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Summation of Series of Binomial Variation

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Abstract

Difference delay and discrete renewal equations provide a rich source of material for the exploration and generation of infinite series. From this vantage point it will be shown that, with certain restrictions, these generated infinite series may be represented in closed form. The analysis will consist of an application of the Z transform together with a detailed description of the location of zeros of polynomial type characteristic functions.

Introduction

In this paper a technique is developed that allows for the exploration and generation of infinite series which in turn may be represented in closed form. Renewal processes provide a rich source of material for such investigations and of critical importance in our work, is the form of the characteristic equation in the denominator of a queue-length generating function including the location of zeros.

In section 1, a discrete renewal equation will be considered and a method developed for the generation of Binomial type infinite series, by considering a similar characteristic function as produced by a bulk service queue. To analyse the discrete renewal equation a generating function approach by the use of the Z transform will be employed. It will be shown that the infinite series may be represented in closed form such that the closed form representation may be expressed in terms of the dominant zero of a characteristic function.

For a particular parameter values of the series, the closed form identity may be confirmed by the WZ pairs method of Wilf and Zeilberger [11].

Section 2 investigates the connection of the Binomial type series with generalized hypergeometric functions. For a particular case the Binomial series will be shown to satisfy the identity of Kummer [7].

In section 3 we consider a modified density function in the discrete renewal equation.

The closed form representation of the generated series may then be expressed in terms of a multiple number of dominant zeros of an associated characteristic function. In section 4, we consider a forcing term in the difference-delay representation of the stationary size probabilities, that yields an interesting Binomial convolution identity.

1. A Volterra Type Discrete Renewal Equation

The analysis is begun by considering a volterra type discrete renewal equation

$$f_n = w_n + \sum_{k=0}^{n} f_{n-k}\,\phi_k \tag{1.1}$$

where f_n may represent the total average number of renewals at epoch n, ϕ_n may represent the probability that a new item that is installed at a given time will fail after exactly n time units and w_n may represent the average number of renewals at time n of the original population. For a derivation of (1.1) one can refer to the books of Saaty [12] or Cohen [5].

Alternatively (1.1) may be represented as, where * is the discrete convolution

$$f_n = w_n + f_n * \phi_n \ . \tag{1.2}$$

Therefore by the use of the Z transform (1.2) may be written as, after rearrangement

$$F(z) = \frac{W(z)}{1-\Phi(z)} \quad , \tag{1.3}$$

where $F(z)$, $W(z)$ and $\Phi(z)$ are the Z transforms of the respective functions f_n, w_n and ϕ_n.

Without loss of generality a convolution type argument of (1.3) will allow for the consideration of the more general transform function,

$$F(z)=\frac{W(z)}{(1-\Phi(z))^R} \tag{1.4}$$

for $R=1,2,3,4,\ldots\ldots$.

A Bulk Service Queue

The simplest Markovian queue to which the characteristic functions in the denominator of (1.4) becomes germane is the bulk service variation of the $M|M^{(a)}|1$ system in which service is in fixed batches of size 'a', irrespective of whether or not the server has to wait for a full batch of size a, see Gross and Harris [9].

To obtain a similar characteristic function as that of the $M|M^{(a)}|1$ system, consider the densities

$$w_n=\binom{n}{R-1}b^{n-(R-1)}c^{R-1} \qquad \text{and} \tag{1.5}$$

$$\phi_n=b^{n-a}\,U(n-(a+1)) \tag{1.6}$$

where $U(n-x)$ is the discrete step function, $a\geq 1$, $c\in\Re$, $b\in\Re$ and $R\geq 1$.

Let $W(z)$ and $\Phi(z)$ be the Z transform of (1.5) and (1.6) respectively, then

$$W(z)=\frac{zc^{R-1}}{(z-b)^R} \qquad \text{and} \tag{1.7}$$

$$\Phi(z)=\frac{bz^{-a}}{z-b} \quad . \tag{1.8}$$

Substituting (1.7) and (1.8) into (1.4) yields, upon simplification, the transformed function

$$\frac{1}{c^{R-1}}F_1(z) = F(z) = \frac{z}{\left(z-b-bz^{-a}\right)^R} = \frac{z^{aR+1}}{\left(z^{a+1}-bz^a-b\right)^R}. \quad (1.9)$$

An equivalent characteristic function as the denominator of (1.9) may be obtained when the stationary system-size probabilities are related in difference - equation form. It has been shown by Sofo and Cerone [14] that the same generating function (1.9) for the case $R=1$ can be arrived at via the use of a related Fibonacci sequence, see also Kelley and Peterson [10].

Expanding the second term of (1.9) in series form results in

$$F(z) = \sum_{r=0}^{\infty}\binom{R+r-1}{r}b^r\frac{z^{1-ar}}{(z-b)^{r+R}} \quad (1.10)$$

and therefore the inverse Z transform of (1.10) is

$$f_n = \sum_{r=0}^{\infty}\binom{R+r-1}{r}\binom{n-ar}{R+r-1}b^{n-ar-(R-1)}\,U(n-ar). \quad (1.11)$$

Equation (1.11) may also be rewritten as

$$f_n = \sum_{r=0}^{\left[\frac{n+1-R}{a+1}\right]}\binom{R+r-1}{r}\binom{n-ar}{R+r-1}b^{n-ar-(R-1)} \quad (1.12)$$

where $[x]$ represents the integer part of x.

The inverse transform of (1.9) may also be expressed as, refer to Elaydi [6],

$$f_n = \frac{1}{2\pi i}\int_C z^n\left(\frac{F(z)}{z}\right)dz = \sum_{j=0}^{a} Res_j\left(\frac{F(z)}{z}\right)z^n \tag{1.13}$$

where C is a smooth Jordan curve enclosing the singularities of (1.9) and the integral is traversed once in an anticlockwise direction around C. It may also be shown that there is no contribution from the integration around the contour C. Res_j is the residue of the poles of (1.9).

From (1.9), the characteristic function (with some restriction)

$$g(z) = z^{a+1} - bz^a - b \tag{1.14}$$

has exactly $(a+1)$ distinct zeros, ξ_j for $j = 0,1,2.......\ a$, of which at least one and at most two are real. See appendix B for a clarification of this statement.

Therefore $\xi_j^{a+1} - b\xi_j^a - b = 0$, and all the singularities in (1.9) are poles of order R.

Now, from (1.9), $F(z)$ has exactly '$(a+1)$' poles of order R at the zeros ξ_j.

Hence from (1.13) a solution of the system (1.2) may be expressed as

$$f_n = \sum_{j=0}^{a}\sum_{\mu=0}^{R-1} Q_{-(R,-\mu)}(\xi_j)\binom{n}{R-1-\mu}\xi_j^{n-(R-1-\mu)} \tag{1.15}$$

where the residue contribution

$$\mu!\,Q_{-(R,-\mu)}(\xi_j) = \lim_{z\to\xi_j}\left[\frac{d^\mu}{dz^\mu}\left\{(z-\xi_j)^R\,\frac{F(z)}{z}\right\}\right] \tag{1.16}$$

for each $j = 0,1,2,.......\ a$.

From (1.11) or (1.12) and using (1.15) it can be seen that

$$\sum_{r=0}^{\infty}\binom{R+r-1}{r}\binom{n-ar}{R+r-1}b^{n-ar+1-R}\,U(n-ar)$$

$$=\sum_{j=0}^{a}\sum_{\mu=0}^{R-1}Q_{(R,-\mu)}(\xi_j)\binom{n}{R-1-\mu}\xi_j^{\,n-(R-1-\mu)}. \tag{1.17}$$

Main Result

The characteristic function (1.14) has at least one real zero. The dominant real zero, ξ_0 of (1.14) is defined as the one with the greatest modulus, and it may be easily shown that $|\xi_0|>\left|\frac{ab}{a+1}\right|$. Details of this statement are given in Appendix B.

The limiting behaviour of (1.17) is such that for n large

$$f_n=\sum_{r=0}^{\left[\frac{n+1-R}{a+1}\right]}\binom{R+r-1}{r}\binom{n-ar}{R+r-1}b^{n-ar+1-R}\sim\sum_{\mu=0}^{R-1}Q_{(R,-\mu)}(\xi_0)\binom{n}{R-1-\mu}\xi_0^{\,n-(R-1-\mu)}. \tag{1.18}$$

The suggestive limiting behaviour of (1.18), leads the authors to the **conjecture** that

$$\sum_{r=0}^{\infty}\binom{R+r-1}{r}\binom{n-ar}{R+r-1}b^{n-ar+1-R}=\sum_{\mu=0}^{R-1}Q_{(R,-\mu)}(\xi_0)\binom{n}{R-1-\mu}\xi_0^{\,n-(R-1-\mu)} \tag{1.19}$$

for <u>all values</u> of n (and not just n large) in the region where the infinite series converges.

The conjectured result (1.19) together with the equation (1.17) implies that

$$\sum_{r=0}^{\left[\frac{n+1-R}{a+1}\right]}\binom{R+r-1}{r}\binom{n-ar}{R+r-1}b^{n-ar+1-R}+\sum_{r=\left[\frac{n+2-R}{a+1}\right]}^{\infty}\binom{R+r-1}{r}\binom{n-ar}{R+r-1}b^{n-ar+1-R}=\sum_{\mu=0}^{R-1}Q_{(R,-\mu)}(\xi_0)\binom{n}{R-1-\mu}\xi_0^{\,n-(R-1-\mu)}$$

which gives the result

$$\sum_{r=\left[\frac{n+2-R}{a+1}\right]}^{\infty} \binom{R+r-1}{r}\binom{n-ar}{R+r-1} b^{n-ar+1-R} = -\sum_{j=1}^{a}\sum_{\mu=0}^{R-1} Q_{(R,-\mu)}(\xi_j)\binom{n}{R-1-\mu}\xi_j^{n-(R-1-\mu)}.$$

An appropriate test indicates that the infinite series (1.19) converges in the region

$$\left|\frac{(a+1)^{a+1}}{(ab)^a}\right| < 1. \tag{1.20}$$

The restriction (1.20) also applies to (1.14).

