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OF TECHNOLOGY

DEPARTMENT OF MATHEMATICS, COMPUTING AND OPERATIONS RESEARCH

INSPECTION INTERVAL FOR MAXIMUM FUTURE RELIABILITY USING THE DELAY TIME MODEL

Peter Cerone

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TECHNICAL REPORT

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Efficiency, Quality and Reliability
Management Centre

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Abstract

The main problem addressed in this article is the determination of an inspection interval T_{max}, given the number of inspections $m-1$, which will result in the maximum reliability at some future point in time $t = t^*$. The reliability model developed by Christer is used in which the notion of delay time is involved, representing the start to eventual failure of an item subject to a fault detectable on inspection. A numerical procedure is used to solve the model for general delay time density f(h) and time to failure from new density g(y).

T_{max} is shown to migrate towards the left hand side of the interval $\left[\frac{t^*}{m}, \frac{t^*}{m-1}\right]$ as the number of inspections increase. If both densities are exponential then the optimal inspection interval is shown to be $T_{max} = \frac{t^*}{m}$.

Keywords: Inspection, Reliability, Decision, Optimisation.

Running Title: Optimal Inspection Using Delay Time.

1. INTRODUCTION

Christer (1987) developed an expression for the reliability of a single component unit which is subject to a detectable fault. He utilises the notion of delay time which is the span of time from when a defect is first detectable upon inspection to when it is considered to have failed. If a defect is found at an inspection then the component is replaced or repaired to an as new condition and thus avoiding a failure. Inspections are assumed to be non-detrimental. The delay time h is governed by the probability density function $f(h)$. The probability that a new component at time $t = 0$ has not failed by time t as a result of a defect at time y from new is subject to a probability density function $g(y)$. Both densities have been obtained experimentally and applied successfully by Christer and Waller (1984 a, b).

The reliability $R_T(t)$ due to a periodic inspection every T time units is derived by Christer (1987) to be

$$R_T(t) = r_T^{(m)}(t) \quad , \quad (m-1)\,T \le t \le mT$$

where,

$$r_T^{(m)}(t) = \sum_{j=1}^{m-1} \kappa_j(T)\, r_T^{(m-j)}(t-jT) + B_T(t) \tag{1}$$

with,

$$\kappa_j(T) = \int_{(j-1)T}^{jT} g(y)M(jT-y)dy,$$

$$B_T(t) = \int_t^{\infty} g(y)dy + \int_{(m-1)T}^{t} g(y)M(t-y)dy \tag{2}$$

and $$M(x) = \int_x^{\infty} f(h)dh = 1-F(x).$$

It should be noted that m is a positive integer and throughout the paper the convention is used that when m = 1 the sum in equation (1), and similar expressions, is zero.

The main problem to be addressed here is to determine, for fixed number of inspections m-1, the optimal inspection interval, T that will result in the maximum reliability at some future point in time $t = t^*$. The type of problem envisaged is that of a mission starting at $t = t^*$ until which time the item may be inspected for a fault. Alternatively we may investigate the optimal inspection interval for a deteriorating item whose time of commencement of a mission has been delayed.

2. THE CONVERSE PROBLEM

Let us assume that it is advantageous for a deteriorating item to be as reliable as possible at some future point in time $t = t^*$. We can inspect the item at periodic intervals of length T and the item is either renewed or repaired to an as good as new condition. The problem we wish to address here is to find the optimal inspection interval T_{max} given a desired number of inspections m - 1.

Thus given $t = t^*$ in (1) we obtain

$$r_T^{(m)}(t^*) = \sum_{j=1}^{m-1} \kappa_j(T)\, r_T^{(m-j)}(t^* - jT) + B_T(t^*)\ , \quad \frac{t^*}{m} \le T \le \frac{t^*}{m-1} \qquad (3)$$

where $\kappa_j(T)$ and $B_T(t^*)$ are as given in (2).

The problem becomes that of finding for each number of inspections m-1, the optimal inspection interval, T over the domain indicated in equation (3). We notice that as m increases then T will decrease since the interval of search is of length $\frac{t^*}{m(m-1)}$ and the bounds on T will become tighter.

One way of obtaining the optimal inspection interval T would be to find $\frac{dr_T^{(m)}(t^*)}{dT}$ and determine where it becomes zero. Further investigation would be needed to be performed to determine whether this was indeed the point at which the global maximum over $\frac{t^*}{m} \le T < \frac{t^*}{m-1}$ occurred.

It is much easier and more practical to either evaluate $r_T^{(m)}(t^*)$ over the interval $\left[\frac{t^*}{m}, \frac{t^*}{m-1}\right]$ or else use some interval bisection or refinement of mesh to find the maximum. It is felt that the most practical method would be to actually plot equation (3) and thus allowing the user the convenience of deciding on a suitable value of T since there may be some flexibility if the reliability does not vary greatly about the maximum.

3. A SIMPLE EXAMPLE

It is instructive to consider a simple example of the problem. Let us examine the problem shown in the diagram of Figure 1. We wish to perform only one inspection so that we need to choose when this inspection is to occur given that the maximum reliability at $t = t^*$ is desired.

From the diagram of Figure 1 it may be noticed that the earliest possible time the inspection can be made is at $T = \frac{t^*}{2}$ in which case another inspection is due at our time of interest t^*. The latest the inspection can be made is at our point of interest $T = t^*$ resulting in no benefit. Since the inspections are assumed to be benign and perfect it follows that $r^{(2)}_{\frac{t^*}{2}}(t^*) > r^{(2)}_{t^*}(t^*) = r^{(1)}_{t^*}(t^*)$. Thus having one inspection is better than having none.

We wish to find when the optimal inspection should occur.

Consider equation (3) with $m = 2$ to give

$$r^{(2)}_T(t^*) = \kappa_1(T) r^{(1)}_T(t^* - T) + B_T(t^*), \quad \frac{t^*}{2} \leq T \leq t^* . \qquad (4)$$

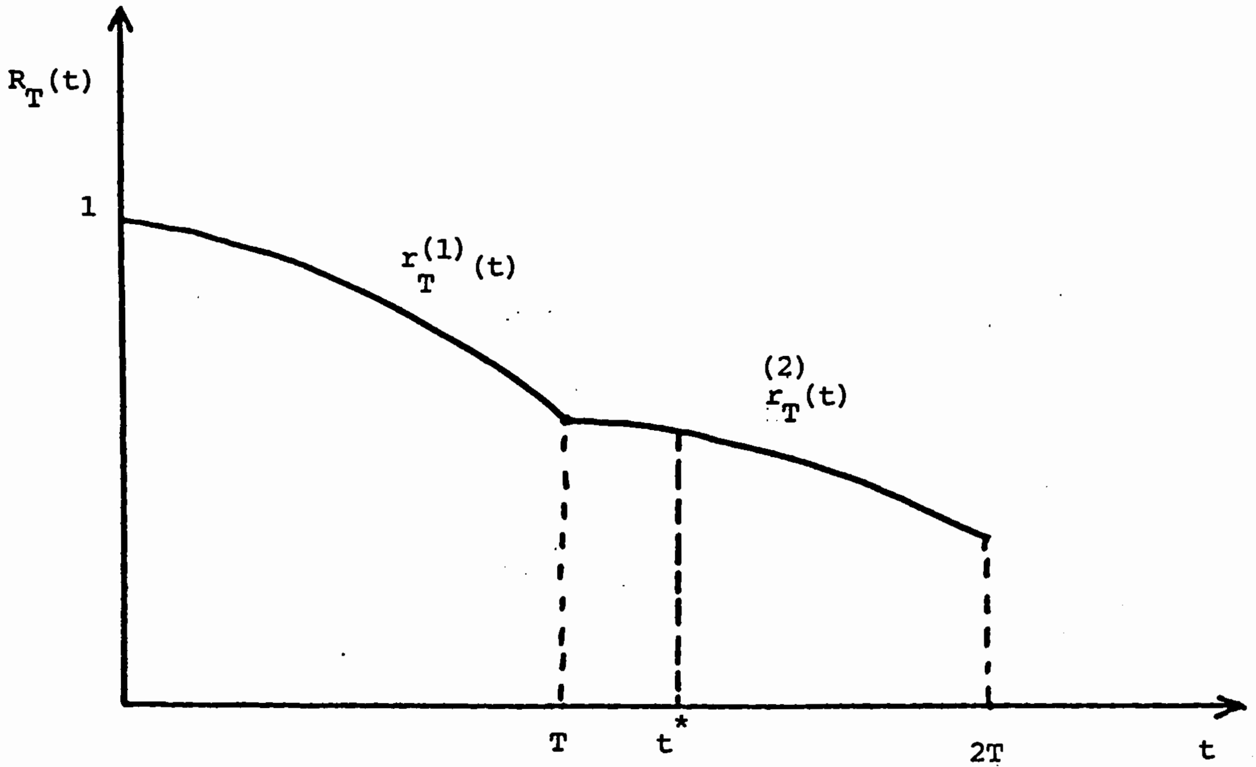
Thus the problem becomes that of determining T such that $r^{(2)}_T(t^*)$ is a maximum. For definitiness we take the densities used by Christer (1987) with the delay time density $f(h) = \alpha e^{-\alpha h}$ and $g(y)$ as uniformly distributed on [0, 10].

