AD = -ABDE. Finally the two replicates I = -BE = -AD = ABDE and I = BE = AD = ABDE, combine to form I = ABDE while the remaining four fractional replicates, I = BE = AD = ABDE, I = BE = -AD =-ABDE, I = -BE = -AD = ABDE and I = -BE = -AD = -ABDE combine to form the full factorial 2^4. Hence no matter which factor we collapse we get a full factorial plus a 2^{4-1} design.

Finally collapsing onto 3 - space, the 2^4 collapses to a replicated 2^3 and the 2^{4-1} collapses to a full 2^3 and hence we have a 2^3 replicated 3 times in every set of 3 factors. This latter result was originally established by Seiden (1955), who showed that by folding over the 12 run Plackett-Burman design an orthogonal array of strength 3 is obtained.

3. IDENTIFICATION OF LIKELY INTERACTIONS

The Plackett-Burman foldover involving 12 factors has 66 possible two factor interactions and if we could not reduce the number of possible interactions then it would be impossible to determine which interactions are operating. However a more manageable system could exist if we could assume that two-factor interactions are not likely unless the corresponding main effects are present. A possible method then is: First decide which main effects are real, and secondly if possible estimate the two factor interactions involving each pair of real main effects. This is in agreement to the general principle outlined by Cox (1984) that large component main effects are more likely to lead to appreciable interactions than small components, and in any case these interactions are likely to be of more practical importance.

It turns out that this method is much more useful for the folded over Plackett-Burman design than for the Plackett-Burman design itself and not only because of the doubling of the number of runs in the experiment. The reason for this is that in the folded over Plackett-Burman design the 12 main effect estimates are orthogonal to the two factor interaction space and therefore the identification of the main effects is unaffected by the number and sizes of the two factor interactions.

On the other hand in the Plackett-Burman design the aliasing of the main effects is so severe that any two-factor interaction affects all but two of the main effect estimates and therefore the identification of the real main effects is difficult.

An example chosen to illustrate this point is taken from Box, Hunter and Hunter (1978, p. 375 to 378) which gives the results of a 2^5 factorial design on a reactor involving five factors, feed rate, catalyst, agitation rate, temperature and concentration. The analysis given by Box, Hunter and Hunter shows that only the factors catalyst, temperature and concentration give rise to effects that are large compared to the noise. Suppose that instead of a 2^5 design a Plackett-Burman experiment on 11 factors in 12 runs had been carried out, where A corresponds to the feed rate, B to the catalyst, D to the agitation rate, E to the temperature, F to the concentration, while C and G to L (excluding I) correspond to factors that do not affect the response. Then the 12 runs of the Plackett-Burman design and the corresponding runs from the 2^5 design involving the factors A, B, D, E and F is given in Table 2 with the response from Table 12.1a of Box, Hunter and Hunter given in column a.

	Factor in PB Design											Factor in 2^5 Design					a	b
Run No.	A	B	C	D	E	F	G	H	J	K	L	FR	Cat	AR	T	Conc	Response	Foldover Response
1	+	+	-	+	+	+	-	-	-	+	-	+	+	+	+	+	82	61
2	-	+	+	-	+	+	+	-	-	-	+	-	+	-	+	+	78	56
3	+	-	+	+	-	+	+	+	-	-	-	+	-	+	-	+	55	94
4	-	+	-	+	+	-	+	+	+	-	-	-	+	+	+	-	95	63
5	-	-	+	-	+	+	-	+	+	+	-	-	-	-	+	+	44	61
6	-	-	-	+	-	+	+	-	+	+	+	-	-	+	-	+	59	93
7	+	-	-	-	+	-	+	+	-	+	+	+	-	-	+	-	61	67
8	+	+	-	-	-	+	-	+	+	-	+	+	+	-	-	+	65	66
9	+	+	+	-	-	-	+	-	+	+	-	+	+	-	-	-	61	49
10	-	+	+	+	-	-	-	+	-	+	+	-	+	+	-	-	54	45
11	+	-	+	+	+	-	-	-	+	-	+	+	-	+	+	-	60	70
12	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	61	82

Table 2. Results from a PB foldover experiment based on an actual 2^5 given in Box, Hunter and Hunter.

