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DEPARTMENT OF COMPUTER AND MATHEMATICAL SCIENCES

Generalisation of a Waiting Time Relation

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Abstract

A generalisation of a waiting time relation is developed by the use of Laplace Transform theory. The generalisation produces an infinite series and it is demonstrated how it may be summed by representation in closed form. Extensions and examples of the waiting time relation are given.

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1. Introduction

It seems that the sum, after a rearrangement to suit the following work,

$$\sum_{n=0}^{\infty}(-1)^n\left(abe^{ab}\right)^n\frac{(t+n)^n}{n!}=\frac{e^{-abt}}{1+ab} \tag{1}$$

first appeared in the work of Jensen [10]. Jensen's work was based on an extension of the Binomial theorem due to Abel and an application of Lagrange's formula.

In the analysis of the delay in answering of telephone calls Erlang [7] obtains an integro-differential-difference equation from which a similar result to (1) is quoted. Likewise a series similar to (1) later appeared in the works of Bruwier [3] and [4] in his analysis of differential-difference equations. In fact, the result (1) arises in a number of areas including the works of Feller [8] on ruin problems, Hall [19] on coverage processes, Smith [13] on renewal theory and Tijms [14] on stochastic modelling, just to name a few. To date no extension to (1) appears to be available. It is therefore the aim of this paper to give a novel technique for the development and generalisation of (1). Recurrence relations will be developed and further extensions to the results indicated.

2. The Differential-Difference Equation

Consider the differential-difference equation

$$\left.\begin{aligned}\sum_{n=0}^{R}\binom{R}{R-n}c^{R-n}\sum_{r=0}^{n}\binom{n}{r}b^{n-r}f^{(r)}\big(t-(R-n)a\big)=0 \quad &; \quad t>Ra\\ \sum_{r=0}^{R}\binom{R}{r}b^{R-r}f^{(r)}(t)=0 \qquad\qquad &; \quad 0<t\le Ra\end{aligned}\right\} \tag{2}$$

with $f^{(R-1)}(0)=1$ and all other initial conditions at rest, where $a, b,$ and c are real constants.

Erlang [2] considered equation (2) in his work on the delay in answering of telephone calls for the case of $R=1$ only. For the case of R servers Erlang derived a differential-difference equation different than (2) and this will be the subject of a forthcoming paper.

It has become common place to analyse differential-difference equations by the use of Laplace Transform Theory. In this paper Laplace Transform techniques will be used to bring out the essential features that are required for the results.

Taking the Laplace transform of (2) results in

$$F(p)=\mathscr{L}\{f(t)\}=\frac{1}{\left(p+b+ce^{-ap}\right)^R}=\sum_{n=0}^{\infty}\binom{n+R-1}{n}(-1)^n\frac{c^n e^{-an(p+b)}e^{anb}}{(p+b)^{n+R}}. \quad (3)$$

The inverse Laplace transform is

$$f(t)=\sum_{n=0}^{\infty}\binom{n+R-1}{n}(-1)^n c^n e^{-b(t-an)}\frac{(t-an)^{n+R-1}}{(n+R-1)!}H(t-an) \quad (4)$$

where $H(x)$ is the unit Heaviside step function.

The solution to (2) by Laplace transform theory may be written as

$$f(t)=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}e^{pt}[g(p)]^{-1}dp,$$

for an appropriate choice of γ such that all the zeros of the characteristic equation

$$g(p)=\left(p+b+ce^{-ap}\right)^R \quad (5)$$

are contained to the left of the line in the Bromwich contour.

Now using the residue theorem

$$f(t)=\sum \text{Residues of}\left\{e^{pt}[g(p)]^{-1}\right\}$$

which suggests the solution of $f(t)$ maybe written in the form

$$f(t)=\sum_r Q_r e^{p_r t}$$

where the sum is over all the characteristic roots p_r of $g(p)=0$ and Q_r is the contribution of the residues in $F(p)$ at $p=p_r$.