From equations (1) - (3) we obtain

$$10\, \alpha\, \kappa_1(T) = 1 - e^{-\alpha T}$$

$$10\, \alpha\, B_T(t^*) = (10 - t^*)\alpha + 1 - e^{-\alpha(t^* - T)}, \; t^* \leq 10$$

$$10\, \alpha\, r^{(1)}_T(u) = (10 - u)\alpha + 1 - e^{-\alpha u}$$



<u>Figure 1:</u> Diagram showing $R_T(t)$ for $0 \le t \le 2T$ and the location of t^* allowing for only one inspection.

so that equation (4) becomes

$$(10\alpha)^2 r_T^{(2)}(t^*) = A + \alpha T - (b + \alpha T) e^{-\alpha T} - Be^{\alpha T}, \ \frac{t^*}{2} \le T \le t^* \le 10 \qquad (5)$$

where

$$a = 10\alpha + 1 \ , \ b = a - \alpha t^*$$

and

$$A = ab + e^{-\alpha t^*}, \ B = ae^{-\alpha t^*}.$$

If we let $Y = (10\alpha)^2 \ r_T^{(2)} (t^*)$ then we obtain from equation (5)

$$\frac{1}{\alpha} \frac{dY}{dT} = 1 + (b + \alpha T - 1)e^{-\alpha T} - Be^{\alpha T}. \qquad (6)$$

Further, a maximum exists since

$$\frac{1}{\alpha^2} \frac{d^2Y}{dT^2} = - \left\{ \left[(10 - (t^* - T))\alpha + 1 \right] e^{-\alpha T} + a e^{-\alpha (t^* - T)} \right\} \le 0$$

over the interval of interest namely, $\frac{t^*}{2} \le T \le t^* \le 10.$

Consider specifically the situation when $\alpha = 0.5$ and $t^* = 8$ then, $4 \le T \le 8$

and, from (5) - (6),

$$\frac{dr_T^{(2)}(8)}{dT} = \frac{1}{50} \left\{ 1 - 6e^{-4} e^{T/2} + (1 + T/2)e^{-T/2} \right\}.$$

The critical point is given by the intersection of the curves

$$y_1(T) = 6e^{-4}e^{T/2} - 1 \text{ and } y_2(T) = (1 + T/2)e^{-T/2} \text{ for } 4 \le T \le 8 \ . \qquad (7)$$

The diagram in Figure 2 shows a sketch of these curves and their intersection gives T_{max} which is obtained by some root finding procedure as $T_{max} = 4.896$.

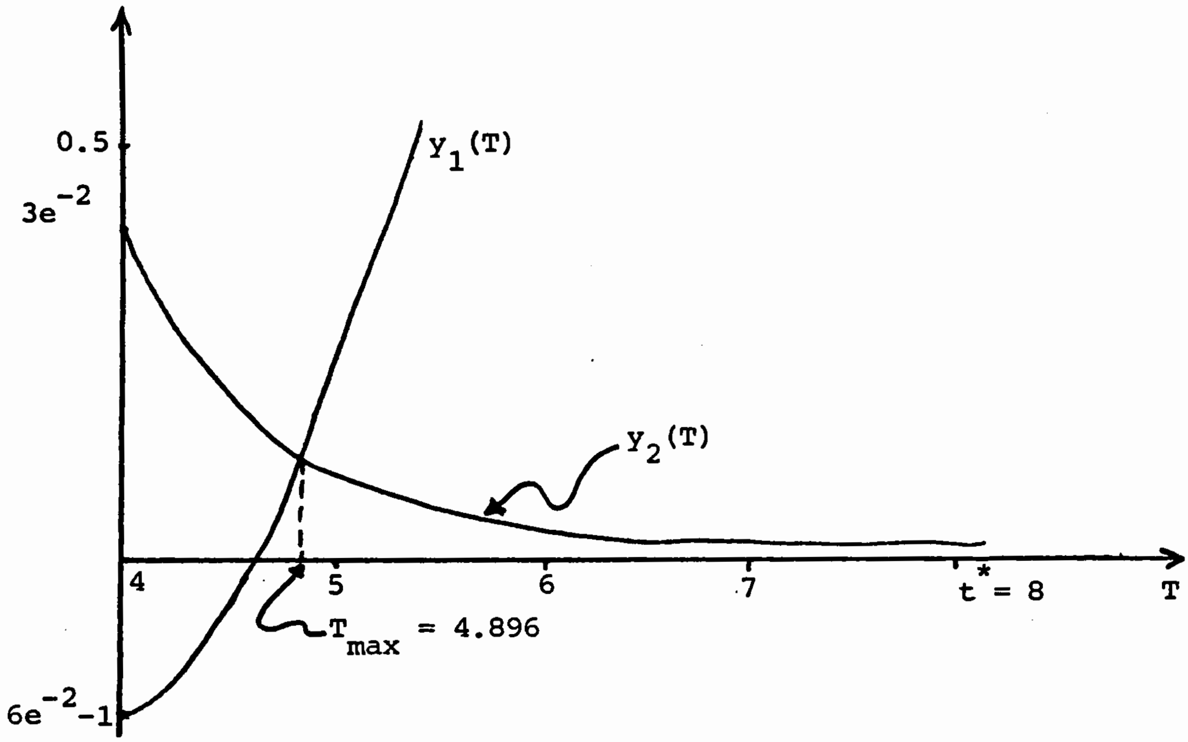


Figure 2: Diagram showing y_1 (T) and y_2 (T) as given by equation (7). The location of T_{max} is obtained at the intersection.

Substitution of $T_{max} = 4.896$ into equation (5) gives (with $\alpha = 0.5$) the maximum possible reliability at $t^* = 8$, given that only one inspection is performed, as $r^{(2)}_{T_{max}}(8) = 0.5124$.

Similarly, $r^{(2)}_{T_{max}}(10) = 0.327$ where $T_{max} = 6.637$ and we notice that the reliability is lower since it relates to a later time of $t^* = 10$.

4. NUMERICAL SOLUTION OF $r^{(m)}_T(t)$

Before proceeding to the solution of the converse problem as represented by equation (3) we investigate the numerical solution of equation (1) for general densities $f(h)$ and $g(y)$. Christer [1987] solved equation (1) for $f(h) = \alpha e^{-\alpha h}$ and $g(y)$ uniform on [0,10].

We may notice from equation (1) that the evaluation of $r^{(m)}_T(t)$ at $t = (m - 1 + \lambda)T$ where $0 \le \lambda \le 1$ requires all previous $r^{(k)}_T(t)$ for $k = (m-1), (m-2), \ldots, 1$ as can be seen from

$$r^{(m)}_T((m-1+\lambda)T) = \sum_{j=1}^{m-1} \kappa_j(T)\, r^{(m-j)}_T((m-j-1+\lambda)T) + B_T((m-1+\lambda)T),\ 0 \le \lambda \le 1. \tag{8}$$

The $r^{(k)}_T(t)$ should be evaluated in the order $k = 1, 2, \ldots, (m-1)$ since successive terms depend on all previous terms. It should further be noted that $\lambda = 0$ represents the evaluation at the left of an inspection interval and $\lambda = 1$ corresponds to the right. The expression for $B_T((m-1+\lambda)T)$ needed in (8) and given in (2) may be written in the following form:

$$B_T((m-1+\lambda)T) = 1 - \sum_{j=1}^{m-1} \int_{(j-1)T}^{jT} g(y)dy - \int_{(m-1)T}^{(m-1+\lambda)T} g(y)F((m-1+\lambda)T-y)dy. \tag{9}$$

Thus the numerical evaluation of (8) with (9) involves the evaluation of integrals over an interval of at most of length T. This gives the ability to control the accuracy of integration. It may further be observed that $\lambda = 0$ corresponding to the evaluation on the left hand side of an inspection interval, eliminates the second integral term in (9) and simplifies the working. This fact allows for a fast determination of the behaviour of the reliability by evaluation at an inspection point.