Figure 2 gives the Daniel plot of the main effect estimates (Note two plotted points are identical). The results are inconclusive since none of the plotted points full off the straight line.

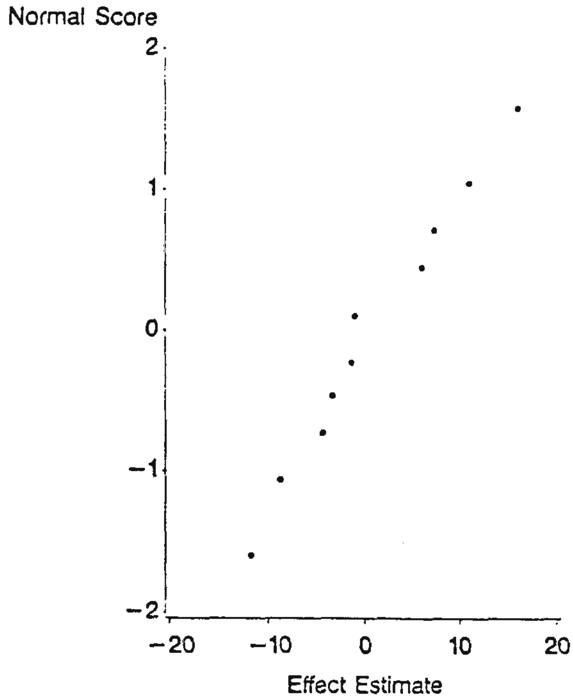

Figure 2 : Daniel plot of results of Plackett-Burman design given in Table 2.

The responses of the foldover runs are given in column b of Table 1. Note that the responses of runs 1 and 12 have been used twice and hopefully this will only introduce a slight distortion. Figure 3 gives the Daniel plot of the main effect estimates (As in Figure 2, two plotted points are identical). The plot shows clearly that the three main effect estimates B, E and F are significant and therefore we would be led to try to estimate BE, BF, and EF.

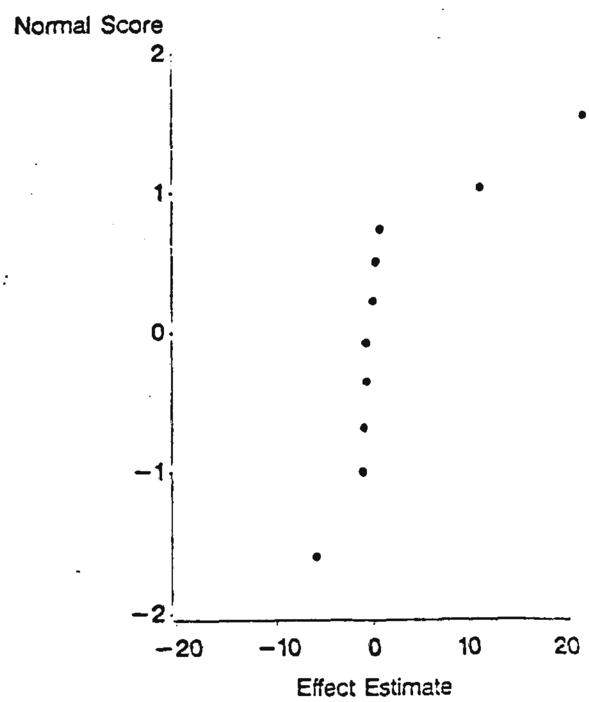

Figure 3 : Daniel plot of results of Plackett-Burman foldover design given in Table 2

4. ESTIMATION OF LIKELY INTERACTIONS

Of course, the estimates of the main effects in the Plackett-Burman foldover design are 100% efficient. This section concerns the question of when there are only 3, 4, or 5 real main effects whether we can estimate the corresponding 3, 6, or 10 two factor interactions respectively, and if we can what are the efficiencies of those estimates.

If 3 main effects are real then since there is a replicated 2^3 in every set of 3 columns, the three two factor interactions are estimated with 100% efficiency.