The poles of (3) depend on the zeros of the characteristic equation (5), the roots of $g(p)=0$. The dominant root p_0 of $g(p)=0$ has the greatest real part and therefore asymptotically

$$f(t)\sim\sum_{k=0}^{R-1}k!\,Q_{-(R-k)}\frac{t^{R-k-1}}{(R-k-1)!}e^{p_0 t}$$

and so from (4)

$$f(t)=\sum_{n=0}^{\infty}(-1)^n \frac{c^n e^{-b(t-an)}(t-an)^{n+R-1}}{n!(R-1)!}H(t-an) \sim \sum_{k=0}^{R-1} k!\,Q_{-(R-k)} \frac{t^{R-k-1}}{(R-k-1)!}e^{p_0 t} \qquad (6)$$

where the contribution $Q_{-(R-k)}$ to the residue

$$k!\,Q_{-(R-k)} = \lim_{p\to p_0}\left\{\frac{d^{(k)}}{dp^{(k)}}\left(\frac{(p-p_0)^R}{g(p)}\right)\right\} \quad , \quad k=0,1,2.....R-1 \qquad (7)$$

since the right hand side of (3) has a pole of order R at the dominant root $p=p_0$.

It seems reasonable to suggest that if in (6), t is large, more and more terms in the expression on the left hand side will be included. For $R=1$, Cerone and Sofo [5] conjectured and then proved, by the use of Burmann's theorem, that (6) is an equality, as represented by a special case of (8), for all real values of t,

$$\sum_{n=0}^{\infty}(-1)^n \frac{c^n e^{-b(t-an)}(t-an)^{n+R-1}}{n!(R-1)!} = \sum_{k=0}^{R-1} k!\,Q_{-(R-k)} \frac{t^{R-k-1}}{(R-k-1)!}e^{p_0 t} \quad . \qquad (8)$$

Therefore it is conjectured that for all positive integer values of R, (8) is an equality, for all real values of t, in the region where the infinite series converges.

Burmann's theorem may again be used to prove (8) once the specific form of the right hand side is evaluated.

By the use of the ratio test it can be seen that the series on the left hand side of (8) converges in the region

$$\left|ace^{1+ab}\right| < 1 .$$

An alternate expansion for the Laplace transform from (3) is

$$F(p)=\frac{1}{p^R}\left[1+\frac{b+ce^{-ap}}{p}\right]^{-R}$$

$$=\sum_{n=0}^{\infty}\sum_{r=0}^{n}\binom{n+R-1}{n}(-1)^n\binom{n}{r}b^{n-r}\frac{c^r e^{-arp}}{p^{n+R}}$$

and using the Inverse Laplace transform gives

$$f(t)=\sum_{n=0}^{\infty}\binom{n+R-1}{n}(-1)^n\sum_{r=0}^{n}\binom{n}{r}b^{n-r}\frac{c^r(t-ar)^{n+R-1}}{(n+R-1)!}H(t-ar)\sim\sum_{k=0}^{R-1}k!Q_{-(R-k)}\frac{t^{R-k-1}}{(R-k-1)!}e^{p_0 t}.$$

Hence, following the above work it is conjectured that

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(R-1)!}\sum_{r=0}^{n}\binom{n}{r}b^{n-r}c^r(t-ar)^{n+R-1}=\sum_{k=0}^{R-1}k!Q_{-(R-k)}\frac{t^{R-k-1}}{(R-k-1)!}e^{p_0 t} \qquad (9)$$

whenever the double series converges.

Without loss of generality, choosing $b+c=0$ and $(1+ab)>0$ allows the dominant root of the characteristic equation (5) to occur at $p_0=0$.

Let $t=-a\tau$ and so from (8)

$$\sum_{n=0}^{\infty}(-1)^n\frac{\left(abe^{ab}\right)^n(\tau+n)^{n+R-1}}{n!}=e^{-ab\tau}\sum_{k=0}^{R-1}Q_{-(R-k))}(-1)^k\binom{R-1}{k}\frac{1}{a^k}\tau^{R-k-1}. \qquad (10)$$

Further, using the same transformation in (9) gives

$$\sum_{n=0}^{\infty}\frac{(ab)^n}{n!}\sum_{r=0}^{n}\binom{n}{r}(-1)^r(\tau+r)^{n+R-1}=\sum_{k=0}^{R-1}Q_{-(R-k)}(-1)^k\binom{R-1}{k}\frac{1}{a^k}\tau^{R-k-1}. \qquad (11)$$

In the case when $1+ab<0$, the dominant root of the characteristic equation (5) occurs at $p_0=\xi$ and therefore the right hand sides of (8) and (9) are multiplied by the factor $e^{\xi t}$. When $1+ab=0$, the right hand side of (3) has a pole of order $2R$ and the left hand side of (8) is divergent. In this case

$$\lim_{t\to\infty}\left[f(t)-\sum_{k=1}^{2R-1}c_k t^k\right]=c_0$$

where the constants c_0 , c_k can be evaluated from residue theory. Details of calculations for the case $R=1$ can be found in [5].