Figures 3, 4 and 5 show the numerical solution of $r_T^{(m)}(t)$ for $0 \leq t \leq 20$ with $T = 10, 5$ and 2.5 using a variety of densities as indicated in the captions. Equispaced points with $\lambda = 0.5$ were taken to produce the figures however a variable λ to take into account the behaviour of $r_T^{(m)}(t)$ could possibly be used.

Figure 3 uses the densities $f(h) = \alpha e^{-\alpha h}$, $\alpha = 0.5$ and g(y) is uniform on [0,10] for which a closed form expression was obtained by Christer [1987] which has enabled a comparison with the numerical procedure.

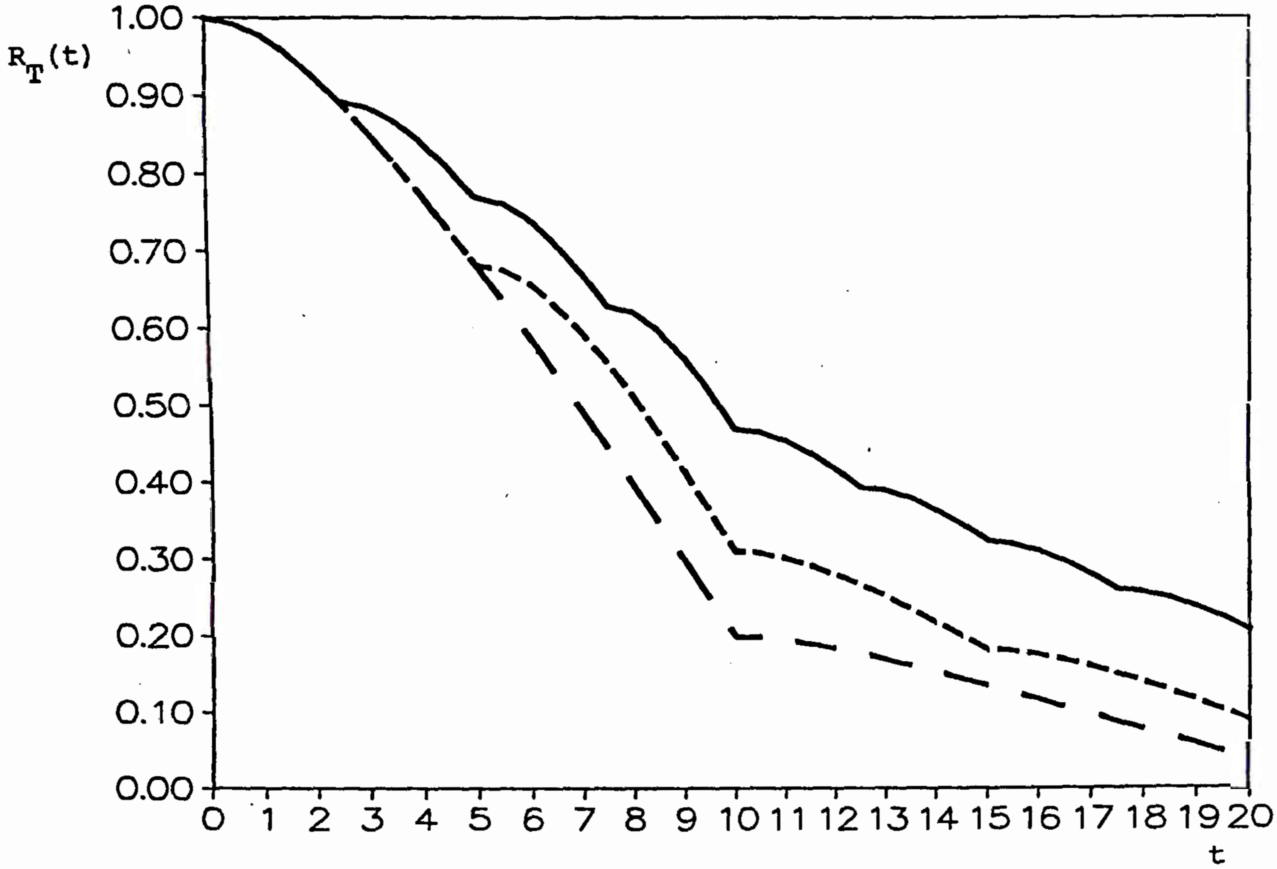
The behaviour of $r_T^{(m)}(t)$ for $(m-1)T \leq t \leq mT$ shown in Figures 3-5 may be expected intuitively.

The monotonic behaviour may also be shown from the differentiation of equation (1) to obtain

$$\dot{r}_T^{(m)}(t) = \sum_{j=1}^{m-1} \kappa_j(T)\, \dot{r}_T^{(m-j)}(t-jT) - b_T(t) \;, \qquad (m-1) \leq t \leq mT, \qquad (10)$$

where

$$b_T(t) = -\dot{B}_T(t) = \int_{(m-1)T}^{t} g(y)\, f(t-y)\, dy \;.$$



<u>Figure 3:</u> The figure shows R_T (t), as defined in equations (1) - (2), for $0 \leq t \leq 20$ and T = 10 (— —), T =5 (– – –) and T = 2.5 (____). The densities are $f(h) = 0.5e^{-0.5\,h}$ and g (y) is uniform on [0, 10].

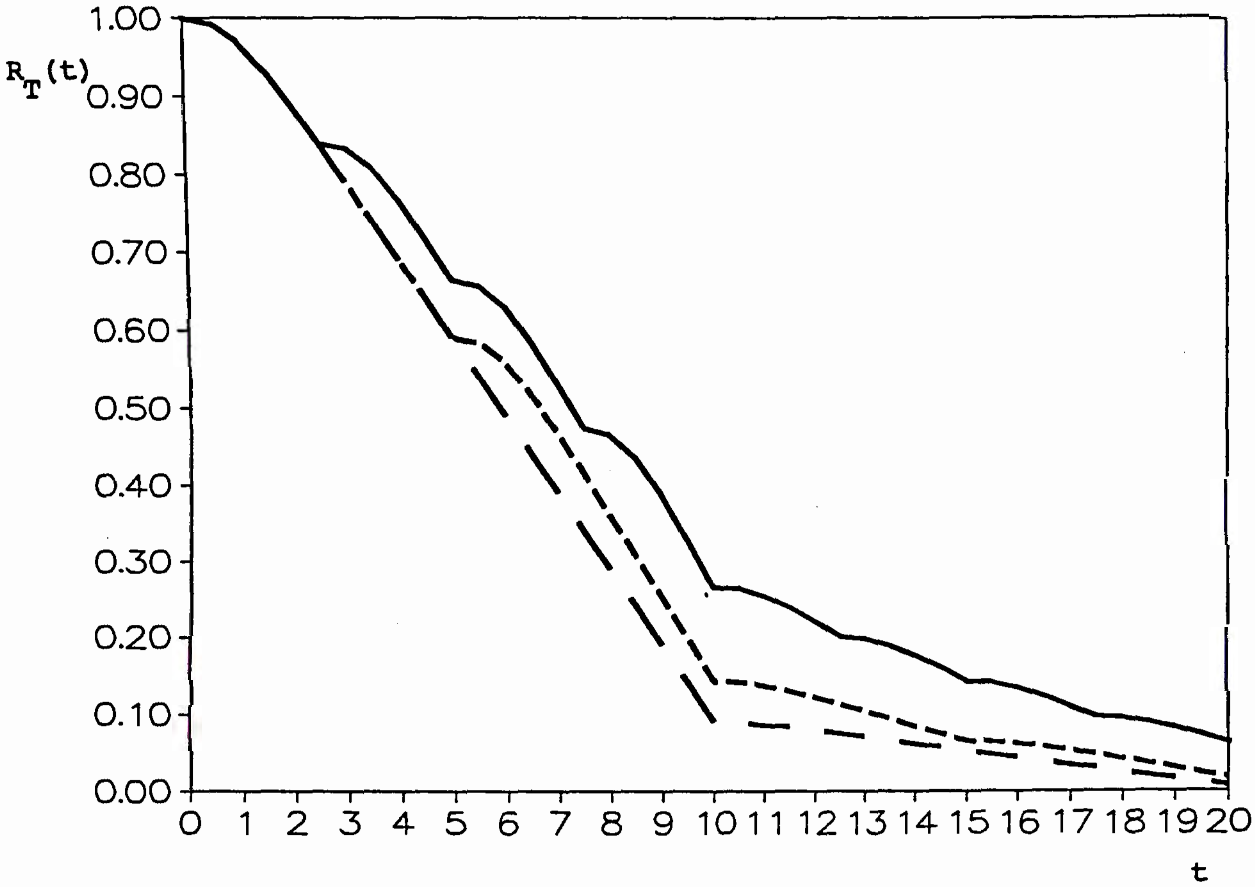


Figure 4: The figure shows R_T (t) for $0 \le t \le 20$ and T = 10 (— —), T = 5 (- - -) and T = 2.5 (____). The densities are $f(h) = \frac{3}{2} h^{\frac{1}{2}} e^{-h^{\frac{3}{2}}}$ and g(y) is uniform on [0, 10].