A diagram of the region of convergence is shown as the shaded area of figure 1

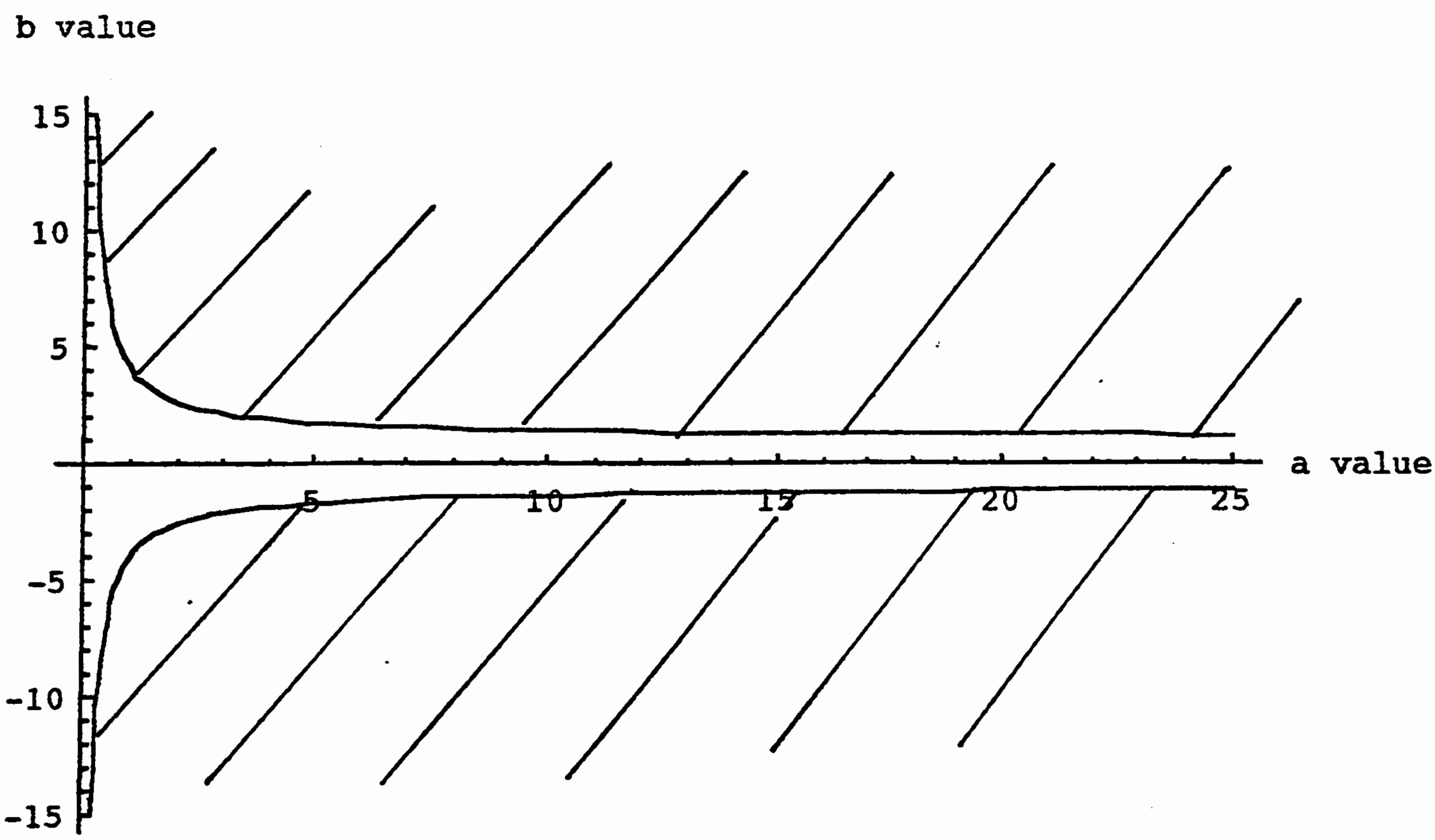


Figure 1: The region of convergence (1.20).

The residue contribution from (1.16), can be evaluated and some results are listed in table 1 below for the dominant zero ξ_0 and various values of μ.

μ	$\mu! Q_{-(R-\mu)}(\xi_0)$
0	$\dfrac{\xi_0^R}{A^R}$
1	$\dfrac{aR\xi_0^{R-1}(A-b)}{2A^{R+1}} = \dfrac{aR(a+1)(\xi_0-b)}{2A^{R+1}}\xi_0^{R-1}$
2	$\dfrac{aR\xi_0^{R-2}}{12A^{R+2}}\left[A^2(3aR-a-8)-2bA(3aR+a-4)+3(R+1)ab^2\right]$
3	$\dfrac{aR\xi_0^{R-3}}{8A^{R+3}}\left[\begin{array}{l}A^3\{a^2R(R-1)+2a(1-4R)+12\}\\+bA^2\{-a^2R(3R+1)+2a(8R+3)-12\}\\+b^2A\{a^2(3R+2)(R+1)-8a(R+1)\}\\-a^2b^3(R+1)(R+2)\end{array}\right]$

Table 1: $\mu! Q_{-(R,-\mu)}(\xi_0)$ terms for $\mu = 0,1,2,3,$ where $A=(a+1)\xi_0 - ab$ and ξ_0 is the dominant zero of (1.14).

Utilizing the terms of table 1 the closed form expressions of (1.19) are listed in table 2, for various R values

R	The closed form expression of (1.19)
1	$\dfrac{\xi_0^{n+1}}{A}$
2	$\dfrac{\xi_0^{n+1}}{A^2}\left[n+\dfrac{a(a+1)(\xi_0-b)}{A}\right]$
3	$\dfrac{\xi_0^{n+1}}{A^3}\left[\binom{n}{2}+\binom{n}{1}\dfrac{3a(A-b)}{2A}+\dfrac{a\left(2(a-1)A^2-(5a-2)bA+3ab^2\right)}{2A^2}\right]$
4	$\dfrac{\xi_0^{n+1}}{A^4}\left[\binom{n}{3}+\binom{n}{2}\dfrac{2a(A-b)}{A}+\binom{n}{1}\dfrac{a\left((11a-8)A^2-2b(13a-4)A+15ab^2\right)}{6A^2} +\dfrac{a}{12A^3}\left\{A^3\left(12a^2-30a+12\right)+bA^2\left(-52a^2+70a-12\right) +b^2A\left(70a^2-40a\right)-30a^2b^3\right\}\right]$

Table 2: The closed form expressions of the infinite sum at (1.19) for the values $R=1,2,3$ and 4. Where $A=(a+1)\xi_0-ab$ and ξ_0 is the dominant zero of (1.14).

The proof of the conjecture at (1.19) can now proceed.

Proof of Conjecture

The proof of (1.19) will involve the application of an induction argument. Firstly a recurrence relation will be developed for the series

$$S_R = \sum_{r=0}^{\infty} \binom{R+r-1}{r} \binom{n-ar}{R+r-1} b^{n-ar+1-R}. \tag{1.21}$$

Lemma

A recurrence relation for (1.21) is

$$(a+1)b\frac{d}{db}S_R - abRS_{R+1} - (n+1-R)S_R = 0 \tag{1.22}$$

Proof

$$\frac{d}{db}S_R = -\frac{a}{b}\sum_{r=0}^{\infty} r\binom{R+r-1}{r}\binom{n-ar}{R+r-1}b^{n-ar-R+1} + \left(\frac{n+1-R}{b}\right)S_R$$

and

$$S_{R+1} = \sum_{r=0}^{\infty}\binom{R+r}{r}\binom{n-ar}{R+r}b^{n-ar-R}$$

$$= -\frac{(a+1)}{bR}\sum_{r=0}^{\infty} r\binom{R+r-1}{r}\binom{n-ar}{R+r-1}b^{n-ar-R+1} + \left(\frac{n+1-R}{bR}\right)S_R$$

From the left hand side of (1.22).

$$(a+1)b\left[\left(\frac{n+1-R}{b}\right)S_R - \frac{a}{b}\sum_{r=0}^{\infty} r\binom{R+r-1}{r}\binom{n-ar}{R+r-1}b^{n-ar-R+1}\right]$$

$$-abR\left[-\frac{(a+1)}{bR}\sum_{r=0}^{\infty} r\binom{R+r-1}{r}\binom{n-ar}{R+r-1}b^{n-ar-R+1} + \left(\frac{n+1-R}{bR}\right)S_R\right] - (n+1-R)S_R$$

$= 0$ which is the right hand side of (1.22)

and the proof of the lemma in complete.

The next step in the procedure of the proof of (1.19) will involve the expansion of the left hand side of (1.19) in inverse powers of the dominant zero ξ_0, therefore showing that this expansion is the same as the right hand side of (1.19), in which case the basis for $R = 1$ is true.

Consider (1.19) and let $n = -aN$ such that

$$\sum_{r=0}^{\infty}\binom{R+r-1}{r}\binom{-aN-ar}{R+r-1}b^{-aN-ar+1-R} = \sum_{\mu=0}^{R-1} Q_{-(R,-\mu)}(\xi_0)\binom{-aN}{R-1-\mu}\xi_0^{-aN-(R-1-\mu)}. \quad (1.23)$$

From the characteristic equation (1.14)

$$b = \frac{\xi_0^{a+1}}{1+\xi_0^a}$$

so that, the left hand side of (1.23) maybe written as

$$\sum_{r=0}^{\infty}\binom{R+r-1}{r}\binom{-a(N+r)}{R+r-1}\left(\frac{1+\xi_0^a}{\xi_0^{a+1}}\right)^{a(N+r)+R-1}$$

$$= \sum_{r=0}^{\infty}\binom{R+r-1}{r}(-1)^{R+r-1}\binom{aN+ar+R+r-2}{R+r-1}\sum_{k=0}^{a(N+r)+R-1}\binom{a(N+r)+R-1}{k}\xi_0^{ak-(a+1)(a(N+r)+R-1)} \quad (1.24).$$

The convergent double sum (1.24) may be written term by term as

$$(-1)^{R-1}\binom{R-1}{0}\binom{aN+R-2}{R-1}\left[\binom{aN+R-1}{0}\xi_0^{-(a+1)(aN+R-1)}+.................+\binom{aN+R-1}{aN+R-1}\xi_0^{-a(N+0)-(R-1)}\right]$$

$$+(-1)^{R}\binom{R}{1}\binom{aN+a+R-1}{R}\left[\binom{aN+a+R-1}{0}\xi_0^{-(a+1)(aN+a+R-1)}+............+\binom{aN+a+R-1}{aN+a+R-1}\xi_0^{-a(N+1)-(R-1)}\right]$$

$$+(-1)^{R+1}\binom{R+1}{2}\binom{aN+2a+R}{R+1}\left[\binom{aN+2a+R-1}{0}\xi_0^{-(a+1)(aN+2a+R-1)}+............+\binom{aN+2a+R-1}{aN+2a+R-1}\xi_0^{-a(N+2)-(R-1)}\right] \quad (1.25)$$

$$+(-1)^{R+2}\binom{R+2}{3}\binom{aN+3a+R-1}{R+2}\left[\binom{aN+3a+R-1}{0}\xi_0^{-(a+1)(aN+3a+R-1)}+............+\binom{aN+3a+R-1}{aN+3a+R-1}\xi_0^{-a(N+3)-(R-1)}\right]$$

+..