The following lemma regarding moments of the convolution of the generator function $\phi(x)$ will be proved and required in the evaluation of a recurrence relation for the contribution $Q_{-(R-m)}$ to the residues.

Lemma: The n^{th} moment of the R^{th} convolution of $\phi(x) = -bH(a-x)$ is $(-ab)^R(-1)^n a^n n! C_n^R$.

Proof: Consider the rectangular wave $\phi(x) = -bH(a-x) = b(-1+H(x-a))$, which has a Laplace Transform of

$$\Phi(p) = \frac{b(-1+e^{-ap})}{p}$$

The R^{th} convolution of $\Phi(p)$ can be expressed as

$$\Phi^R(p) = b^R\left(\frac{-1+e^{-ap}}{p}\right)^R$$

$$= b^R\left(\frac{-1+\sum_{r=0}^{\infty}(-1)^r\frac{(ap)^r}{r!}}{p}\right)^R \qquad ; \quad R = 1,2,3,....$$

$$= b^R\left(-\sum_{r=0}^{\infty}a(-1)^r\frac{(ap)^r}{(r+1)!}\right)^R$$

$$= (-ab)^R\sum_{r=0}^{\infty}(-1)^r C_r^R a^r p^r . \tag{12}$$

The convolution constants, C_r^R in (12) can be evaluated recursively as follows:

$$\left.\begin{aligned} C_r^1 &= \beta_r = \frac{1}{(r+1)!} && ; \quad R = 1 \\ C_r^R &= \sum_{j=0}^{r}\beta_{r-j}\,C_j^{R-1} && ; \quad R = 2,3,4..... \end{aligned}\right\} . \tag{13}$$

Moreover, the convolution constants are polynomials in R of degree r, so that

$$C_0^R = 1 \ , \ C_1^R = \frac{R}{2} \ , \ C_2^R = R(3R+1)/24 \text{ and so on.}$$

These convolution constants are related to Stirling polynomials and details may be found in the work of Cerone, Sofo and Watson [6].

The n^{th} moment of the R^{th} convolution can be obtained by differentiating (12) n times with respect to p, so that

$$\frac{d^n}{dp^n}\left[\Phi^R(p)\right] = (-ab)^R \sum_{r=n}^{\infty} (-1)^r C_r^R a^r r(r-1).......(r-n+1)p^{r-n}.$$

Upon letting $r = n$, gives the constant relative to p as,

$$\lim_{p\to 0} \frac{d^n}{dp^n}\left[\Phi^R(p)\right] = (-ab)^R (-1)^n a^n n! C_n^R.$$

This result will now be used in the determination of a recurrence relation for Q_r

3. A Recurrence relation for Q_r

The contribution $Q_{-(R-m)}$ to the residue can be obtained from (7). In this section a recurrence relation for $Q_{-(R-m)}$ is developed which, it is argued to be more computationally efficient than using (7) directly and better allows for an induction type proof of (10).

From (7)

$$m! Q_{-(R-m)} = \lim_{p\to 0} \frac{d^m}{dp^m}\left[\left(\frac{1}{1-\Phi(p)}\right)^R\right] \qquad m = 0, 1, 2, ..., (R-1)$$

$$= \lim_{p\to 0} \frac{d^m}{dp^m}\left[\left(1 + \frac{\Phi(p)}{1-\Phi(p)}\right)^R\right]$$

$$= \lim_{p\to 0} \sum_{k=0}^{R} \frac{d^m}{dp^m}\left[\binom{R}{k}\Phi^k(p) \cdot \frac{1}{(1-\Phi(p))^k}\right]$$

$$= \lim_{p\to 0}\sum_{k=0}^{R}\binom{R}{k}\sum_{r=0}^{m}\binom{m}{r}\frac{d^{m-r}}{dp^{m-r}}\left[\Phi^k(p)\right].\frac{d^r}{dp^r}\left\{\frac{1}{(1-\Phi(p))^k}\right\}$$