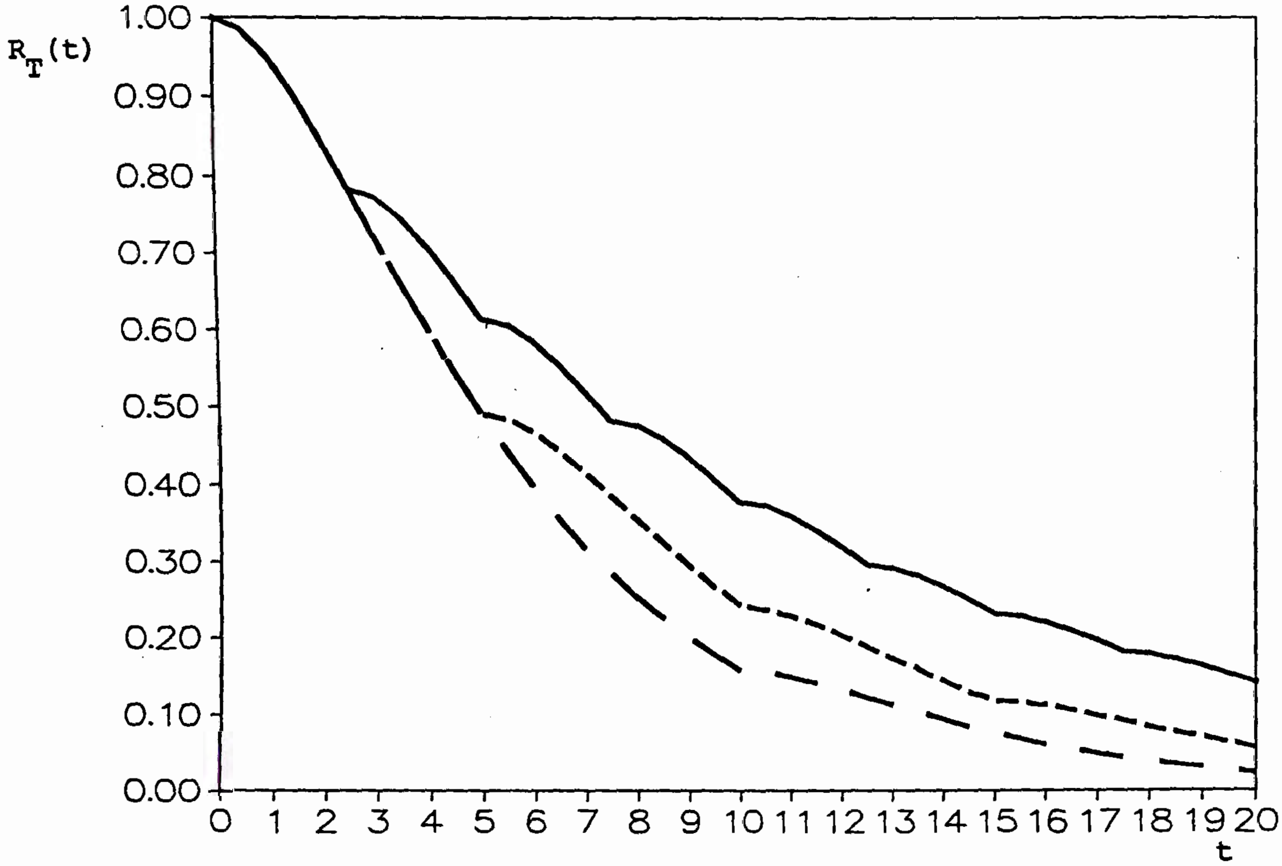


Figure 5. The graphs depict $R_T(t)$ for $0 \le t \le 20$ and T = 10 (— —), T = 5 (-- --) and T = 2.5 (____). The densities are $f(h) = 0.5\,e^{-0.5h}$ and $g(y) = 0.25e^{-0.25h}$.

We may deduce from equation (10) that $r_T^{(m)}(t)$ is a continuous monotonically decreasing function of t with a zero right hand slope at t = (m-1) T with possible discontinuities of derivatives at multiples of T.

We may notice that the graphs in Figure 4 decrease at a faster rate than those in Figure 3 since it takes on average shorter delay time for a fault to become serious enough for action to be taken on the unit. The delay time density f(h) in Figure 4 is Weibull with the shape parameter $\beta = 1.5$ and the characteristic life $\eta = 1.0$ rather that $f(h) = 0.5e^{-0.5h}$ used to produce Figure 3. Further, the graphs is Figure 5 also decrease at a faster rate than those in Figure 3 where the delay time density is the same but $g(y) = 0.25e^{-0.25y}$ rather than uniform on [0, 10] so that failures are occuring more frequently on average.

5. SOLUTION OF THE GENERAL CONVERSE PROBLEM

Returning now to the solution of equation (3) we note that in order to evaluate

$r_T^{(m)}(t^*)$ for $\frac{t^*}{m} \leq T \leq \frac{t^*}{m-1}$ we need $r_T^{(k)}(t^* - (m-k)T)$

for k = 1, 2, ..., m-1.

Thus we may use the procedure outlined in the previous section to evaluate.

$$r_T^{(k)}(t) = \sum_{j=1}^{k-1} \kappa_j(T)\, r_T^{(k-j)}(t - jT) + B_T(t) \qquad (11)$$

at $t = t^* - (m-k)T$ for k = 1,2, ...m to give $r_T^{(m)}(t^*)$.

Here in equation (11), $\kappa_j(T)$ and $B_T(t)$ are given by equation (2).

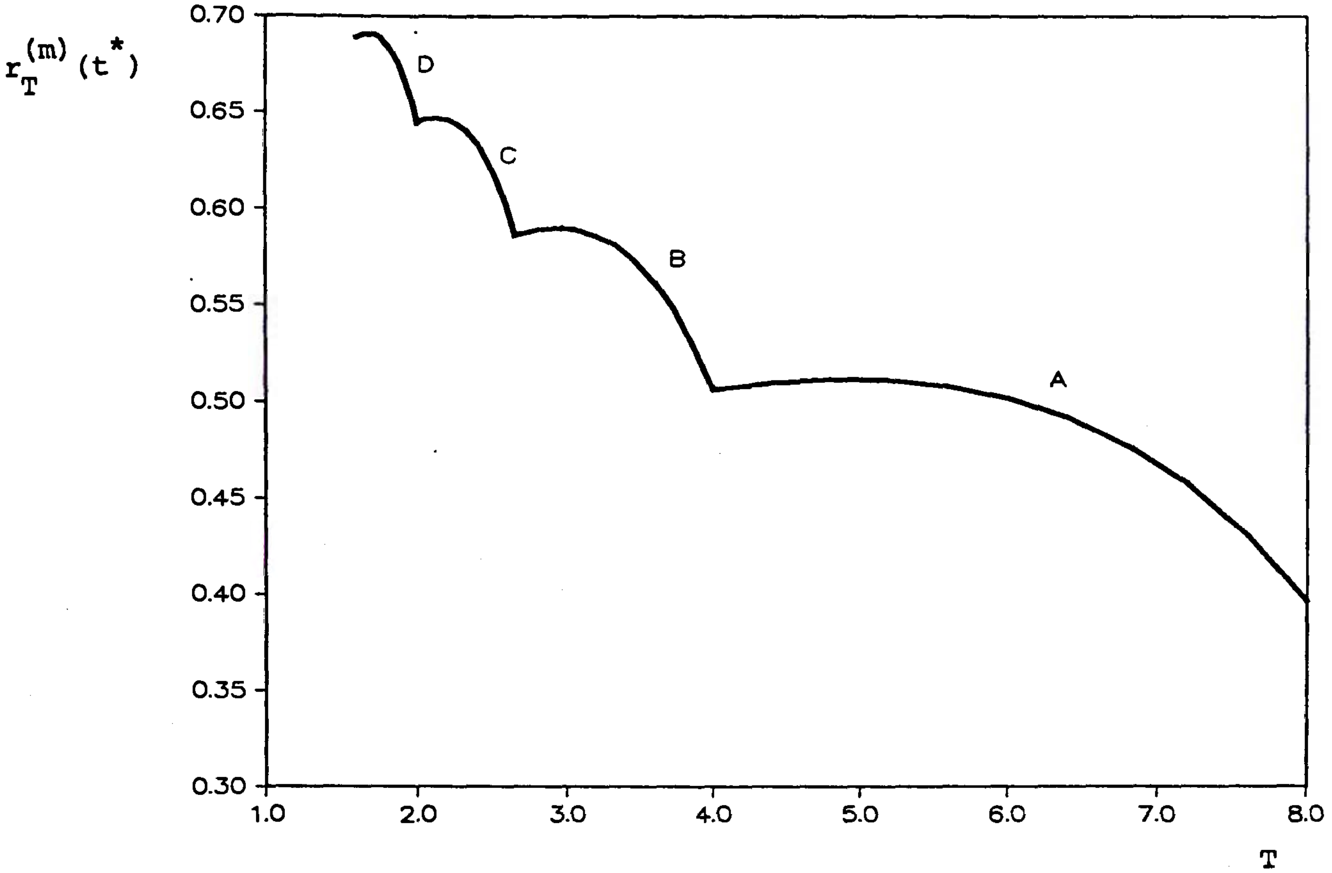
Figure 6 - 8 are based on the exponential delay time density $f(h) = \alpha e^{-\alpha h}$, $\alpha = 0.5$ and g (y) is uniform on [0, 10]. The three figures show the reliability at $t^* = 8, 10, 12$ respectively with varying inspection interval T for one inspection (A) through to four inspections (D). The possible range of T clearly depends upon t^* and m. In each of the sections of the graphs A-D we notice that there is a point T_{max}, depending on the number of inspections, which results in the maximum reliability a $t = t^*$.