Summing (1.25) diagonally from the top right hand corner and gathering the coefficients of inverse powers of ξ_0 gives

$$\sum_{r=0}^{\infty}\xi_0^{-a(N+r)-(R-1)}(-1)^{R-1+r}\sum_{k=0}^{r}(-1)^k\binom{R+r-k-1}{r-k}\binom{a(N+r-k)+R+r-k-2}{R+r-k-1}\binom{a(N+r-k)+R-1}{a(N+r-k)+R-k-1}. \quad (1.26)$$

In the case that R=1, (1.26) is again expanded to allow for the collection of the ξ_0^{-ar} terms such that

$$\sum_{r=0}^{\infty}\xi_0^{-a(N+r)}\sum_{k=0}^{r}(-1)^{r+k}\binom{a(N+r-k)}{a(N+r-k)-k}\binom{a(N+r-k)+r-k-1}{r-k}$$

$$=\xi_0^{-aN}\left[1+a\sum_{r=1}^{\infty}(-1)^r(1+a)^{r-1}\xi_0^{-ar}\right]$$

$$=\xi_0^{-aN}\left[\frac{1+\xi_0^{a}}{(a+1)+\xi_0^{a}}\right]$$

$$= \xi_0^{-aN+1}\left[\frac{1}{(a+1)\xi_0 - a\left(\dfrac{\xi_0^{a+1}}{1+\xi_0^a}\right)}\right]$$

$$= \frac{\xi_0^{-aN+1}}{(a+1)\xi_0 - ab} \qquad , \quad \text{putting } n = -aN \text{ gives}$$

$$= \frac{\xi_0^{n+1}}{(a+1)\xi_0 - ab} \; . \tag{1.27}$$

From the right hand side of (1.19) and using (1.16) for $R = 1$

$$\sum_{\mu=0}^{R-1} Q_{(R,-\mu)}(\xi_0)\binom{n}{R-1-\mu}\xi_0^{n-(R-1-\mu)} = \frac{\xi_0^{n+1}}{(a+1)\xi_0 - ab} \tag{1.28}$$

and comparing (1.27) and (1.28) indicates that (1.19) is proved for $R = 1$.

Now, we consider the right hand side of (1.19) for the general case R

$$S_R = \sum_{\mu=0}^{R-1} Q_{(R,-\mu)}(\xi_0)\binom{n}{R-1-\mu}\xi_0^{n-(R-1-\mu)}$$

and utilize the recurrence relation (1.22).

$$S_{R+1} = \frac{1}{abR}\left[(a+1)b\frac{d}{db}S_R - (n+1-R)S_R\right]. \tag{1.29}$$

Firstly, from $\dfrac{d}{db}S_R = \dfrac{\xi_0^2}{Ab}\dfrac{d}{d\xi_0}S_R$,

$$\frac{d}{db}S_R = \frac{\xi_0^2}{Ab}\sum_{\mu=0}^{R-1}\binom{n}{R-1-\mu}\xi_0^{n-(R-1-\mu)}\left[\frac{d}{d\xi_0}Q_{(R,-\mu)} + \frac{(n-(R-1-\mu))}{\xi_0}Q_{(R,-\mu)}\right]$$

and substitute into (1.29) such that

$$S_{R+1}=\frac{1}{abR}\left[\frac{(a+1)b\xi_0^2}{Ab}\sum_{\mu=0}^{R-1}\binom{n}{R-1-\mu}\xi_0^{n-(R-1-\mu)}\left\{\frac{d}{d\xi_0}Q_{-(R,-\mu)}+\frac{(n-R+1+\mu)}{\xi_0}Q_{-(R,-\mu)}\right\}\right.$$

$$\left.-(n+1-R)\sum_{\mu=0}^{R-1}\binom{n}{R-1-\mu}\xi_0^{n-(R-1-\mu)}Q_{-(R,-\mu)}\right]$$

$$=\frac{1}{abRA}\sum_{\mu=0}^{R-1}\binom{n}{R-1-\mu}\xi_0^{n-(R-1-\mu)}\left\{(a+1)\xi_0^2\frac{d}{d\xi_0}Q_{-(R,-\mu)}+(a+1)\xi_0(n-R+1+\mu)Q_{-(R,-\mu)}-(n+1-R)AQ_{-(R,-\mu)}\right\}$$

$$=\frac{1}{abRA}\sum_{\mu=0}^{R-1}\binom{n}{R-1-\mu}\xi_0^{n-(R-1-\mu)}\left\{(A+ab)\xi_0\frac{d}{d\xi_0}Q_{-(R,-\mu)}+\mu AQ_{-(R,-\mu)}+ab(n-R+1+\mu)Q_{-(R,-\mu)}\right\}$$

$$=\frac{1}{abRA}\left[\sum_{\mu=0}^{R-1}ab(R-\mu)\binom{n}{R-\mu}\xi_0 Q_{-(R,-\mu)}\xi_0^{n-(R-\mu)}\right.$$

$$\left.+\sum_{\mu=0}^{R-1}\binom{n}{R-1-\mu}\xi_0^{n-(R-1-\mu)}\left\{(A+ab)\xi_0\frac{d}{d\xi_0}Q_{-(R,-\mu)}+\mu AQ_{-(R,-\mu)}\right\}\right]$$

$$=\frac{1}{abRA}\left[abR\xi_0\binom{n}{R}Q_{-(R,-0)}\xi_0^{n-R}+\sum_{\mu=1}^{R-1}ab(R-\mu)\binom{n}{R-\mu}\xi_0^{n-(R-1-\mu)}Q_{-(R,-\mu)}\right.$$

$$\left.+\sum_{\mu=0}^{R-1}\binom{n}{R-1-\mu}\xi_0^{n-(R-1-\mu)}\left\{(A+ab)\xi_0\frac{d}{d\xi_0}Q_{-(R,-\mu)}+\mu AQ_{-(R,-\mu)}\right\}\right]. \qquad (1.30)$$

In the second sum rename $\mu^*=\mu+1$ (and let $\mu^*=\mu$ again), so that (1.30) may be written as

$$\binom{n}{R}\left(\frac{\xi_0}{A}\right)Q_{-(R,-0)}\xi_0^{n-R}+\frac{1}{abRA}\left[\sum_{\mu=1}^{R}ab(R-\mu)\xi_0\binom{n}{R-\mu}Q_{-(R,-\mu)}\xi_0^{n-(R-\mu)}\right.$$

$$\left.+\sum_{\mu=1}^{R}\binom{n}{R-\mu}\xi_0^{n-(R-\mu)}\left\{(A+ab)\xi_0\frac{d}{d\xi_0}Q_{-(R,-(\mu-1))}+(\mu-1)AQ_{-(R,-(\mu-1))}\right\}\right]$$

$$= Q_{(1,-0)}Q_{(R,-0)}\binom{n}{R}\xi_0^{n-R} + \frac{1}{abRA}\sum_{\mu=1}^{R}\binom{n}{R-\mu}\xi_0^{n-(R-\mu)}\Big[ab(R-\mu)\xi_0 Q_{(R,-\mu)}$$

$$+(A+ab)\xi_0\frac{d}{d\xi_0}Q_{(R,-(\mu-1))} + (\mu-1)AQ_{(R,-(\mu-1))}\Big]$$

$$= Q_{(R+1,-0)}\binom{n}{R}\xi_0^{n-R} + \frac{1}{abRA}\sum_{\mu=1}^{R}\binom{n}{R-\mu}\xi_0^{n-(R-\mu)}\Big[ab(R-\mu)\xi_0 Q_{(R,-\mu)}$$

$$+(A+ab)\xi_0\frac{d}{d\xi_0}Q_{(R,-(\mu-1))} + (\mu-1)AQ_{(R,(\mu-1))}\Big]. \tag{1.31}$$

Now utilizing the lemma in appendix A, and after replacing μ with $\mu-1$ we have

$$abRAQ_{(R+1,-\mu)} = ab(R-\mu)\xi_0 Q_{(R,-\mu)} + (A+ab)\xi_0\frac{d}{d\xi_0}Q_{(R,-(\mu-1))} + (\mu-1)AQ_{(R,-(\mu-1))}$$

so that, from (1.31) we have

$$\binom{n}{R}\xi_0^{n-R}Q_{(R+1,-0)} + \sum_{\mu=1}^{R}\binom{n}{R-\mu}\xi_0^{n-(R-\mu)}Q_{(R+1,-\mu)}$$

$$= \sum_{\mu=0}^{R}\binom{n}{R-\mu}\xi_0^{n-(R-\mu)}Q_{(R+1,-\mu)}$$

which completes the proof of (1.19).

Some numerical results are now given for various parameter values of (1.19).