Now utilizing the lemma for the $(m-r)th$ moment of $\Phi^k(p)$ implies that

$$m!Q_{-(R-m)} = \sum_{k=0}^{R}\binom{R}{k}\sum_{r=0}^{m}\binom{m}{r}(-ab)^k(-a)^{m-r}(m-r)!C_{m-r}^k\, r!Q_{-(k-r)}$$

$$Q_{-(R-m)} = \sum_{k=0}^{R}\binom{R}{k}\sum_{r=0}^{m}(-a)^{m-r}(-ab)^k\, C_{m-r}^k\, Q_{-(k-r)}$$

Using the fact that $C_0^R = 1$ and taking the term at $k = R,\ r = m$ to the left hand side results in,

$$Q_{-(R-m)} = \frac{1}{\left(1-(-ab)^R\right)}\left[\sum_{k=0}^{R}\binom{R}{k}\sum_{r=0}^{m}(-a)^{m-r}(-ab)^k C_{m-r}^k Q_{-(k-r)} - (-ab)^R Q_{-(R-m)}\right].(14)$$

Equation (14) allows for the recursive evaluation of the contribution to the residues, $Q_{-(R-m)}$, with the initial values $C_0^0 = 1$, $Q_{-(0-0)} = 1$.

Its instructive to follow an example through so that the flavour of the calculations for $Q_{-(R-m)}$ can be gleaned.

Consider (14) and let $m = 1$, then

$$Q_{-(R-1)} = \frac{1}{\left(1-(-ab)^R\right)}\left[\sum_{k=0}^{R}\binom{R}{k}(-ab)^k(-a)C_1^k Q_{-(k-0)} + \sum_{k=0}^{R-1}\binom{R}{k}(-ab)^k C_0^k Q_{-(k-1)}\right].$$

Since $C_1^0 = 0$, $Q_{-(0-1)} = 0$ and from previous recursive calculations $C_0^k = 1$, $C_1^k = \frac{k}{2}$,

$$Q_{-(k-0)} = \frac{1}{(1+ab)^k}\ ,\ Q_{-(k-1)} = \frac{-ka(-ab)}{2(1-(-ab))^{k+1}}\ \text{then}$$

$$Q_{-(R-1)} = \frac{-a}{\left(1-(-ab)^R\right)}\left[\sum_{k=1}^{R}\binom{R}{k}\frac{(-ab)^k k}{2(1-(-ab))^k} + \sum_{k=1}^{R-1}\binom{R}{k}(-ab)^{k+1}\frac{k}{2(1-(-ab))^{k+1}}\right]$$

$$= \frac{-a(-ab)R}{2\left(1-(-ab)^R\right)(1-(-ab))^R}\sum_{k=0}^{R-1}\binom{R}{k}(-ab)^k(1-(-ab))^{R-k-1}.$$

Using the definition of the Bernstein polynomial [1], $B^n(x)=\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}$ then $Q_{-(R-1)}$ may be expressed as

$$Q_{-(R-1)}=\frac{a^2bR\left[B^{R-1}(-ab)\right]}{2\left(1-(-ab)\right)^{R+1}\sum_{k=0}^{R-1}(-ab)^k}$$ and so

$$Q_{-(R-1)}=\frac{a^2bR}{2(1+ab)^{R+1}}.$$

If the value of m is specified at the outset, equation (14) may be simplified to produce the following recurrence relations on R only, so that;

for $m=0$

$$Q_{-((R+1)-0)}=\left[Q_{-(1-0)}\right]^{R+1},$$

for $m=1$

$$Q_{-((R+1)-1)}=(R+1)Q_{-(1-1)}\left[Q_{-(1-0)}\right]^{R},$$

for $m=2$

$$Q_{-((R+1)-2)}=(R+1)\left[Q_{-(1-0)}Q_{-(1-2)}+R\left\{Q_{-(1-1)}\right\}^2\right]\left[Q_{-(1-0)}\right]^{R-1},$$

and for $m=3$

$$Q_{-((R+1)-3)}=(R+1)\left[Q_{-(2-0)}Q_{-(1-3)}+3RQ_{-(1-0)}Q_{-(1-1)}Q_{-(1-2)}+R(R-1)\left\{Q_{-(1-1)}\right\}^3\right]\left[Q_{-(1-0)}\right]^{R-2}.$$

Utilizing some of the above ideas, section 6 will detail a functional relationship of (10).