There are a number of observations that can be made from Figures 6 -8. We may notice the effect of the uniform density on [0, 10] coming through in Figure 8 in which a defect exisiting in an original component will lead to a renewal if it has not caused a component failure. This effect does not manifest itself in either Figure 6 or 7 since $t^* < 10$.

Further, it is interesting to note that the inspection period T_{max} occurs closer to the left of the interval $\left[\frac{t^*}{m}, \frac{t^*}{m-1} \right]$ the smaller t^* is. We may also observe that T_{max} migrates towards the left hand side of the interval of interest, namely towards $\frac{t^*}{m}$, which increasing m.

This observation begs the question as to how many inspections are needed prior to $r^{(m)}_{T_{max}}(t^*)$ being as close as we wish to $r^{(m)}_{\frac{t^*}{m}}(t^*)$.

We use two measures L_m and R_m to demonstrate the approach of T_{max} towards $\frac{t^*}{m}$. L_m shows the relative difference between these values and R_m shows the relative effect on the reliability at $t=t^*$ if the inspection period was taken as $\frac{t^*}{m}$ rather than T_{max}.



<u>Figure 6</u>: Graph showing $r_T^{(m)}(t^*)$, $\frac{t^*}{m} \le T \le \frac{t^*}{m-1}$, for $t^* = 8$

with m-1 = 1 (A), 2(B), 3(C), 4(D) inspections where $f(h) = \alpha e^{-\alpha h}$, $\alpha = 0.5$ and g(y) is uniform on [0,10].

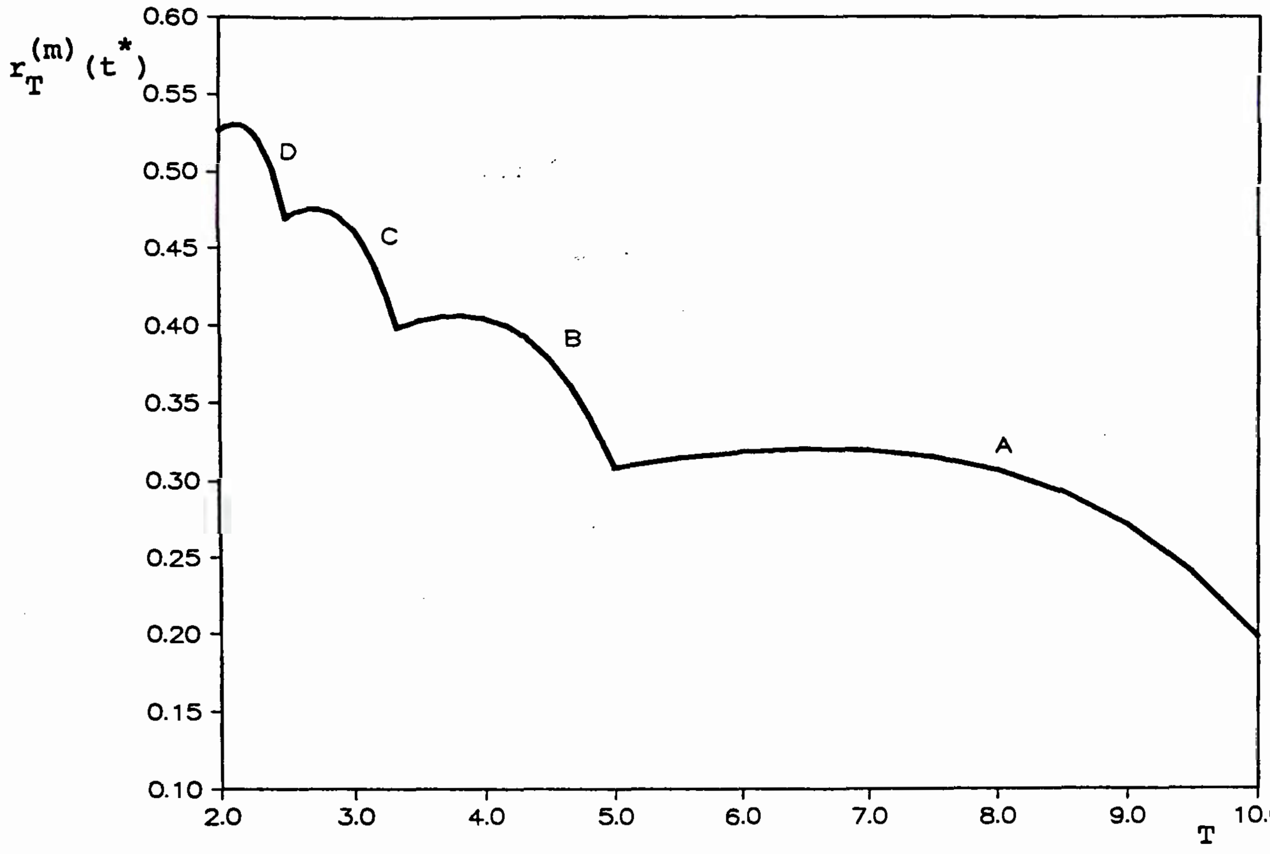


Figure 7: Graph showing $r_T^{(m)}(t^*)$, $\frac{t^*}{m} \leq T \leq \frac{t^*}{m-1}$ for $t^* = 10$ with $m-1 = 1$ (A), 2(B), 3(C), 4(D) inspections where $f(h) = \alpha e^{-\alpha h}$, $\alpha = 0.5$ and $g(y)$ is uniform on [0, 10].