If 4 main effects are large then, since there is a full 2^4 and a 2^{4-1} in every set of columns, we can estimate all six two-factor interactions. Consider AB which is aliased with CD in the 2^{4-1}. Then from the 2^4 we have estimates of AB, say AB_1 and CD, say CD_1, with variances $\sigma^2/4$. From the 2^{4-1} we have an estimate of (AB + CD), say $(AB + CD)_1$ with variance $\sigma^2/2$. Hence we can obtain another estimate of AB, say AB_2, as $(AB + CD)_1 - CD_1$ with variance $3\sigma^2/4$. The best estimate of AB is given by $(3AB_1 + AB_2)/4$ with variance $(9(\sigma^2/4) + (3\sigma^2/4))/16 = 12\sigma^2/64$. Therefore the efficiency is given by $(1/6)/(12/64) = 64/72 = 88.8\%$.

If 5 main effects are large then we must estimate the 10 two-factor interactions in the corresponding factors. It turns out that the property of the design that it is the union of three resolution II designs, which was used in Section 3, does not turn out to be useful in deciding whether interactions can be estimated and if so with what efficiency.

In an orthogonal fractional replicate the AB interaction, say, is given by half the average of all the trials where factors A and B are at the same levels less half the average of all the trials where factors A and B are at different levels. We can do the same calculation for this design but because of the design's unusual nature we would not expect this to estimate AB without bias. We will call the estimate obtained in this manner AB_N, where N denotes naive.

The trials where A and B are at the same levels are {(1), (1), ab, abc, cd, abd, ce, abe, de, cde, abcde, abcde}. Using the Yates table of signs given in Table 3, we can see the sum of the responses is (12μ + 12AB + 4CD + 4CE + 4DE). The trials where A and B are at different levels are {ac, bc, ad, bd, acd, bcd, ae, be, ace, bce, ade, bde}. Using the Yates table of signs given in Table 4 we can see the sum of the responses is (12μ - 12AB - 4CD - 4CE - 4DE). Combining these two we obtain 24AB + 8CD + 8CE + 8DE and therefore AB_N = AB + (CD + CE + DE)/3. Note that only the interactions not involving A and B bias the estimate.

	(1)	(1)	ab	abc	cd	abd	ce	abe	de	cde	abcde	abcde	Sum
μ	+	+	+	+	+	+	+	+	+	+	+	+	12μ
A	-	-	+	+	-	+	-	+	-	-	+	+	0
B	-	-	+	+	-	+	-	+	-	-	+	+	0
AB	+	+	+	+	+	+	+	+	+	+	+	+	12AB
C	-	-	-	+	+	-	+	-	-	+	+	+	0
AC	+	+	-	+	-	-	-	-	+	-	+	+	0
BC	+	+	-	+	-	-	-	-	+	-	+	+	0
D	-	-	-	-	+	+	-	-	+	+	+	+	0
AD	+	+	-	-	-	+	+	-	-	-	+	+	0
BD	+	+	-	-	-	+	+	-	-	-	+	+	0
CD	+	+	+	-	+	-	-	+	-	+	+	+	4CD
E	-	-	-	-	-	-	+	+	+	+	+	+	0
AE	+	+	-	-	+	-	-	+	-	-	+	+	0
BE	+	+	-	-	+	-	-	+	-	-	+	+	0
CE	+	+	+	-	-	+	+	-	-	+	+	+	4CE
DE	+	+	+	+	-	-	-	-	+	+	+	+	4DE

Table 3: Yates Table of Signs for runs in the Plackett-Burman Foldover where A and B are both at the same levels.

	ac	bc	ad	bd	acd	bcd	ae	be	ace	bce	ade	bde	Sum
μ	+	+	+	+	+	+	+	+	+	+	+	+	12μ
A	+	-	+	-	+	-	+	-	+	-	+	-	0
B	-	+	-	+	-	+	-	+	-	+	-	+	0
AB	-	-	-	-	-	-	-	-	-	-	-	-	-12AB
C	+	+	-	-	+	+	-	-	+	+	-	-	0
AC	+	-	-	+	+	-	-	+	+	-	-	+	0
BC	-	+	+	-	-	+	+	-	-	+	+	-	0
D	-	-	+	+	+	+	-	-	-	-	+	+	0
AD	-	+	+	-	+	-	-	+	-	+	+	-	0
BD	+	-	-	+	-	+	+	-	+	-	-	+	0
CD	-	-	-	-	+	+	+	+	-	-	-	-	-4CD
E	-	-	-	-	-	-	+	+	+	+	+	+	0
AE	-	+	-	+	-	+	+	-	+	-	+	-	0
BE	+	-	+	-	+	-	-	+	-	+	-	+	0
CE	-	-	+	+	-	-	-	-	+	+	-	-	-4CE
DE	+	+	-	-	-	-	-	-	-	-	+	+	-4DE

Table 4 : Yates Table of Signs for runs in the Plackett-Burman Foldover where A and B are at different levels.