The following table 1 is given for some of the $Q_{-(R-m)}$, the contribution to the residues from using the recurrence relation (14), or those below it.

m	$Q_{-(R-m)}$
0	$\dfrac{1}{(1+ab)^{R}}$
1	$\dfrac{Ra^{2}b}{2(1+ab)^{R+1}}$
2	$\dfrac{-Ra^{3}b(4-ab(3R-1))}{12(1+ab)^{R+2}}$
3	$\dfrac{Ra^{4}b\left[2-4abR+a^{2}b^{2}R(R-1)\right]}{8(1+ab)^{R+3}}$
4	$\dfrac{-Ra^{5}b\left[48-ab(56+200R)+a^{2}b^{2}\left(-16+40R+120R^{2}\right)-a^{3}b^{3}\left(2+5R-30R^{2}+15R^{3}\right)\right]}{240(1+ab)^{R+4}}$
5	$\dfrac{Ra^{6}b\left[16-ab(64+128R)+a^{2}b^{2}\left(-8+36R+140R^{2}\right)+a^{3}b^{3}\left(16R+40R^{2}-40R^{3}\right)+a^{4}b^{4}\left(2R+5R^{2}-10R^{3}+3R^{4}\right)\right]}{96(1+ab)^{R+5}}$

Table 1 : Values of $Q_{-(R-m)}$

4. Examples

From (10) and using the residue calculations at (14), or from table 1, the following results for the right hand side of (10) are listed in table 2.

R	The right hand side of (10)
1	$e^{-ab\tau}\left[\dfrac{1}{1+ab}\right]$
2	$e^{-ab\tau}\left[\dfrac{\tau}{(1+ab)^{2}}-\dfrac{ab}{(1+ab)^{3}}\right]$
3	$e^{-ab\tau}\left[\dfrac{\tau^{2}}{(1+ab)^{3}}-\dfrac{3ab\tau}{(1+ab)^{4}}-\dfrac{ab(1-2ab)}{\left(1+ab^{5}\right)}\right]$
4	$e^{-ab\tau}\left[\dfrac{\tau^{3}}{(1+ab)^{4}}-\dfrac{6ab\tau^{2}}{(1+ab)^{5}}-\dfrac{ab(4-11ab)\tau}{(1+ab)^{6}}-\dfrac{ab\left(1-8ab+6a^{2}b^{2}\right)}{(1+ab)^{7}}\right]$

Table 2 : The closed form expression of (10) for R= 1, 2, 3 and 4.

These elegant results, expressing the infinite series in closed form, can be generated form (10) for any positive integer value of R.

The results at (8), (9), (10) or (11) can be used as a basis for the generation of other infinite series which may be expressed in closed form. This will be investigated in the next section.

5. Generating Function

The basic equations at (8), (9) or (10) (11) can be differentiated and integrated to produce more identities in closed form.

Integrating the result at (8) or (9) will yield the same result as when considering the differential-difference equation (2) with an exponential or polynomial type forcing term respectively. The analysis can also be achieved with a polynomial-exponential type forcing term.

From (8) with $b+c=0$

$$\frac{d^j}{dt^j}\left[\sum_{n=0}^{\infty} b^n e^{-b(t-an)}\frac{(t-an)^{n+R-1}}{n!}\right]=\frac{d^j}{dt^j}\left[(R-1)!\sum_{k=0}^{R-1} Q_{-(R-k)}\frac{t^{R-k-1}}{(R-k-1)!}\right]\;;\qquad 0<j\le R-1$$

So that

$$\sum_{n=0}^{\infty} b^n(n+R-1)(n+R-2)\ldots\ldots\ldots(n+R-j)e^{-b(t-an)}\frac{(t-an)^{n+R-(j+1)}}{n!} \tag{15}$$

$$=\sum_{r=0}^{j}\binom{j}{r}b^{j-r}\frac{d^r}{dt^r}\left[(R-1)!\sum_{k=0}^{R-1} Q_{-(R-k)}\frac{t^{R-k-1}}{(R-k-1)!}\right].$$

Differentiation is permissible within the radius of convergence of the infinite series, which for (15) is $\qquad \left|-abe^{1+ab}\right|<1.$ (16)

For $R=2$ and $j=1$ then

$$\sum_{n=0}^{\infty}(n+1)b^n e^{-b(t-an)}\frac{(t-an)^n}{n!}=\frac{b}{(1+ab)^2}\left[t+\frac{a^2b}{1+ab}+\frac{1}{b}\right].$$