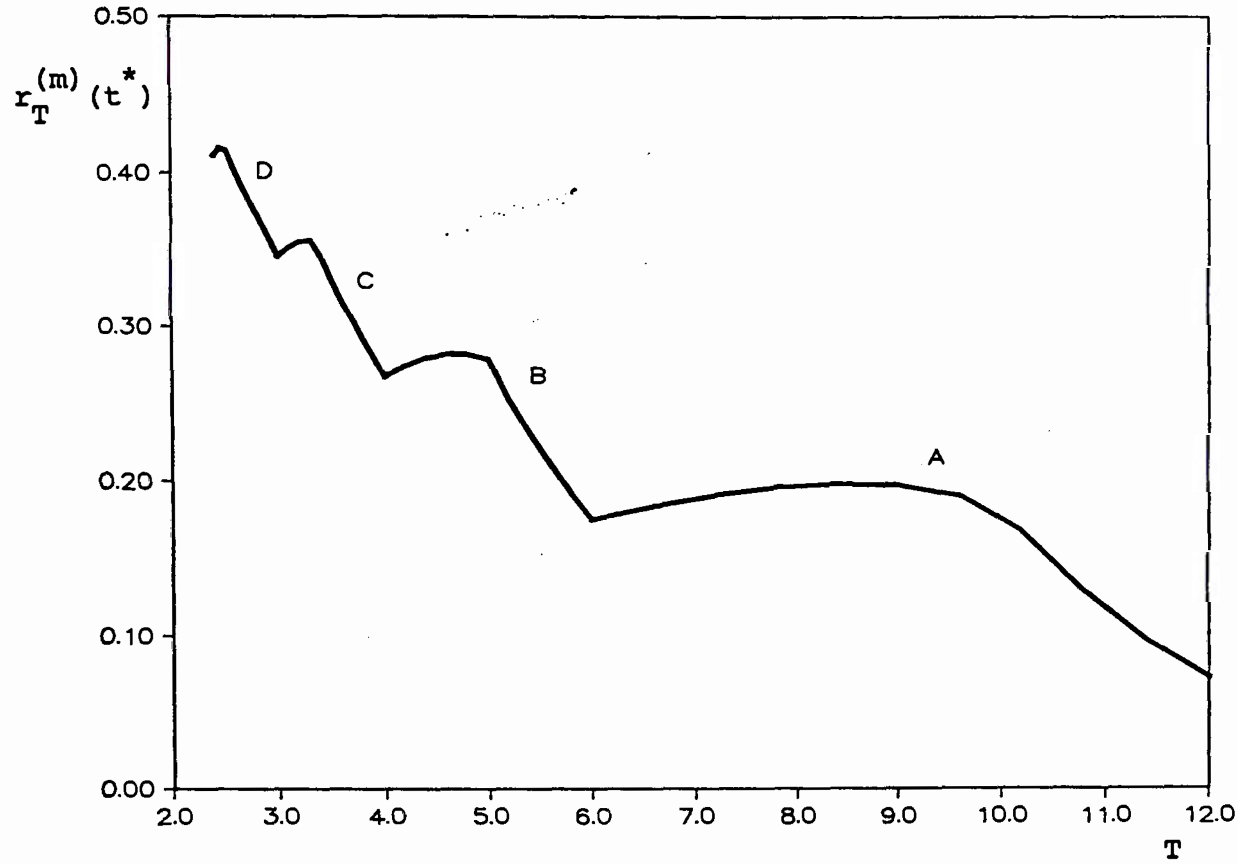


Figure 8: Graph showing $r_T^{(m)}(t^*)$, $\frac{t^*}{m} \leq T \leq \frac{t^*}{m-1}$ for $t^* = 12$ with $m-1 = 1$ (A), 2(B), 3(C), 4(D) inspections where $f(h) = \alpha = 0.5$ and $g(y)$ is uniform on [0, 10].

Table 1 shows L_m the ratio of the distance between T_{max} and t^*/m to the total length of the interval of observation D_m viz

$$L_m = \frac{T_{max} - \frac{t^*}{m}}{D_m} \tag{12}$$

where $D_m = \dfrac{t^*}{m(m-1)}$.

The table also shows

$$R_m = \frac{r^{(m)}_{T_{max}}(t^*) - r^{(m)}_{\frac{t^*}{m}}(t^*)}{r^{(m)}_{T_{max}}(t^*)} \tag{13}$$

demonstrating that the maximum reliability at $t = t^*$ very quickly approaches from the right $T = \frac{t^*}{m}$ the smaller t^* is and the greater the number of inspections m-1.

Determining the value of T_{max} is more crucial the larger t^* and the fewer number of inspections required.

It is interesting to observe the effect of inspections on $r^{(m)}_{T_{max}}(t^*)$ from Table 1 for $t^* = 8$, 10 and 12. These may be compared with the reliability values of 0.3963, 0.1986 and 0.0731 respectively when no inspection takes place. These values correspond to the rightmost points in Figure 6,7 and 8 respectively.

t^*	m	T_{max}	$\frac{t^*}{m}$	L_m	$r^{(m)}_{T_{max}}(t^*)$	$r^{(m)}_{\frac{t^*}{m}}(t^*)$	R_m
8	2	4.8960	4	0.224	0.5124	0.5066	0.011
	3	2.9397	2.66•	0.051	0.5902	0.5865	0.006
	4	2.1344	2	0.022	0.6476	0.6450	0.004
	5	1.6640	1.6	0.010	0.6912	0.6894	0.003
10	2	6.6368	5	0.327	0.3221	0.3091	0.0404
	3	3.7973	3.33•	0.070	0.4072	0.3989	0.0204
	4	2.7133	2.5	0.028	0.4757	0.4699	0.0122
	5	2.1280	2	0.016	0.5309	0.5266	0.0081
12	2	8.5056	6	0.418	0.1995	0.1757	0.119
	3	4.6880	4	0.086	0.2834	0.2678	0.055
	4	3.3507	3	0.037	0.3588	0.3457	0.036
	5	2.5098	2.4	0.011	0.4199	0.4113	0.020

Table 1: The table shows the approach of T_{max} towards $\frac{t^*}{m}$ for increasing number of inspection (m-1) as signified by L_m . R_m shows the approach of the reliability at T_{max} towards that at $\frac{t^*}{m}$. L_m and R_m are as defined in equations (12) and (13) respectively. The densities are $f(h) = \alpha e^{-\alpha h}$, $\alpha = 0.5$ and g(y) is uniform on [0, 10].

It may further be observed from Figures 6 on, that:

(i) $r_{T_{max}}^{(m_1)}(t^*) < r_{T_{max}}^{(m_2)}(t^*)$ for $m_2 > m_1$,

(ii) $r_{T_1}^{(m)}(t^*) = r_{T_2}^{(m)}(t^*)$ for some $\frac{t^*}{m} \leq T_1, T_2, \leq \frac{t^*}{m-1}$,

(iii) $r_{\frac{t^*}{m-1}}^{(m)}(t^*) < r_T^{(m)}(t^*)$ for some $\frac{t^*}{m} \leq T < \frac{t^*}{m-1}$,

(iv) $r_{\tau}^{(m+1)}(t^*) \leq r_T^{(m)}(t^*)$ for some $\delta_m > 0, \frac{t^*}{m} - \delta_m \leq \tau \leq \frac{t^*}{m}$.

The first observation states that if the best possible inspection period is chosen then increasing the number of inspections improves the reliability. Increasing the number of inspections in itself, is not reasonably enough to guarantee an improvement in the reliability as demonstrated by obervation (iv). Point (ii) follows immediately from the fact that since there is an optimal impection interval T_{max} then there are points T_1 and T_2 with which the reliability at t^* is equal. This point only holds provided T_{max} does not occur at an end point. (It will be shown subsequently that this situation arises when both densities are exponential). Observation (iii) results from the fact that the last inspection is made at the time of interest t^*. It represents the worst case situation.