We can do a similar calculation for all the other interactions and we therefore get the matrix equation:

$$\begin{bmatrix} AB_N \\ AC_N \\ AD_N \\ AE_N \\ BC_N \\ BD_N \\ BE_N \\ CD_N \\ CE_N \\ DE_N \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & 1 & 0 & 0 & 0 & 0 & \frac{1}{3} \\ 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 1 & 0 & 0 & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 1 & \frac{1}{3} & 0 & 0 \\ \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} & 1 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 & 0 & 1 & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \widehat{AB} \\ \widehat{AC} \\ \widehat{AD} \\ \widehat{AE} \\ \widehat{BC} \\ \widehat{BD} \\ \widehat{BE} \\ \widehat{CD} \\ \widehat{CE} \\ \widehat{DE} \end{bmatrix}$$

We need the inverse of the matrix in the above equation, which will follow a similar pattern with 1 replaced by α, 0 replaced by β and 1/3 replaced by γ. We therefore obtain the following equations:

$$\alpha + \gamma = 1$$

$$\beta + 2\beta/3 + \gamma/3 = 0$$

$$\gamma + 2\beta/3 + \alpha/3 = 0$$

The solution is $\alpha = 13/8$, $\beta = 1/8$ and $\gamma = -5/8$. Hence $\hat{AB} = 13AB_N/8 + (AC_N + AD_N + AE_N + AE_N + BC_N + BC_N + BE_N)/8 - 5\,(CD_N + CE_N + DE_N)/8$.

These results show that, remarkably at least to the author, the Plackett-Burman Foldover is resolution V in every set of 5 factors.

In appendix 1 it is shown that the variance of the two factor interactions is $13\sigma^2/48$ with relative efficiency of 61.54%, while the covariance of a pair of two factor interactions with one letter in common is 1/48, while that for a pair with no letters in common is -5/48. Despite the fact that the estimates of the two factor interactions are not fully efficient and are correlated it is somewhat surprising that estimates can be made at all. Of course if we knew which five factors were the active ones we could do a 2^{5-1} resolution V design in 16 runs. However what the Plackett-Burman Foldover design is effectively doing is a 2^{5-1} design in every set of 5 factors in only 24 runs, since the variances of the interaction effects are only slightly less efficient compared to those from the 2^{5-1} design.

5. CRITICISM OF THE FITTED MODEL

In the previous section it was shown that if the number of real main effects is less than or equal to five then it is possible to estimate the two factor interaction corresponding to each pair of real main effects.

In practice a statistician would be surprised if the two factor interaction involving two factors, say A and B, was large, while the A and B main effects were small. However, if AB and A were large but B was small this would not be unduly surprising. Hence when checking the tentative model, identified as in section 3 and estimated as in section 4, it would be necessary to consider the possibility that interactions involving only one of the identified main effects could be real.

This can be put into the framework of search designs, developed by Srivastava (1975, 1976). He pointed out that the factorial effects in an experiment can be divided into three categories:

(1) effects that we are sure are negligible, in this case the higher order interactions as well as the two factor interactions involving none of the identified real main effects,

(2) effects that we want to estimate, in this case the mean and all the two factor interactions involving pairs of identified real main effects, and

(3) the remaining effects, most of which are negligible, but a few of which may be non-negligible, in this case the two factor interactions involving only one of the identified real mean effects.

If there is no error, then in order to estimate all the effects of type (2) and search over at most k non-negligible effects of type (3), a necessary and sufficient condition is that every submatrix consisting of all the columns corresponding to the effects of type (2) and 2k of the columns corresponding to the effects of type (3) must be of full rank, and the design is said to be strongly resolvable of order k. If error is present this condition is necessary but no longer sufficient and therefore the k non-negligible effects can only be identified with hopefully a high probability.