Integrating (8) v-times with $b+c=0$, results in

$$\sum_{n=0}^{\infty}\frac{b^n e^{abn}(t-an)^{n+R-1+v}}{n!(n+R-1+1)(n+R-1+2)\ldots..(n+R-1+v)}=\underbrace{\int\ldots..\int}_{v\text{-times}} e^{bt}(R-1)!\sum_{k=0}^{R-1} Q_{-(R-k)}\frac{t^{R-k-1}}{(R-k-1)!}dt$$

$v=1,2,3,\ldots..$

This integration is permissible within the radius of convergence (16).

For $R=2$ and $v=1$ then,

$$\sum_{n=0}^{\infty}(n+1)b^{n}e^{-b(t-an)}\frac{(t-an)^{n+2}}{(n+2)!}=\frac{1}{(1+ab)^{2}}\left[\frac{t}{b}+\frac{a^{2}b^{2}-1-ba}{b^{2}(1+ab)}\right]+\frac{e^{-bt}}{\left(be^{ab}\right)^{2}}$$

By integrating (9), with $b+c=0$, in its radius of convergence will yield similar results as above. For $R=2$ and integrating once results in,

$$\sum_{n=0}^{\infty}(-1)^{n}b^{n}(n+1)\sum_{r=0}^{n}\binom{n}{r}(-1)^{r}\frac{(t-ar)^{n+2}}{(n+2)!}=\frac{1}{(1+ab)^{2}}\left[\frac{t^{2}}{2}+\frac{a^{2}bt}{(1+ab)}+\frac{a^{3}b(5ab-4)}{12(1+ab)^{2}}\right].$$

Further results may be obtained by considering forcing terms of a specific type. Using previous techniques and choosing the forcing term $w(t)=e^{-dt}\frac{t^{m-1}}{(m-1)!}$ in the right hand side of the differential-difference equation (2), with all initial conditions at rest, results in, for $R=2$ and $m=2$,

$$\sum_{n=0}^{\infty}(-1)^{n}\frac{(n+1)}{(n+3)!}\sum_{r=0}^{n}\binom{n}{r}(-1)^{r}(b-d)^{n-r}b^{r}e^{-d(t-ar)}(t-ar)^{n+3}$$

$$=\frac{t}{d^{2}(1+ab)^{2}}+\frac{a^{2}bd-2ab-2}{d^{3}(1+ab)^{3}}+\frac{e^{-dt}}{\left(b-d-be^{ad}\right)^{2}}\left[t-\frac{2\left(1+abe^{ad}\right)}{b-d-be^{ad}}\right],$$

$$0<a<1\ ;\quad |b|<|d|.$$

In the case when (2) has an impulsive type forcing term of the form $w(t)=\delta(t-d)$, and all initial conditions at rest, then by a change of variable $t-d=T, d\in\boldsymbol{R}^{+}$, the relation (8) holds with t replaced by T.

The following section develops specific functional relationships for the equation (10).

6. Functional Relations

For the case of $R=1$, Pyke and Weinstock (12) gave a functional relationship of (10). The following lemma states a functional relationship for (10) in the general case with $R-1=\nu$.

Lemma

Given that $f_\nu(\tau)=\sum_{n=0}^{\infty}(-1)^n\gamma^n\frac{(\tau+n)^{n+\nu}}{n!}$ $\quad \nu=0,1,2,\ldots\ldots$ then

$$f_\nu(\tau)+\gamma f_\nu(\tau+1)=\tau f_{\nu-1}(\tau) \text{ and}$$

$$f_\nu(\tau)=q_\nu(\tau)\,Exp\left[-\frac{\tau\gamma^\alpha f_\nu(\alpha)}{(ab)^{\alpha-1}q_\nu(\alpha)}\right] \qquad \alpha=1,2,3,\ldots\ldots$$