Numerical Results

The following numerical results, to five significant digits, are given for various parameter values of the conjecture (1.19)

R	n	a	b	ξ_0	The left and right hand side of (1.19)
2	3	2	9	9.10848	242.97104
2	3	2	-9	-9.10848	242.97104
2	3	3	9	9.01230	242.99553
2	3	3	-9	-8.98760	242.99532
3	3	2	9	9.10848	27.00525
3	3	2	-9	-9.10848	-27.00525
3	3	3	9	9.01230	27.00147
3	3	3	-9	-8.98760	-27.00158

For a even and $b > 0$ or $b < 0$ the modulus of the zero, $|\xi_0|$, of (1.14) is identical, hence the modulus of the sum (1.19) is identical. Details of the zeros of (1.14) are given in appendix B.

The Degenerate Case

From (1.19), for $a = 0$ the degenerate case can now be noted.

From (1.14) $\xi_0 = 2b$, and from (1.16)

$\mu! Q_{(R,-\mu)}(2b) = 1$ for $\mu = 0$ and zero otherwise such that (1.19) becomes

$$\sum_{r=0}^{\infty}\binom{R+r-1}{r}\binom{n}{R+r-1}b^{n+1-R}=\binom{n}{R-1}(2b)^{n+1-R}$$

and upon simplification yields the familiar result

$$\sum_{r=0}^{n-R+1}\binom{n-R+1}{r}=2^{n+1-R}. \qquad (1.32)$$

The series on the left hand side of (1.32) is not Gosper summable, as defined by Petkovsek et al. [11], however the closed form solution of (1.32) can be verified by the *WZ* pairs method, see Petkovsek et al. [11]. A sketch of the procedure of *WZ* pairs method follows. Let

$$\sum_{r=0}^{n-R+1}F(n,r)=1 \qquad \text{where } F(n,r)=\binom{n-R+1}{r}2^{R-1-n}.$$

The rational function

$$R(n,r)=\frac{r}{2(r-n+R-2)}$$

can be obtained by the *WZ* pairs method, a package which is available for use an MATHEMATICA. Now let

$$G(n,r)=R(n,r)\,F(n,r)=-\binom{n-R+1}{r-1}2^{R-n-2}$$

and the recurrence

$$F(n+1,r)-F(n,r)=G(n,r+1)-G(n,r)$$

including the initial condition $F(0,r)=1$ holds, hence the identity (1.32) is verified.

In the next section we investigate the connection of the series (1.21) with generalised hypergeometric functions.

2. Generalized Hypergeometric Functions

It is known that a series

$$\sum_{\eta} \alpha(\eta)$$

is called geometric if the ratio of consecutive terms are constant, and it is called hypergeometric if the ratio of consecutive terms is a rational function of η.

From the left hand side of (1.19) let

$$T_r = \binom{R+r-1}{r}\binom{n-ar}{R+r-1} b^{n-ar-R+1} \qquad (2.1)$$

$$T_0 = \binom{n}{R-1} b^{n-R+1}$$

and n may be relaxed to be a real number. The ratio of consecutive terms, using (2.1), is

$$\frac{T_{r+1}}{T_r} = \frac{\prod_{j=0}^{a}\left(r+\frac{j+R-n-1}{a+1}\right)}{(r+1)\prod_{j=0}^{a-1}\left(r+\frac{j-n}{a}\right)}(s) \qquad (2.2)$$

Since (2.2) is a quotient of rational functions in r, then the left hand side of (1.19) may be expressed as a generalized hypergeometric function,

$$T_0\, {}_{a+1}F_a\left[\begin{matrix} \frac{R-n-1}{a+1}, \frac{R-n}{a+1}, \frac{R-n+1}{a+1}, \ldots\ldots\ldots, \frac{R+a-n-1}{a+1} \\ -\frac{n}{a}, \frac{1-n}{a}, \frac{2-n}{a}, \ldots\ldots\ldots, \frac{a-1-n}{a} \end{matrix} \middle| s \right] \tag{2.3}$$

$$= T_0 \sum_{k=0}^{\infty} \frac{\left(\frac{R-n-1}{a+1}\right)_k \left(\frac{R-n}{a+1}\right)_k \left(\frac{R-n+1}{a+1}\right)_k \cdots\cdots\cdots \left(\frac{R+a-n-1}{a+1}\right)_k}{\left(-\frac{n}{a}\right)_k \left(\frac{1-n}{a}\right)_k \left(\frac{2-n}{a}\right)_k \cdots\cdots\cdots \left(\frac{a-1-n}{a}\right)_k} \frac{s^k}{k!} \tag{2.4}$$

where $(x)_m = x(x+1)\ldots\ldots\ldots(x+m-1)$ is known as Pochhammer's symbol or as rising factorial powers of m, as mentioned by Graham et al. [8], and

$$s = -\frac{(a+1)^{a+1}}{(ab)^a}. \tag{2.5}$$

Generalizing the expression 15.1.1 given on page 556 of Abromowitz and Stegun [1], allows (2.3) or (2.4) to be written as

$$T_0 \frac{\prod_{j=0}^{a-1} \Gamma\left(\frac{j-n}{a}\right)}{\prod_{j=R-1}^{a+R-1} \Gamma\left(\frac{j-n}{a+1}\right)} \sum_{k=0}^{\infty} \frac{\prod_{j=R-1}^{a+R-1} \Gamma\left(\frac{j-n}{a+1}+k\right)}{\prod_{j=0}^{a-1} \Gamma\left(\frac{j-n}{a}+k\right)} \frac{s^k}{k!} \tag{2.6}$$

where $\Gamma(x)$ is the classical Gamma function. It is evident that (2.4) is a divergent series for n a positive integer. Many relations of the generalized hypergeometric function (2.3) exist in terms of the special functions of mathematical physics, and some of these may be seen in the classical works of Slater [13] and Gaspar and Rahman [7].

Some special cases of (2.3) are worthy of mention, since for $a=1$, (2.3) reduces to the classical Gauss series. Thus letting $a=1$ in (2.3), we have that

$$T_0 \; {}_2F_1\left[\begin{matrix} \frac{R-n-1}{2}, \frac{R-n}{2} \\ -n \end{matrix}\middle| \left(-\frac{4}{b}\right)\right] \tag{2.7}$$

$$= T_0\left[1+\sum_{k=1}^{\infty}\frac{\prod_{j=0}^{2k-1}(R-n-1+j)}{k!b^k \prod_{j=0}^{k-1}(n-j)}\right].$$

The Gauss hypergeometric series (2.6) may be expressed as an integral. From Abramowitz and Stegun, let $\alpha=-n\in\Re\backslash J^-$, so that (2.7) may be written as

$$\frac{T_0}{B\left(\frac{R+\alpha}{2},\frac{\alpha-R}{2}\right)}\int_{t=0}^{1}(1-t)^{(\alpha-R-2)/2}t^{(R+\alpha-2)/2}(1-st)^{(1-R-\alpha)/2}dt \tag{2.8}$$

which is valid for $|s|<1$, $s=-4/b$ and $B(x,y)$ is the Beta function.

Since the difference in the two top terms of the hypergeometric function (2.7) is ½, there exists a quadratic transformation [13] connected with the Legendre functions, P_ν^μ. A definition of the Legendre function P_ν^μ may be seen in [1]. From (2.7) and using identity 15.4.11 of [1] we have that

$$T_0 \; {}_2F_1\left[\begin{matrix} \frac{R}{2}+\frac{\alpha}{2}-\frac{1}{2}, \frac{R}{2}+\frac{\alpha}{2} \\ \alpha \end{matrix}\middle| s\right] = T_0 . 2^{\alpha-1}\Gamma(\alpha)(-s)^{(1-\alpha)/2}(1-s)^{-R/2}P_{R-1}^{1-\alpha}\left\{(1-s)^{-1/2}\right\} \tag{2.9}$$

whre $s=-\frac{4}{b}$ and $s\in(-\infty,0)$.

Since the left hand side of (1.19) converges for $b>4$, in the case that $a=1$, then we may write from (2.9) that

$$\sum_{r=0}^{\infty}\binom{R+r-1}{r}\binom{-\alpha-r}{R+r-1}b^{-\alpha-r-R+1}=T_0\,2^{\alpha-1}\Gamma(\alpha)\left(\frac{4}{b}\right)^{\frac{1}{2}-\frac{\alpha}{2}}\left(1+\frac{4}{b}\right)^{-\frac{R}{2}}P_{R-1}^{1-\alpha}\left\{\left(1+\frac{4}{b}\right)^{-\frac{1}{2}}\right\}$$

$$=\sum_{\mu=0}^{R-1}Q_{-(R,-\mu)}(\xi_0)\binom{-\alpha}{R-1-\mu}\xi_0^{-\alpha-R+1+\mu} \qquad (2.10)$$

for $b>4$, $2\xi_0=b+\sqrt{b^2+4b}$ and $Q_{-(R,-\mu)}(\xi_0)$ is defined by (1.16).