The models with three, four and five identified real main effects were checked for resolvability as described in Appendix 2. With three main effects the design is resolvable of order 2. With four main effects the design is resolvable of order 1 but not of order 2. With five main effects the design is not even resolvable of order 1.

In fact in this latter case all interactions involving one particular factor which does not have a real main effect and the factors that do are contained in the model space and hence cannot be checked.The particular factor depends on the set of factors that had real main effects.

These results are useful since it allows us to determine what assumptions can be confirmed from the data and what assumptions need to be verified from information external to the data.

With the example given in section 3, since only three main effects are identified as real, we are led to search all possible models with B, E, F, BE, BF and EF plus an additional two interactions each involving any one of the factors B, E or F. The 23 best fitting subsets turn out to involve the interaction DE and thus it is worthwhile considering whether this interaction is real. Using the approximation to the distribution of the maximum F-to-enter statistic given by Miller (1984) it is found that DE is not quite significant at the 5% level. Interestingly if for example A and K also had real main effects then we would be unable to check for the existence of the DE interaction since it would be totally confounded with the interactions corresponding to all pairs of the factors A, B, E, F and K.

6. CONCLUSION

This paper has covered the exploitation of some interesting properties of the Plackett-Burman foldover design. A Daniel plot of the main effect estimates is used to identify a tentative model involving a small number of main effects and associated two factor interactions. The design is shown to be resolution V in every set of five variables although in this case the estimates of the interaction effects are correlated and not fully efficient. Checks to the fitted model can be applied particularly when the number of real main effects is less than five. The design should be useful to experimenters with between 9 and 12 potential influential factors of which only a few are expected to have any effect.

Appendix 1 : Calculations of variances and covariances of the estimates given in Section 4.

To calculate the variances and covariances of these estimates, we need to calculate the variances and covariances of AB_N, AC_N ... DE_N. Examination of the table of signs given in Table 5 shows that $\text{Var}(AB_N) = \sigma^2/6$ and $\text{Cov}(AB_N, AC_N) = \ldots = \text{Cov}(AB_N, BE_N) = 0$ and $\text{Cov}(AB_N, CD_N) = \text{Cov}(AB_N, CE_N) = \text{Cov}(AB_N, DE_N) = \sigma^2/18$. These latter results follow since $\text{Cov}(AB_N, AC_N) = \left\{\text{Var}(AB_N + AC_N) - \text{Var}(AB_N - AC_N)\right\}/4$ and $\text{Cov}(AB_N, CD_N)$ $= \left\{\text{Var}(AB_N + CD_N) - \text{Var}(AB_N - CD)\right\}/4$ and $\text{Var}(AB_N + AC_N) = \text{Var}(AB_N - AC_N)$ $= \text{Var}(AB_N + CD_N) = 64\sigma^2/144$ and $\text{Var}(AB_N - CD_N) = 32\sigma^2/144$.

Then $\widehat{AB} = a'x$ with $8a' = \left[13, 1, 1, 1, 1, 1, 1, -5, -5, -5\right]$ and $x' = [AB_N \; AC_N \; AD_N \; AE_N \; BC_N \; BD_N \; BE_N \; CD_N \; CE_N \; DE_N]$. The variance covariance matrix of x is given by $\Sigma = \sigma^2 G/18$ where

$$G = \begin{bmatrix} 3 & . & . & . & . & . & . & 1 & 1 & 1 \\ . & 3 & . & . & . & 1 & 1 & . & . & 1 \\ . & . & 3 & . & 1 & . & 1 & . & 1 & . \\ . & . & . & 3 & 1 & 1 & . & 1 & . & . \\ . & . & 1 & 1 & 3 & . & . & . & . & 1 \\ . & 1 & . & 1 & . & 3 & . & . & 1 & . \\ . & 1 & 1 & . & . & . & 3 & 1 & . & . \\ 1 & . & . & 1 & . & . & 1 & 3 & . & . \\ 1 & . & 1 & . & . & 1 & . & . & 3 & . \\ 1 & 1 & . & . & 1 & . & . & . & . & 3 \end{bmatrix}$$