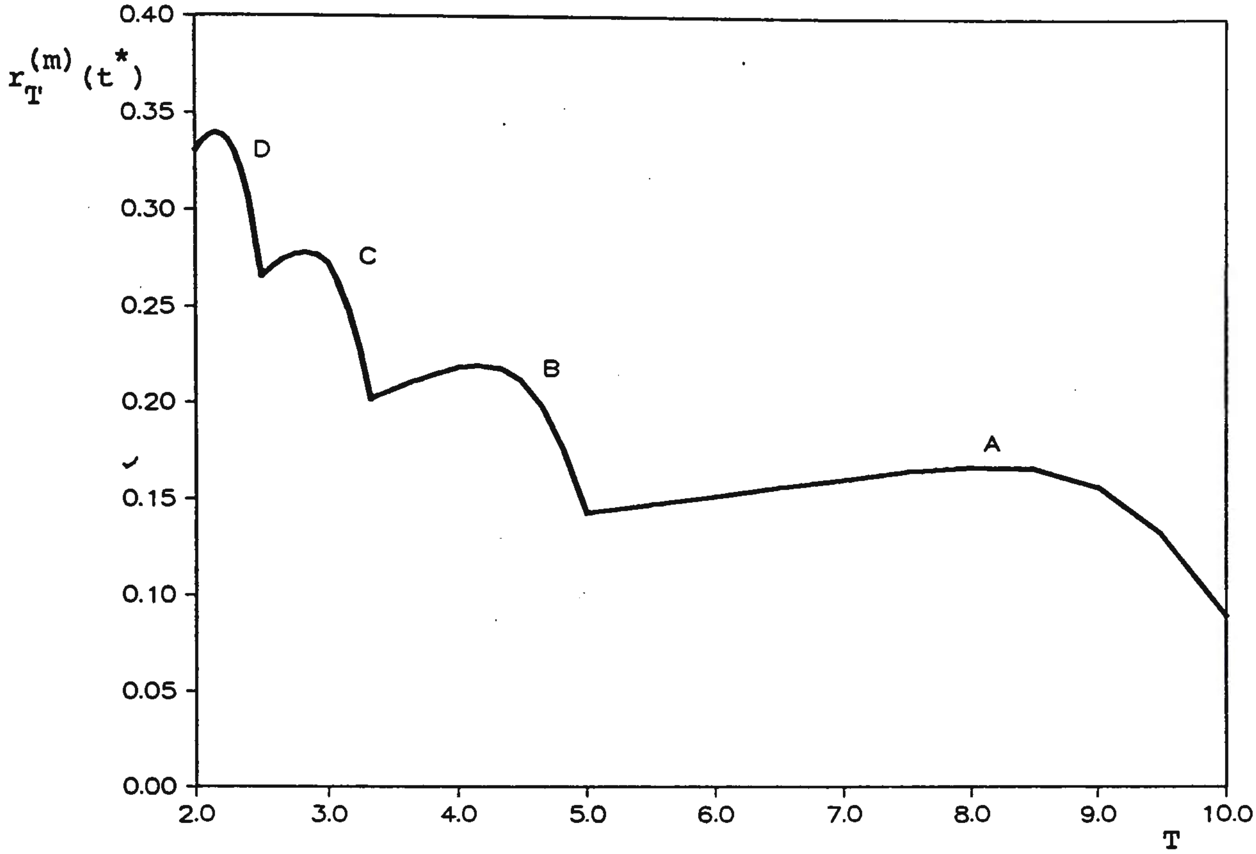


Figure 9: Graph of $r_T^{(m)}(t^*)$, $\frac{t^*}{m} \leq T \leq \frac{t^*}{m-1}$, for $t^* = 10$ with m-1 = 1 (A), 2(B), 3(C), 4(D) inspections where $f(h) = \frac{3}{2} h^{\frac{1}{2}} e^{-h^{\frac{3}{2}}}$ and g(y) uniform on [0, 10].

We now return to looking at the converse problem corresponding to the densities used to produce Figures 4 and 5. The future point of interest $t^* = 10$ is used to produce Figure 9 demonstrating when compared with Figure 7, the effect of a change to a Weibull delay time density which has a shortening of the average delay time when compared to the exponential density with α=0.5. The optimal inspection intervals T_{max} are further to the right in $\left[\frac{t^*}{m}, \frac{t^*}{m-1}\right]$.

Figure 10 shows the graph of $r_T^{(m)}(t^*)$ for $\frac{t^*}{m} \leq T \leq \frac{t^*}{m-1}$ with t^*=10 and m = 2, 3, 4, 5 where both densities are exponential. We notice that the optimal inspection interval occurs at the left hand limit, namely

$$T_{max} = \frac{t^*}{m} . \qquad (14)$$

That is, the maximum reliability is obtained if we choose T in such a way that an inspection is due at our point of interest $t = t^*$ but is not carried out. This is due to the memoryless property of the exponential.

It is interesting to demonstrate equation (14) analytically when the densities are both exponential. This is done in the Appendix.

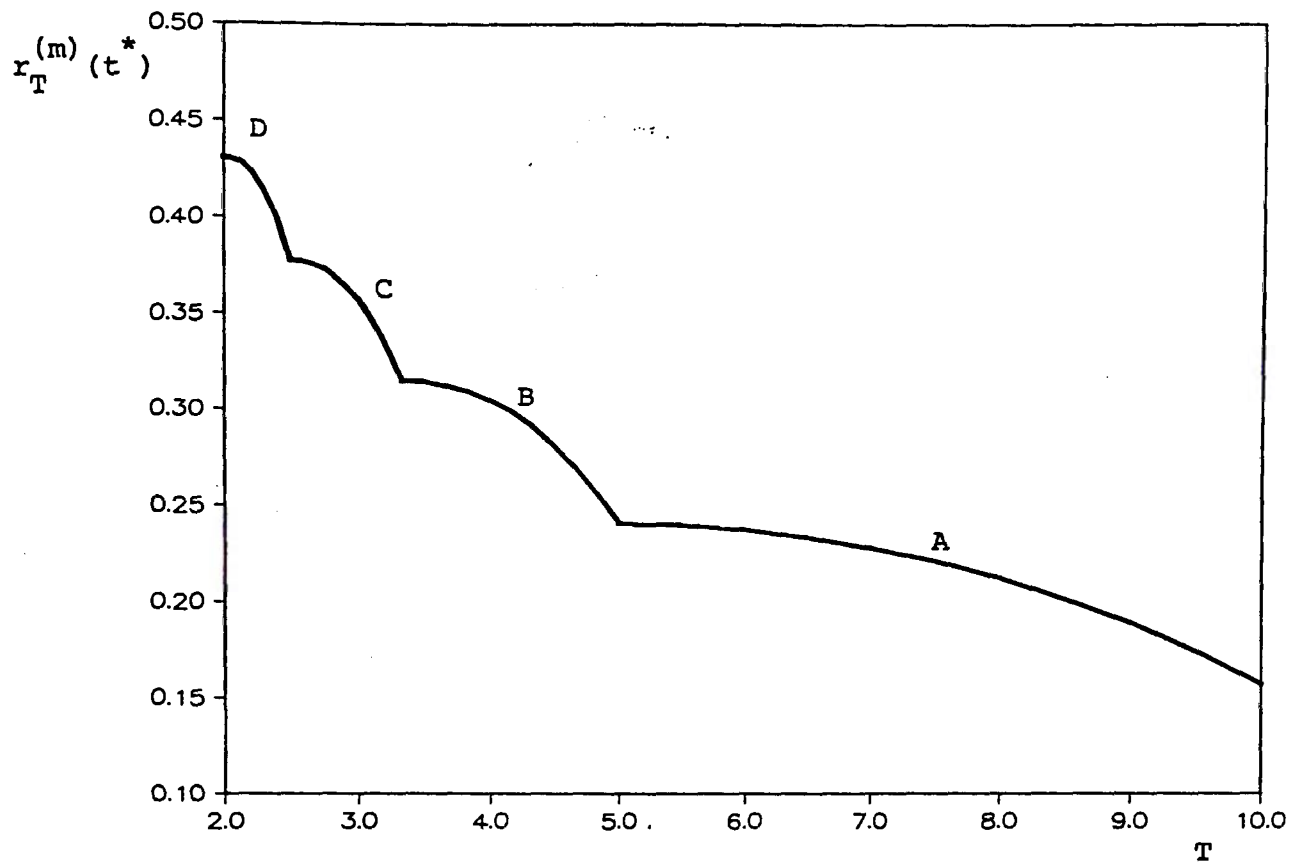


Figure 10: Graph of $r_T^{(m)}(t^*)$, $\frac{t^*}{m} \leq T \leq \frac{t^*}{m-1}$, for $t^* = 10$ with m-1 = 1 (A), 2(B), 3(C), 4(D) inspections where $f(h) = \alpha e^{-\alpha h}$, $\alpha = 0.5$ and $g(y) = \beta e^{-\beta h}$, $\beta = 0.25$.

6. CONCLUSION

The paper has addressed the problem of determing the optimal regular inspection period for maximum reliability at some future point in time for a given number of inspections. The general reliability model developed by Christer, which includes the notion of delay time, has been solved in principle for any two densities of delay time and time to failure from new. The numerical method used has been further developed to allow a solution of the converse problem stated above and the approach of T_{max} towards t^*/m for increasing number of inspections m-1 has been demonstrated.

The work may be developed to take into account the use of cost models giving a trade-off between cost of mission failure and inspection cost. Such a cost model may be to determine m^* and T^* such that

$$K_{m^*,T^*} = \min_{m,T} K_{m,T} \qquad (15)$$

where $K_{m,T} = (m-1)\,c\;\; r_T^{(m)}(t^*) + C\,(1 - r_T^{(m)}(t^*)).$

with c being the inspection cost and C the cost of mission failure.

Equation (15) may be written in a slightly different form from which a number of observations may be made easily.