Other specific cases of (2.7) and (2.9) are as follows

(i) For $b=4$, $s=-1$ and $\alpha=\dfrac{3}{2}$, we have from (2.9) and 15.1.22 of [1],

$$T_0\,{}_2F_1\left[\begin{matrix}\frac{2R+1}{4},\frac{2R+3}{4}\\ 3/2\end{matrix}\middle|-1\right]=\frac{T_0\,2^{-\left(\frac{R}{2}+\frac{1}{4}\right)}\sqrt{\pi}\,\Gamma\left(\frac{3}{2}\right)}{\left(\frac{R}{2}-\frac{1}{4}\right)}\left[\frac{1}{\Gamma\left(\frac{R}{4}+\frac{1}{8}\right)\Gamma\left(\frac{7}{8}-\frac{R}{4}\right)}-\frac{1}{\Gamma\left(\frac{R}{4}+\frac{5}{8}\right)\Gamma\left(\frac{3}{8}-\frac{R}{4}\right)}\right] \qquad (2.11)$$

Moreover, since the parameters in the hypergeometric function (2.11) gives the result

$\dfrac{R}{2}+\dfrac{1}{4}-\dfrac{R}{2}-\dfrac{3}{4}+\dfrac{3}{2}=1$ then the left hand side of (2.11) satisfies Kummer's identity [11],

and we may write

$$\sum_{r=0}^{\infty}\binom{R+r-1}{r}\binom{-\frac{3}{2}-r}{R+r-1}4^{-\frac{1}{2}-r-R}=T_0\frac{\Gamma\left(\frac{2R+11}{8}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{2R+7}{4}\right)\Gamma\left(\frac{9-2R}{8}\right)}$$

$$=\sum_{\mu=0}^{R-1}Q_{-(R,-\mu)}(\xi_0)\binom{-\frac{3}{2}}{R-1-\mu}\xi_0^{\mu-R-\frac{1}{2}}$$

$$=T_0\,{}_2F_1\left[\begin{matrix}\frac{R}{2}+\frac{1}{4},\frac{R}{2}+\frac{3}{4}\\ \frac{3}{2}\end{matrix}\middle|-1\right]. \qquad (2.12)$$

In equation (2.10), with $b=4$ the characteristic function

$$g(z)=z^2-4z-4$$

gives the dominant zero, $\xi_0=2\left(1+\sqrt{2}\right)$. Some values of (2.12) are

$R=1$	$R=2$	$R=3$	$R=4$
$\dfrac{1}{8\left(1+\sqrt{2}\right)^{\frac{1}{2}}}$	$\dfrac{-\left(2+\sqrt{2}\right)}{2.4^3\left(1+\sqrt{2}\right)^{\frac{1}{2}}}$	$\dfrac{3\left(2+\sqrt{2}\right)}{2\sqrt{2}\,4^5\left(1+\sqrt{2}\right)^{\frac{1}{2}}}$	$\dfrac{-5}{\sqrt{2}\,4^7\left(1+\sqrt{2}\right)^{\frac{1}{2}}}$

(ii) Another elegant identity may be obtained from (2.10). Upon putting $b=4$ and $\alpha=\frac{1}{2}$, we have that

$$T_0\,{}_2F_1\left[\left.\begin{matrix}\frac{R}{2}-\frac{1}{4},\ \frac{R}{2}+\frac{1}{4}\\ \frac{1}{2}\end{matrix}\right| -1\right]=\frac{T_0\,2^{\left(\frac{1}{4}-\frac{R}{2}\right)}\sqrt{\pi}\,\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{5}{8}-\frac{R}{4}\right)\Gamma\left(\frac{3}{8}+\frac{R}{4}\right)} \tag{2.13}$$

which may be extracted from identity 15.1.21 of Abramowitz and Stegun. We may now conclude that, from (2.13) and (1.19)

$$\frac{T_0 2^{\left(\frac{1}{4}-\frac{R}{2}\right)}\sqrt{\pi}\,\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{5}{8}-\frac{R}{4}\right)\Gamma\left(\frac{3}{8}+\frac{R}{4}\right)} = \sum_{\mu=0}^{R-1} Q_{(R,-\mu)}(\xi_0)\binom{-\frac{1}{2}}{R-1-\mu}\xi_0^{\mu-R+\frac{1}{2}}$$

$$= \sum_{r=0}^{\infty}\binom{R+r-1}{r}\binom{-\frac{1}{2}-r}{R+r-1}4^{\frac{1}{2}-r-R} \tag{2.14}$$

where $\xi_0 = 2\left(1+\sqrt{2}\right)$.

The infinite series in (2.12) and (2.14) are on the boundary of convergence, $s=-1$. It may be shown by Leibniz's theorem and Stirling's approximation of the Gamma function that (2.12) and (2.14) converge, albeit very slowly.

Some values of (2.14) are

$R=1$	$R=2$	$R=3$	$R=4$
$\frac{\left(1+\sqrt{2}\right)^{\frac{1}{2}}}{4}$	$\frac{\left(1+\sqrt{2}\right)^{\frac{1}{2}}\left(\sqrt{2}-2\right)}{4^3}$	$\frac{3\sqrt{2}\left(1+\sqrt{2}\right)^{\frac{1}{2}}\left(\sqrt{2}-2\right)}{2.4^5}$	$\frac{5\sqrt{2}\left(1+\sqrt{2}\right)^{\frac{1}{2}}}{4^7}$

If on the other hand $b=-4$, and $s=1$ equation (2.8) does not hold. Equation (2.8) requires the condition that $R<\frac{1}{2}$. In this instance the identities of Gauss and Vandermonde [7] cannot be ascertained from (2.7). A further illustrative case is obtained by taking $a=2$, $b=-\frac{\sqrt{27}}{2}$ and $s=1$ in (2.3) to give, without the coefficient To,

$$ {}_3F_2\left[\begin{matrix}\frac{R-n-1}{3}, & \frac{R-n}{3}, & \frac{R-n+1}{3} \\ -\frac{n}{2}, & \frac{1-n}{2} & \end{matrix}\middle| 1\right] \tag{2.15}$$

Following the preceeding argument relating to ${}_2F_1$ it may be stated that (1.19) diverges and (2.15) will not yield the identities of Dixon, Saalschutz, Watson or Whipple, see Gaspar and Rahman [7].

The result of equation (1.19) can now be extended by varying the form of the characteristic function (1.14). In the next section the case of more than one dominant zero of a characteristic function affecting the closed form solution of an associated infinite series will be investigated. For this case a modified probability density function will be considered in the application of a discrete renewal equation.

3. A Modified Density Function

In this section a modified probability density function will be considered such that more than one dominant zero of a resulting characteristic function will affect the closed form representation of an infinite sum generated by the consideration of the transformed function $F(z)$ in (1.9).

To achieve this end, consider the average number of renewals,

$$w_n = \binom{n}{k-1} b^{n-(k-1)} c^{k-1} \tag{3.1}$$

and the modified probability density function

$$\phi_n = b\binom{n-(ak+1)}{k-1} b^{n-(ak+1)} U(n-(ak+1)) \tag{3.2}$$

where $c \in \Re$, $b \in \Re$, $a \geq 1$ and $k \geq 1$.

Defining $W(z)$ and $\Phi(z)$ as the Z transform of w_n and ϕ_n respectively, results in, upon using (3.1) and (3.2)

$$\left.\begin{aligned} W(z) &= \frac{zc^{k-1}}{(z-b)^k} \\ \text{and} \qquad \Phi(z) &= \left(\frac{bz^{-a}}{z-b}\right)^k \end{aligned}\right\}. \qquad (3.3)$$

Substituting (3.3) into the transformed discrete renewal equation (1.3) gives

$$\frac{1}{c^{k-1}} F_1(z) = F(z) = \frac{z}{(z-b)^k - (bz^{-a})^k} = \frac{z^{ak+1}}{(z^a(z-b))^k - b^k}. \qquad (3.4)$$

Expanding the second term of (3.4) in series form, results in

$$F(z) = \sum_{r=0}^{\infty} \frac{b^{kr} z^{1-akr}}{(z-b)^{k(1+r)}}. \qquad (3.5)$$

The inverse Z transform of (3.5) is

$$f_n = \sum_{r=0}^{\infty} \binom{n-akr}{kr+k-1} b^{n-akr-k+1} U(n-akr) \qquad (3.6)$$

which may be written as

$$f_n = \sum_{r=0}^{\left[\frac{n-k+1}{k(a+1)}\right]} \binom{n-akr}{kr+k-1} b^{n-akr-k+1} \qquad (3.7)$$

where $[x]$ represents the integer part of x.

The inverse Z transform of (3.5) may also be written as

$$f_n = \frac{1}{2\pi i} \int_C z^n \left(\frac{F(z)}{z}\right) dz = \sum_{j=0}^{k-1} \sum_{v=0}^{a} z^n \, Res_{j,v} \left(\frac{F(z)}{z}\right)$$

where $Res_{j,v}$ is the residue of the poles of (3.5) and C is a smooth Jordan curve enclosing the singularities of (3.5).

From (3.5), the characteristic function (with some restriction)

$$g(z) = \left(z^a(z-b)\right)^k - b^k \tag{3.8}$$

has exactly $k(a+1)$ distinct zeros $\xi_{j,v}$ for $j = 0, 1, 2, \ldots, (k-1)$ and $v = 0, 1, 2, \ldots, a$, of which at least one and at most four are real. See appendix B for an explanation of this statement.

Therefore $\left(\xi_{j,v}^a(\xi_{j,v} - b)\right)^k - b^k = 0$, and all the singularities in (3.4) are simple poles.

Hence from (3.4), $F(z)$ has exactly $k(a+1)$ simple poles, so that

$$f_n = \sum_{j=0}^{k-1} \sum_{v=0}^{a} Q(\xi_{j,v}) \xi_{j,v}^n \tag{3.9}$$

where the residue contribution

$$Q(\xi_{j,v}) = \lim_{z \to \xi_{j,v}} \left[(z - \xi_{j,v}) \frac{F(z)}{z} \right]$$

$$= \frac{\xi_{j,v}}{k(\xi_{j,v} - b)^{k-1} \{(a+1)\xi_{j,v} - ab\}} \tag{3.10}$$

From (3.6) or (3.7) and using (3.10)

$$\sum_{r=0}^{\left[\frac{n-k+1}{k(a+1)}\right]} \binom{n-akr}{kr+k-1} b^{n-akr-k+1} = \sum_{j=0}^{k-1} \sum_{v=0}^{a} \frac{\xi_{j,v}^{\;n+1}}{k\left(\xi_{j,v}-b\right)^{k-1}\left\{(a+1)\xi_{j,v}-ab\right\}}. \tag{3.11}$$

The Infinite Series

Define $\xi_{j,0}$, $j = 0,1,2,\ldots\ldots.(k-1)$, as the k dominant zeros (the ones with the greatest modulus) of the characteristic function (3.8). The limiting behaviour of (3.11) is such that for n large, asymptotically

$$f_n = \sum_{r=0}^{\left[\frac{n-k+1}{k(a+1)}\right]} \binom{n-akr}{kr+k-1} b^{n-akr-k+1} \sim \sum_{j=0}^{k-1} \frac{\xi_{j,0}^{n+1}}{k\left(\xi_{j,0}-b\right)^{k-1}\left\{(a+1)\xi_{j,0}-ab\right\}}.$$

This suggestive limiting behaviour leads to the **conjecture** that

$$\sum_{r=0}^{\infty} \binom{n-akr}{kr+k-1} b^{n-akr-k+1} = \sum_{j=0}^{k-1} \frac{\xi_{j,0}^{n+1}}{k\left(\xi_{j,0}-b\right)^{k-1}\left\{(a+1)\xi_{j,0}-ab\right\}} \tag{3.12}$$

for all values of n, in the region where the infinite series converges.