Hence the variance of $\widehat{AB}$ is given by $a'\Sigma a$, the covariance of $\widehat{AB}$ and $\widehat{AC}$ is given by $a'\Sigma b$ and the covariance of $\widehat{AB}$ and $\widehat{CD}$ is given by $a'\Sigma c$, where $8b' = [1, 13, 1, 1, 1, -5, -5, 1, 1, -5]$ and $8c' = \left[-5, 1, 1, -5, 1, 1, -5, 13, 1, 1\right]$. Now $a'\Sigma = \left[1/6, 0, 0, 0, 0, 0, 0, 0, 0, 0,\right]\sigma^2$ and therefore variance $(\widehat{AB}) = 13\sigma^2/48$ which has relative efficiency of $(1/6)/(13/48) = 8/13 = 61.54\%$ and covariance $(\widehat{AB}, \widehat{AC}) = 1/48$ and covariance $(\widehat{AB}, \widehat{CD}) = -5/48$.

Run	AB_N	AC_N	$AB_N + AC_N$	$AB_N - AC_N$	CD_N	$AB_N + CD_N$	$AB_N - CD_N$
(1)	+	+	2	0	+	2	0
(1)	+	+	2	0	+	2	0
ab	+	-	0	2	+	2	0
abc	+	+	2	0	-	0	2
cd	+	-	0	2	+	2	0
abd	+	-	0	2	-	0	2
ce	+	-	0	2	-	0	2
abe	+	-	0	2	+	2	0
de	+	+	2	0	-	0	2
cde	+	-	0	2	+	2	0
abcde	+	+	2	0	+	2	0
abcde	+	+	2	0	+	2	0
ac	-	+	0	-2	-	-2	0
bc	-	-	-2	0	-	-2	0
ad	-	-	-2	0	-	-2	0
bd	-	+	0	-2	-	-2	0
acd	-	+	0	-2	+	0	2
bcd	-	-	-2	0	+	0	2
ac	-	-	-2	0	+	0	2
be	-	+	0	-2	+	0	2
ace	-	+	0	-2	-	-2	0
bce	-	-	-2	0	-	-2	0
ade	-	-	-2	0	-	-2	0
bde	-	+	0	-2	-	-2	0

Table 5. Table of signs used in the calculation of var (AB_N) and the covariance of AB_N and AC_N, and the covariance of AB_N and CD_N.

Appendix 2 : Determination of the Resolvability of the Designs.

When there are five main effects then the design is not resolvable of order 1 since the dimension of the interaction space is 11, with 10 degrees of freedom taken up by the estimates of the two factor interactions, leaving only one remaining degree of freedom and not the two that would necessary.

When there are four main effects then we need to show that every submatrix of size 24 x (7 + 2k) of a 24 x 39 matrix is of full rank. The columns of this latter matrix are those of the design matrix corresponding to the mean and the six two factor interactions in the tentative model plus the 32 interactions involving one factor from the tentative model and one factor not from the tentative model. We use the method in Srivastava (1976) to reduce the problem to one of requiring every submatrix of size 5 x 2k of a 5 x 32 to be of full rank.

Likewise when there are three main effects we need to show that every submatrix of size 24 x (4 + 2k) of a 24 x 31 matrix is of full rank and we can also reduce this to the problem of requiring every submatrix of size 8 x 2k of a 8 x 27 matrix to be of full rank.

When both 5 x 32 and 8 x 27 matrices are calculated and displayed it can easily be seen that none of the columns are multiples of other columns and therefore the design is resolvable of at least order one in both cases. To check if the design is resolvable of order 2 is easily achieved using the option RSQUARE in the REG procedure of the SAS/STAT module of the package SAS (1988, p.773), which fits all possible regressions of a number of explanatory variables on a response variable. The matrix columns are initially centred and then augmented by zeros to overcome an unnecessary restriction in the package requiring the number of rows to exceed the number of columns. Taking each column in turn as the response variable, all possible regressions of size 3 are run with the remaining columns as the explanatory variables. A dependency is indicated by at least one regression with a $R^2 = 100\%$. This does not occur when the number of identified main effects is three but does when the number is four.

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