Viz,

$$K_{m,T} = C + \left[(m-1)c - C\right] r_T^{(m)}(t^*) \;. \qquad (16)$$

Firstly, we note that (m-1)c $<$ C so that the cost of m-1 inspections is less than the cost of mission failure making the term in the square brackets negative. This observation gives us a bound on the number of inspections and so

$$m-1 = 1, 2, \ldots\ldots, \left[\frac{C}{c}\right]. \qquad (17)$$

with $\left[\frac{C}{c}\right]$ meaning the smallest integer part of $\frac{C}{c}$.

A second observation which may be made is that since the term in square brackets in equation (16) is negative, then for fixed m we have that $K_{m,T}$ is minimal where $r_T^{(m)}(t^*)$ is maximal. That is at $r_{T_{max}}^{(m)}(t^*)$ so that $T^* = T_{max}$.

A search through m as given by equation (17) into equation (16) with $T = T^* = T_{max}$ will give m^*. T_{max} does of course change with m even though it is not explicity shown.

Thus the problem of solving (15) becomes that of finding m^* in

$$\min_m \left\{ C + \left[(m-1)\,c - C\right] r_{T_{max}}^{(m)}(t^*) \right\}, \quad m = 2, 3, \ldots, \left[\frac{C}{c}\right] + 1 . \qquad (18)$$

As a simple example consider the problem with $t^*=12$, c =1 and C = 3.5 then, using the data in Table 1, $m^* = 2$ and T^* (corresponding to m=2) = 8.5056. If C=4.5 then, $m^*=3$ and $T^* = 4.6880$.

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APPENDIX: The Double Exponential Problem

We wish to show that $r_T^{(m)}(t^*)$ is maximal when $T = \frac{t^*}{m}$ if both densities are exponential.

Suppose the delay time density to be $f(h) = \alpha e^{-\alpha h}$ and the time to the appearance

of a fault from new to be governed by $g(y) = \beta e^{-\beta y}$ then, from (1), (2)

$$B_T(t) = \frac{1}{\alpha-\beta}\left[\alpha e^{-\beta t} - \beta e^{-\alpha t} . e^{(\alpha-\beta)(m-1)T}\right]$$

$$\kappa_j(T) = \kappa B^{j-1}$$

$$\text{(A1)}$$

with $\kappa = \kappa_1(T) = \frac{\beta}{\alpha-\beta}\left[B-A\right]$

and $A = e^{-\alpha T}, \quad B = e^{-\beta T}.$

Now from (3) and (A1) we obtain

$$\frac{d\, r_T^{(m)}(t^*)}{dT} = \sum_{j=1}^{m-1}\left[\kappa_j'(T)\; r_T^{(m-j)}(t^* - jT) - j\,\kappa_j(T)\,\frac{d}{du}\, r_T^{(m-j)}(u)\right]$$

$$- (m-1)\,\beta e^{-\alpha t^*} e^{(\alpha-\beta)(m-1)T}, \quad \frac{t^*}{m} \le T \le \frac{t^*}{m-1} \qquad \text{(A2)}$$

where $u = t^* - jT$.

Evaluation of (A2) at $T = \frac{t^*}{m}$ gives

$$\frac{dr_T^{(m)}(t^*)}{dT} = \sum_{j=1}^{m-1}\left[\kappa_j'(T)\, r_T^{(m-j)}\left((m-j)\frac{t^*}{m}\right) - j\;\kappa_j(T)\;\dot{r}_T^{(m-j)}\left((m-j)\frac{t^*}{m}\right)\right]$$

$$- (m-1)\;\beta\, a\, B^{m-1} \qquad \text{(A3)}$$

$$\text{where } a = \frac{\alpha B - \beta A}{\alpha - \beta} \qquad \text{(A4)}$$

$$\kappa_j^{'}(T) = \beta\, B^{j-1} \left[A - j\,\kappa \right]$$

and A, B and κ are given in (A1) with $T = \frac{t^*}{m}$.

We now notice that to evaluate (A3) we need

$$r^{(k)}_{\frac{t^*}{m}} \left(k \frac{t^*}{m}\right) \text{ and } \dot{r}^{(k)}_{\frac{t^*}{m}} \left(k \frac{t^*}{m}\right) \quad \text{for } k = 1,2, \ldots, m-1.$$

To this end putting $T = \frac{t^*}{m}$ in equation (3) and using (A1) we obtain the recurrence

relation

$$u_m = \kappa \sum_{j=1}^{m-1} B^{j-1}\, u_{m-j} + a\, B^{m-1} \qquad \text{(A5)}$$

where $u_m = r^{(m)}_{\frac{t^*}{m}} \left(m \frac{t^*}{m}\right) = r^{(m)}_{\frac{t^*}{m}} (t^*)$.

Equation (A5) can be shown to have a solution given by

$$u_m = a^m \qquad \text{(A6)}$$

with $\qquad a = \kappa + B$ from (A1) and (A4).

Further, differentiating equation (1) with respect to t, using equation (A1) and putting

$T = \frac{t^*}{m}$ and $t = t^*$ we obtain

$$v_m = \kappa \sum_{j=1}^{m-1} B^{j-1}\, v_{m-j} - \alpha\, \kappa B^{m-1}$$

which has solution

$$v_m = -\alpha \kappa a^{m-1} \tag{A7}$$

with $v_m = \dot{r}^{(m)}_{\frac{t^*}{m}}\left(m \cdot \frac{t^*}{m}\right) = \dot{r}^{(m)}_{\frac{t^*}{m}}(t^*)$.

Substitution of (A4), (A6) and (A7) into equation (A3) gives after some algebra

$$\frac{dr_T^{(m)}}{dT}(t^*) = \beta A \left\{ \sum_{j=1}^{m-1} B^{j-1}\ a^{m-j-1} \left[a - j\,(a-B)\right] - (m-1)\ B^{m-1} \right\}$$

$$= \beta A \left\{ \sum_{j=1}^{m-1} \left[j\, B^j\, a^{m-j-1} - (j-1)\, B^{j-1}\, a^{m-j}\right] - (m-1) B^{m-1} \right\}$$

$$= \beta A \left\{ \sum_{j=1}^{m-1} j\, B^j\, a^{m-j-1} - \left[\sum_{j=1}^{m-2} j\, B^j\, a^{m-j-1} + (m-1) B^{m-1} \right] \right\}$$

$$= 0.$$

Thus $\frac{dr_T^{(m)}}{dT}(t^*) = 0$ when $T = \frac{t^*}{m}$ and so is a critical point. It remains to show

that it is maximal. To this end, we need to show that

$$r^{(m)}_{\frac{t^*}{m}}(t^*) > r^{(m)}_{\frac{t^*}{m-1}}(t^*) . \tag{A8}$$

Now, from (A6) and using (A1) and (A4) we obtain

$$r^{(m)}_{\frac{t^*}{m}}(t^*) = a^m = \left[\frac{\alpha\, e^{-\beta \frac{t^*}{m}} - \beta\, e^{-\alpha \frac{t^*}{m}}}{\alpha - \beta} \right]^m . \tag{A9}$$

Further, using equation (3) and (A1) with $T = \frac{t^*}{m-1}$ results in

$$w_m = r^{(m)}_{\frac{t^*}{m-1}} \left((m-1) \cdot \frac{t^*}{m-1}\right)$$

satisfying

$$w_m = \kappa \sum_{j=1}^{m-1} B^{j-1} w_{m-j} + B^{m-1}$$

which has solution

$$w_m = a^{m-1}$$

with a given by (A4) and $A = e^{-\alpha \frac{t^*}{m-1}}$, $B = e^{-\beta \frac{t^*}{m-1}}$.

Thus,

$$r^{(m)}_{\frac{t^*}{m-1}} (t^*) = \left[\frac{\alpha e^{-\beta \frac{t^*}{m-1}} - \beta e^{-\alpha \frac{t^*}{m-1}}}{\alpha - \beta} \right]^{m-1} . \qquad \text{(A10)}$$

The inequality given by (A8) can be shown to hold since from equation (A9) and (A10) we have with $D = e^{-\beta t^*}$ and $C = e^{-\alpha t^*}$

$$\left(\frac{\alpha D^{\frac{1}{m}} - \beta C^{\frac{1}{m}}}{\alpha D^{\frac{1}{m-1}} - \beta C^{\frac{1}{m-1}}} \right) \left(\frac{\alpha D^{\frac{1}{m}} - \beta C^{\frac{1}{m}}}{\alpha - \beta} \right)$$

$$\geq \frac{\alpha D^{\frac{1}{m}} - \beta C^{\frac{1}{m}}}{\alpha - \beta} \geq 1.$$

Hence the maximum occurs when $T = \frac{t^*}{m}$ if both densities are exponential.

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