The conjectured result (3.12) together with (3.11) implies that

$$\sum_{r=0}^{\left[\frac{n-k+1}{k(a+1)}\right]} \binom{n-akr}{kr+k-1} b^{n-akr-k+1} + \sum_{r=\left[\frac{n+ak+1}{k(a+1)}\right]}^{\infty} \binom{n-akr}{kr+k-1} b^{n-akr-k+1}$$

$$= \sum_{j=0}^{k-1} \frac{\xi_{j,0}^{n+1}}{k\left(\xi_{j,0}-b\right)^{k-1}\left\{(a+1)\xi_{j,0}-ab\right\}}$$

which gives that result

$$\sum_{r=\left[\frac{n+ak+1}{k(a+1)}\right]}^{\infty} \binom{n-akr}{kr+k-1} b^{n-akr-k+1} = -\sum_{j=0}^{k-1}\sum_{v=1}^{a} \frac{\xi_{j,v}^{n+1}}{k\left(\xi_{j,v}-b\right)^{k-1}\left\{(a+1)\xi_{j,v}-ab\right\}}.$$

Application of the ratio test indicates that the infinite series (3.12) converges in the region

$$\left|\left\{\frac{(a+1)^{a+1}}{(ab)^{a}}\right\}^{k}\right| < 1 ,$$

which is the same as (1.20), and this restriction also applies to equation (3.8).

Proof of result (3.12)

The characteristic function (3.8) may be expressed as the product of factors such that

$$g(z) = \left\{z^{a}(z-b)\right\}^{k} - b^{k} = \prod_{j=0}^{k-1}\left(z^{a+1} - bz^{a} - be^{2\pi ij/k}\right) = \prod_{j=0}^{k-1} q_j(z) \qquad (3.13)$$

Now, consider each of the factors in (3.13) and write

$$F_j(z) = \frac{z^{a+1}}{z^{a+1} - bz^{a} - be^{2\pi ij/k}} \qquad (3.14)$$

for each $j = 0, 1, 2, \ldots\ldots (k-1)$.

The singularities in (3.14) with restriction (1.20) are all simple poles and therefore for each j , $F_j(z)$ has exactly $'(a+1)'$ simple poles of which $\alpha_{j,0}$ shall indicate the dominant zero of each of the factors $q_j(\alpha_{j,0})$ in (3.13).

From the work of the previous section and utilizing the result (1.19) for $R=1$, it follows from (3.14) that

$$\sum_{r=0}^{\infty} e^{2\pi ijr/k} \binom{n-ar}{r} b^{n-ar} = W(\alpha_{j,0})\, \alpha_{j,0}^{n} \tag{3.15}$$

where from (1.16) the residue contribution, $W(\alpha_{j,0})$ from (3.14) is

$$W(\alpha_{j,0}) = \lim_{z \to \alpha_{j,0}} \left[(z-\alpha_{j,0}) \frac{z^{a}}{q_j(z)} \right]$$

$$= \frac{\alpha_{j,0}}{(a+1)\alpha_{j,0} - ab}$$

and $\alpha_{j,0}$ are the dominant zeros of each of the factors $q_j(\alpha_{j,0})$. See appendix B for an explanation of this statement.

Substituting $W(\alpha_{j,0})$ into (3.15) results in

$$\sum_{r=0}^{\infty} e^{2\pi ijr/k} \binom{n-ar}{r} b^{n-ar} = \frac{\alpha_{j,0}^{n+1}}{(a+1)\alpha_{j,0} - ab} \tag{3.16}$$

for each $j = 0, 1, 2, \ldots\ldots (k-1)$.

Notice that (3.16) implies that the sum may in fact be a complex number.

The summation of (3.16) for all $j = 0, 1, 2, \ldots\ldots (k-1)$ gives the result, that

$$\sum_{j=0}^{k-1} \sum_{r=0}^{\infty} e^{2\pi ijr/k} \binom{n-ar}{r} b^{n-ar} = \sum_{j=0}^{k-1} \frac{\alpha_{j,0}^{n+1}}{(a+1)\alpha_{j,0} - ab}, \tag{3.17}$$

Rescaling the left hand side of (3.17) by $r=(r^*+1)k$ and then replacing r^* by r results in, after changing the order of summation

$$\sum_{r=-1}^{\infty}\sum_{j=0}^{k-1} e^{2\pi i j(r+1)}\binom{n-ak(r+1)}{k(r+1)} b^{n-ak(r+1)} = \sum_{j=0}^{k-1}\frac{\alpha_{j,0}^{n+1}}{(a+1)\alpha_{j,0}-ab}$$

$$\sum_{r=0}^{\infty}\binom{n-akr-ak}{kr+k} b^{n-akr-ak} = \sum_{j=0}^{k-1}\frac{\alpha_{j,0}^{n+1}}{k\{(a+1)\alpha_{j,0}-ab\}} - b^{n} \qquad (3.18)$$

Now make the substitution $n-ak=m$ in (3.18) such that

$$\sum_{r=0}^{\infty}\binom{m-akr}{kr+k} b^{-akr} = \sum_{j=0}^{k-1}\frac{b^{-m}\alpha_{j,0}^{m+ak+1}}{k\{(a+1)\alpha_{j,0}-ab\}} - b^{ak}. \qquad (3.19)$$

Newton's k^{th} forward difference formula of a function $h(x_j)=h_j$ at x_j is defined as

$$\Delta^{k}h_j = \Delta^{k-1}h_{j+1} - \Delta^{k-1}h_j \quad ; \quad (k=1,2,3,........)$$

and taking the first difference of (3.19) with respect to m results in, from the left hand side,

$$\sum_{r=0}^{\infty}\binom{m+1-akr}{kr+k} b^{-akr} - \sum_{r=0}^{\infty}\binom{m-akr}{kr+k} b^{-akr} = \sum_{r=0}^{\infty}\binom{m-akr}{kr+k-1} b^{-akr}. \qquad (3.20)$$

Similarly from the right hand side of (3.19), gives the result that

$$\sum_{j=0}^{k-1}\frac{\left(\alpha_{j,0}^{m+2+ak}b^{-(m+1)} - \alpha_{j,0}^{m+1+ak}b^{-m}\right)}{Ak} = \sum_{j=0}^{k-1}\frac{b^{-m}\alpha_{j,0}^{m+1+ak}}{Ak}\left(\frac{\alpha_{j,0}-b}{b}\right) \qquad (3.21)$$

where $A=(a+1)\alpha_{j,0}-ab$.

From (3.14) using the characteristic equation

$$\alpha_{j,0}^{a}\left(\alpha_{j,0}-b\right)-be^{2\pi ij/k}=0$$

it may be seen that (3.21) becomes

$$\sum_{j=0}^{k-1}\frac{b^{-m+k-1}\alpha_{j,0}^{m+1}}{kA\left(\alpha_{j,0}-b\right)^{k-1}}. \qquad (3.22)$$

Combining (3.20) with (3.22) one obtains after simplification

$$\sum_{r=0}^{\infty}\binom{m-akr}{kr+k-1}b^{m-akr-k+1}=\sum_{j=0}^{k-1}\frac{\alpha_{j,0}^{m+1}}{k\left(\alpha_{j,0}-b\right)^{k-1}\left\{(a+1)\alpha_{j,0}-ab\right\}}. \qquad (3.23)$$

Since the dominant zeros $\alpha_{j,0}$ $j=0,1,2,3,\ldots\ldots(k-1)$ of $q_j(z)=0$ are the same as the dominant zeros $\xi_{j,0}$ of (3.8) then upon renaming m as n in (3.23) proves the conjecture, since (3.12) and (3.23) are identical.

Putting $k=1$ in (3.23) yields the result (1.19) for $R=1$.

The degenerate case, $a=0$, of (3.23) yields the result

$$\sum_{r=0}^{\left[\frac{m-k+1}{k}\right]}\binom{m}{kr+k-1}=\sum_{j=0}^{k-1}\frac{\left(1+e^{2\pi ij/k}\right)^{m}}{ke^{2\pi ij(k-1)/k}}=\frac{2^{m}}{k}\sum_{j=0}^{k-1}e^{\frac{\pi ij}{k}(m+2)}.\,Cos^{m}\left(\frac{\pi j}{k}\right) \qquad (3.24)$$

Using the *WZ* pairs method of Wilf and Zeilberger [11] a rational function proof certificate $R_k(m,r)$ for $k=1$ and 2 of (3.24) is respectively

$$R_1(m,r)=\frac{r}{2(r-1-m)}$$

and

$$R_2(m,r)=\frac{(r-1)(2r-1)}{m(2r-m-2)}.$$

By definition

$$G_k(m,r)=R_k(m,r)\,F_k(m,r)$$

where

$$F_k(m,r)=\binom{m}{kr+k-1}2^{k-1-m}$$

and therefore the identity

$$\sum_{r=0}^{\left[\frac{m-k-1}{k}\right]} F_k(m,r)=1$$

is certified by the pair (F_k, G_k) with the conditions

$$F_k(m+1,r)-F_k(m,r)=G_k(m,r+1)-G_k(m,r)$$

and $\lim_{r\to\pm\infty} G_k(m,r)=0$ satisfied.

In particular from (3.24) for $k = 4$, we have

$$\sum_{r=0}^{\left[\frac{m-3}{4}\right]} \binom{m}{4r+3} = \frac{1}{4}\left[2^m - 2^{\frac{m}{2}+1} Sin\frac{m\pi}{4}\right]$$

Some numerical results are now given for various parameters values of (3.12).