Proof From (10) let $\gamma=abe^{ab}$ then

$$f_\nu(\tau)=\sum_{n=0}^{\infty}(-1)^n\gamma^n\frac{(\tau+n)^{n+\nu}}{n!}=e^{-ab\tau}\sum_{k=0}^{\nu}Q_{-((\nu+1)-k)}{}^{(-1)^k}\binom{\nu}{k}\frac{1}{a^k}\tau^{\nu-k}$$

$$f_\nu(\tau)+\gamma f_\nu(\tau+1)=\sum_{n=0}^{\infty}(-1)^n\gamma^n\frac{(\tau+n)^{n+\nu}}{n!}+\gamma\sum_{n=0}^{\infty}(-1)^n\gamma^n\frac{(\tau+1+n)^{n+\nu}}{n!}$$

$$=\tau\sum_{n=0}^{\infty}(-1)^n\gamma^n\frac{(\tau+n)^{n+\nu-1}}{n!}$$

$$=\tau f_{\nu-1}(\tau).$$

From the right hand side of (10)

$$f_\nu(\tau)=e^{-ab\tau}\left[\sum_{k=0}^{\nu}Q_{-((\nu+1)-k)}(-1)^k\binom{\nu}{k}\frac{1}{a^k}\tau^{\nu-k}+ab\sum_{k=0}^{\nu}Q_{-((\nu+1)-k)}(-1)^k\binom{\nu}{k}\frac{1}{a^k}(\tau+1)^{\nu-k}\right]$$

$$=e^{-ab\tau}\left[\sum_{k=0}^{\nu}Q_{-((\nu+1)-k)}(-1)^k\binom{\nu}{k}\frac{1}{a^k}\left\{\tau^{\nu-k}+ab(\tau+1)^{\nu-k}\right\}\right]$$

$$=e^{-ab\tau}q_\nu(\tau)$$

and it follows, after some algebraic manipulation, that

$$f_\nu(\tau)=q_\nu(\tau)Exp\left[-\tau\gamma^\alpha f_\nu(\alpha)/(ab)^{\alpha-1}q_\nu(\alpha)\right] \quad , \quad \text{for } \alpha=1,2,3,\ldots\ldots$$

Conclusion

A novel technique has been developed and utilized in the summing of infinite series. A host of infinite series can be expressed in closed form by the use of this procedure. This generalisation of a waiting time relation apart from the case of $R = 1$, does not seem to appear in the literature, such as the work of Gradshteyn and Ryzhik [9].

In a subsequent paper the authors will extend the techniques developed here to consider non-integer values of R and other cases in which more than one dominant root of the characteristic equation will affect the closed form solution of the infinite series.

References

[1] E. J. Barbeau. *Polynomials*, (Springer Verlag, New York, 1989).

[2] E. Brockmeyer and H. L. Halstrom. *The life and works of A. K. Erlang*, (Copenhagen, 1948).

[3] L. Bruwier. 'Sur L'equation fonctionelle $y^{(n)}(x)+a_1y^{(n-1)}(x+c)+......+a_{n-1}y'(x+(n-1)c)+a_ny(x+nc)=0$', *Comptes Rendus Du Congres National Des Sciences, Bruxelles* (1930), (1931) p91-97.

[4] Bruwier L. 'Sur Une Equation Aux Derivees et aux Differences Melees'. *Mathesis*, **47**, (1933) 96-105.

[5] P. Cerone and A. Sofo. 'Summing Series Arising from Integro-Differential-Difference Equations'. (1995) Submitted.

[6] P. Cerone, A. Sofo and D. Watson. 'Stirling Related Polynomials'. (1995). Submitted.

[7] A. K. Erlang. 'Telefon-Ventetider. Et Stykke Sandsynlighedsregning'. *Matematisk Tidsskrift B*, **31**, (1920) 25.

[8] W. Feller. *'An Introduction to Probability Theory and its Applications'*. (John Wiley and Sons, 1971).

[9] I. S. Gradshteyn and I. M. Ryzhik. 'Table of Integrals, Series and Products'. (Academic Press, 1994).

[10] P. Hall. *Introduction to the Theory of Coverage Processes.* (John Wiley and Sons, 1988).

[11] J. L. W. V. Jensen. 'Sur Une Identite D'Abel et sur D'autres Formules Analogues'. *Acta Mathematica*, **26**, (1902) 307-318.

[12] R. Pyke and R. Weinstock. 'A Special Infinite Series'. *American Mathematical Monthly*, **67**, (1960) 704.

[13] W. L. Smith. 'Renewal Theory and its Ramifications', *Royal Statistical Society Journal Series B*, **20**, (1958) 243-302.

[14] H. C. Tijms. *'Stochastic Modelling and Analyses : A Computational Approach'*. (John Wiley and Sons, 1986).