Numerical Results

The following numerical results, to five significant digits, are given for various parameter values of (3.12).

n	k	a	b	$\xi_{j,0}$	The left and right hand sides of (3.12)
3	2	1	-10	$\xi_{0,0} = -8.87298$ $\xi_{1,0} = -10.91671$	299.98554
3	2	2	10	$\xi_{0,0} = 9.89791$ $\xi_{1,0} = 10.09807$	299.98988
3	3	1	-10	$\xi_{0,0} = -8.87298$ $\xi_{1,0} = -10.53286 + 0.78262i$ $\xi_{2,0} = -10.53286 - 0.78262i$	-30.00050
3	3	2	10	$\xi_{0,0} = 10.09807$ $\xi_{1,0} = 9.95107 + 0.08833i$ $\xi_{2,0} = 9.95107 - 0.08833i$	29.99979

Notice that for $k \geq 3$ some dominant zeros of the characteristic function (3.8) occur in complex conjugate pairs.

In the next section the average number of renewals function, w_n, will be modified so as to reflect a non homogeneous difference delay form of the stationary system size probabilities. The effect of this non homogeneity is to allow for distinct factors, therefore producing distinct multiple order poles in an associated distribution function.

4. Forcing Terms

Consider, in this section, the densities

$$w_n = \binom{n}{m+R-1} b^{n-(m+R-1)} c^{m+R-1} \tag{4.1}$$

and

$$\phi_n = b.\, b^{n-(a+1)} U(n-(a+1)) \tag{4.2}$$

where m is a natural number, R, a, n c and b are defined in section 1.

Choosing w_n as given by (4.1) is equivalent to putting a forcing term of the type b^m in the difference delay form of the stationary system size probabilities.

The Z transform of (4.1) and (4.2) is respectively

$$W(z) = \frac{zc^{m+R-1}}{(z-b)^{m+R}} \quad \text{and} \quad \Phi(z) = \frac{bz^{-a}}{z-b} \tag{4.3}$$

and substituting (4.3) into (1.4) results in

$$\frac{1}{c^{m+R-1}} F_1(z) = F(z) = \frac{z}{(z-b)^m (z-b-bz^{-a})^R} = \frac{z^{aR+1}}{(z-b)^m (z^{a+1}-bz^a-b)^R} \tag{4.4}$$

Expanding (4.4) into series form, results in

$$F(z) = \sum_{r=0}^{\infty} \binom{R+r-1}{r} b^r \frac{z^{1-ar}}{(z-b)^{m+R+r}} \tag{4.5}$$

and the inverse Z transform of (4.5) is

$$f_n = \sum_{r=0}^{\infty} \binom{R+r-1}{r} \binom{n-ar}{r+m+R-1} b^{n-ar-(m+R-1)} U(n-ar) \qquad (4.6)$$

Now (3.4) has a simple pole of order m at the singularity $z = b$ and a simple pole of order R at the dominant zero, ξ_0, of (1.14) so that $\xi_0^{a+1} - b\xi_0^a - b = 0$.

Following the procedure as described in section 1, we can define

$$\nu!\, P_{-(m,-\nu)}(b) = \lim_{z \to b} \left[\frac{d^\nu}{dz^\nu} \left\{ (z-b)^m \frac{F(z)}{z} \right\} \right]$$

and

$$\mu!\, Q_{-(R,-\mu)}(\xi_0) = \lim_{z \to \xi_0} \left[\frac{d^\mu}{dz^\mu} \left\{ (z-\xi_0)^R \frac{F(z)}{z} \right\} \right]$$

where $F(z)$ is given by (4.4) so that we arrive at the **conjecture**

$$\sum_{r=0}^{\infty} \binom{R+r-1}{r} \binom{n-ar}{r+m+R-1} b^{n-ar-(m+R-1)} = \sum_{\nu=0}^{m-1} P_{-(m,-\nu)}(\mathrm{b}) \binom{n}{m-1-\nu} b^{n-(m-1-\nu)}$$

$$+ \sum_{\mu=0}^{R-1} Q_{-(R,-\mu)}(\xi_0) \binom{n}{R-1-\mu} \xi_0^{n-(R-1-\mu)}. \qquad (4.7)$$

Note that for $m = 0$, (4.7) reduces to the previous result (1.19).

For $R=2$ and $m=1$, then from (4.7)

$$\sum_{r=0}^{\infty}\binom{r+1}{r}\binom{n-ar}{r+2}b^{n-ar-2}=b^{n+2a-2}+\frac{\xi_0^{n+1}}{(\xi_0-b)A^3}\left[\binom{n}{1}A+a(a+1)(\xi_0-b)\right]$$

where $A=(a+1)\xi_0-ab$.

The degenerate case of (4.7), for $a=0$ gives the interesting binomial convolution identity

$$\sum_{r=0}^{n-m-R+1}\binom{R+r-1}{r}\binom{n}{r+m+R-1}=\sum_{\nu=0}^{m-1}(-1)^R\binom{R+\nu-1}{\nu}\binom{n}{m-1-\nu}$$
$$+\sum_{\mu=0}^{R-1}(-1)^{\mu}\binom{m+\mu-1}{\mu}\binom{n}{R-1-\mu}2^{n-(R-1-\mu)} \qquad (4.8)$$

which is in the spirit of the identities given by Chu [4].

For various specific values of m and R the *WZ* pairs method of Wilf and Zeilberger may be used to verify the identity (4.8).

Appendix A

In section 1, equation (1.31), the identity

$$abRAQ_{-(R+1,-\mu)} = ab(R-\mu)\xi_0 Q_{-(R,-\mu)} + (A+ab)\xi_0 \frac{d}{d\xi_0} Q_{-(R,-(\mu-1))}$$

$$+(\mu-1)AQ_{-(R-(\mu-1))} \tag{A1}$$

was required.

The identity (A1) can be arrived at in the following way.

From equation (1.16)

$$\mu! Q_{-(R,\mu)} = \lim_{z\to\xi_0}\left[\frac{d^\mu}{dz^\mu}\left\{(z-\xi_0)^R \frac{z^{aR}}{(g(z))^R}\right\}\right] \tag{A2}$$

where $g(z)$ is defined by equation (1.14) and ξ_0 is the dominant zero of $g(z)$.

The equation (A2) can be differentiated with respect to b, such that

$$\mu!\frac{d}{db}Q_{-(R,-\mu)} = -\frac{R}{b}\lim_{z\to\xi_0}\left[\frac{d^\mu}{dz^\mu}\left\{\frac{\xi_0^2(z-\xi_0)^{R-1}z^{aR}}{A(g(z))^R} + \frac{(z-\xi_0)^R z^{aR}}{(g(z))^R} - \frac{(z-\xi_0)^R z^{a(R+1)+1}}{(g(z))^{R+1}}\right\}\right] \tag{A3}$$

where $A = (a+1)\xi_0 - ab$.

Simplifying (A3), by adjusting the third term, we obtain

$$\mu!\frac{d}{db}Q_{(R,-\mu)}=\frac{-R}{b}\mu!Q_{(R,-\mu)}+\frac{R}{b}\mu!Q_{(R+1,-\mu)}$$

$$-\frac{R\xi_0}{bA}\lim_{z\to\xi_0}\frac{d^\mu}{dz^\mu}\left[\frac{(z-\xi_0)^{R-1}z^{a(R+1)}}{(g(z))^{R+1}}\{\xi_0 g(z)z^{-a}-A(z-\xi_0)\}\right]. \tag{A4}$$

Let $h(z)=z^{-a}g(z)\xi_0-A(z-\xi_0)$, and obtain a Taylor series expansion of $h(z)$ about the dominant zero ξ_0 giving

$$h(z)=(z-\xi_0)^2\sum_{j=2}(-1)^j\binom{a+j-1}{j}(b-\xi_0)\xi_0^{-j+1}(z-\xi_0)^{j-2}. \tag{A5}$$

Substituting (A5) into (A4) we obtain,

$$\mu!\frac{d}{db}Q_{(R,-\mu)}=\frac{-R\mu!}{b}Q_{(R,-\mu)}+\frac{R\mu!}{b}Q_{(R+1,-\mu)}$$

$$-\frac{R\xi_0}{bA}\lim_{z\to\xi_0}\frac{d^\mu}{dz^\mu}\left[\frac{(z-\xi_0)^{R+1}z^{a(R+1)}}{(g(z))^{R+1}}.B_j\right] \tag{A6}$$

where $B_j=\frac{h(z)}{(z-\xi_0)^2}=\sum_{j=2}(-1)^j\binom{a+j-1}{j}(b-\xi_0)\xi_0^{-j+1}(z-\xi_0)^{j-2}.$ (A7)

Expanding (A6) by the Leibniz differentiation rule gives the result

$$\mu!\frac{d}{db}Q_{(R,-\mu)}=\frac{-R\mu!}{b}Q_{(R,-\mu)}+\frac{R\mu!}{b}Q_{(R+1,-\mu)}$$

$$-\frac{\mathrm{R}\xi_0}{\mathrm{bA}}\sum_{k=0}^{\mu}\binom{\mu}{k}(\mu-k)!Q_{(R+1,-(\mu-k))}\lim_{z\to\xi_0}(B_j)^{(k)} \tag{A8}$$

After evaluating the $\lim_{z\to\xi_0}(B_j)^{(k)}$ from (A7), and substituting in (A8) we obtain

$$\frac{d}{db}Q_{-(R,-\mu)} = \frac{R}{b}\left[Q_{-(R+1,-\mu)} - Q_{-(R,-\mu)} - \frac{(b-\xi_0)}{A}\sum_{k=0}^{\mu}(-1)^k\binom{a+k+1}{k+2}\xi_0^{-k}Q_{-(R+1,-(\mu-k))}\right]. \qquad \text{(A9)}$$

In a similar fashion, we can evaluate

$$(\mu+1)!\,Q_{-(R,-(\mu+1))} = \lim_{z\to\xi_0}\left[\frac{d^\mu}{dz^\mu}\left\{\frac{d}{dz}\left(\frac{(z-\xi_0)^R z^{aR}}{(g(z))^R}\right)\right\}\right]$$

which may be written as

$$(\mu+1)!\,Q_{-(R,-(\mu+1))} = R\lim_{z\to\xi_0}\frac{d^\mu}{dz^\mu}\left[\frac{(z-\xi_0)^{R+1}z^{a(R+1)}h_1(z)}{(g(z))^{R+1}(z-\xi_0)^2}\right] \qquad \text{(A10)}$$

where
$$\frac{h_1(z)}{(z-\xi_0)^2} = \frac{(a(z-\xi_0)+z)g(z) - z(z-\xi_0)g'(z)}{z^{a+1}(z-\xi_0)^2}.$$

Further, expansion in a Taylor series about ξ_0 gives the representation

$$\frac{h_1(z)}{(z-\xi_0)^2} = 1B_j = \sum_{j=2}(-1)^j\binom{a+j-1}{a-1}\frac{(j-1)(\xi_0-b)(z-\xi_0)^{j-2}}{\xi_0^j}. \qquad \text{(A11)}$$

From (A10) and utilizing (A11), we obtain after some simplification

$$(\mu+1)Q_{-(R,-(\mu+1))} = \frac{R(\xi_0-b)}{\xi_0^2}\sum_{k=0}^{\mu}\frac{(-1)^k(k+1)}{\xi_o^k}\binom{a+k+1}{a-1}Q_{-(R+1,-(\mu-k))}. \qquad \text{(A12)}$$

Now (A9) and (A12) suggest that the $Q^{'s}$ may be related by an expression of the form

$$Q_{-(R+1,-(\mu+1))} = c_1\frac{d}{db}Q_{-(R,-\mu)} + c_2\mu Q_{-(R,-\mu)} + c_3(R-(\mu+1))Q_{-(R,-(\mu+1))} \qquad \text{(A13)}$$

for $\mu = 0,1,2,.....(R-1)$. The constants c_1, c_2 and c_3 can be evaluated by forming three simultaneous equations and using the Q values given in table 1 of section 1, such that

$$Q_{-(R+1,-(\mu+1))} = \frac{a+1}{aR}\frac{d}{db}Q_{-(R,-\mu)} + \frac{\mu}{abR}Q_{-(R,-\mu)} + \frac{\xi_0(R-(\mu+1))}{AR}Q_{-(R,-(\mu+1))}$$

which upon rearrangement and allowing $A = (a+1)\xi_0 - ab$ gives

$$\frac{d}{db}Q_{-(R,-\mu)} = \frac{\xi_0}{Ab(A+ab)}\left[abRAQ_{-(R+1,-(\mu+1))} - \mu AQ_{-(R,-\mu)} - ab\xi_0(R-(\mu+1))Q_{-(R,-(\mu+1))}\right]. \quad (A14)$$

Now, since $\frac{d}{d\xi_0}Q = \frac{d}{db}Q.\frac{Ab}{\xi_0^2}$, (A13) can be written as, after rearrangement

$$\xi_0(A+ab)\frac{d}{d\xi_0}Q_{-(R,-\mu)} = abRAQ_{-(R+1,-(\mu+1))} - \mu AQ_{-(R,-\mu)} - ab\xi_0(R-(\mu+1))Q_{-(R,-(\mu+1))}$$

for $\mu = 0,1,2,......(R-1)$

which is the required identity (A1).

Appendix B

Some properties of the zeros of the characteristic functions

$$g(z) = z^{a+1} - bz^a - b \qquad \text{and} \qquad \text{(B1)}$$

$$g_1(z) = z^{ak}(z-b)^k - b^k \qquad \text{(B2)}$$

will be discussed in this appendix.

Let b be a real constant , a and $k \in N$, and z is a complex variable, and firstly we shall consider (B1).

Theorem 1

(i) The equation (B1) has at least one and at most two real zeros and a dominant zero, the one with the greatest modulus, ξ_0 such that $\xi_0 > b$ for $b > 0$ and $|\xi_0| > \left|\frac{ab}{a+1}\right|$ for $b < 0$ and the restriction

$$\left|\frac{(a+1)^{a+1}}{(ab)^a}\right| < 1. \qquad \text{(B3)}$$

(ii) The equation (B2) has at least one and at most four real zeros.

Proof

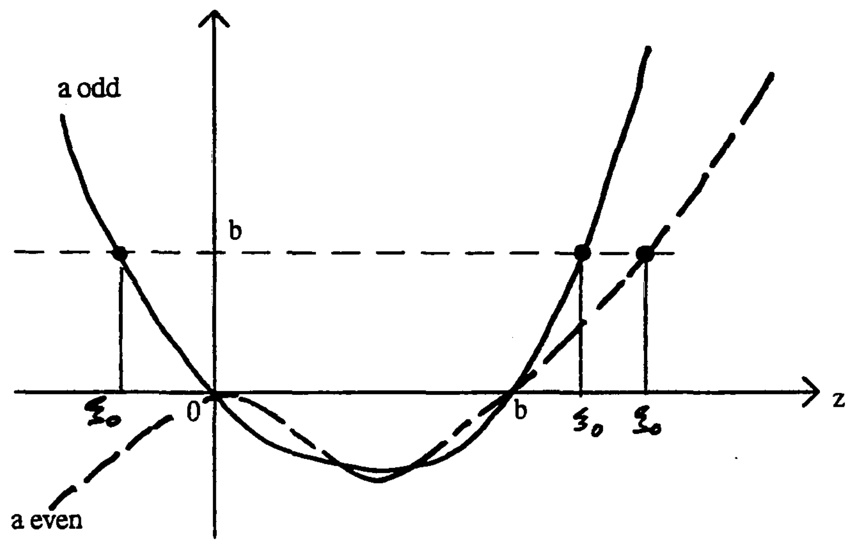
(i) The characteristic equation (B1) with restriction (B3) has $(a+1)$ distinct zeros, for the derivative of $g(z)$ cannot vanish coincidentally with $g(z)$.

The fact that a related equation to (B1) has distinct zeros appears to have been reported first by Bailey [2].

Let $G(z) = z^a(z-b)$, hence $G(z) = b$, and the turning point of $G(z)$, away from the origin occurs at $z = \dfrac{ab}{a+1}$.

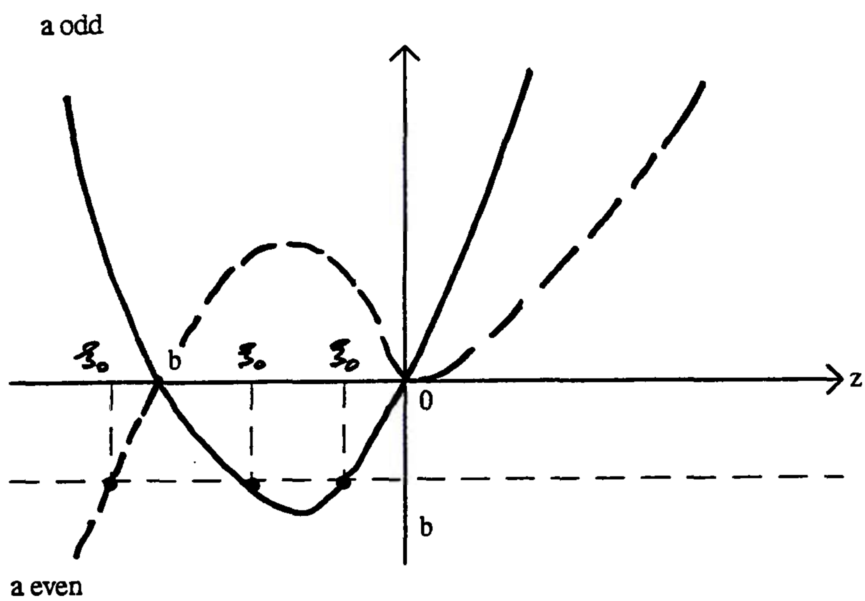
Now consider the graphs of $G(z)$,

For $b > 0$:



Graph 1: The graph of $G(z)$ for a odd or even and $b > 0$.

For $b < 0$:



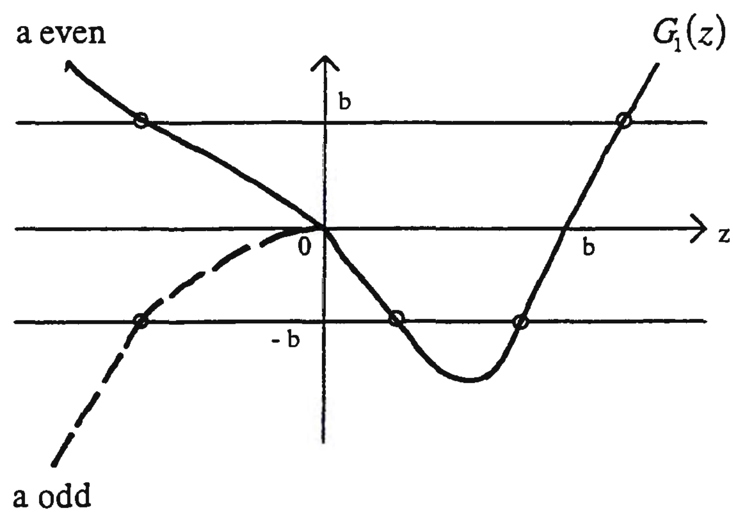
Graph 2: The graph of $G(z)$ for a odd or even and $b < 0$.

The two graphs of $G(z)$ indicate therefore that $(B1)$ has at least one and at most two real zeros. In both cases of $b>0$ and $b<0$ it will be shown in the next theorem that the dominant zero, ξ_0, the one with the greatest modulus, of $(B1)$ is always real, such that $\xi_0 > b$ for $b>0$ and all a values, and that $|\xi_0| > \left|\frac{ab}{a+1}\right|$ for $b<0$ and all values of a with restriction (B3).

(ii) In a similar fashion it may be seen that $g_1(z)$ has at most four and at least one real zeros.

Let $G_1(z) = \left(z^a(z-b)\right)^k$, hence $G_1(z) = b^k$ and the turning point of $G_1(z)$, away from the origin occurs at $z = \frac{ab}{a+1}$.

Now consider the graph of $G_1(z)$ for $b>0$ (the case $b<0$ follows in a similar fashion).



Graph 3: The graph of $G_1(z)$ indicating at least one and at most four real zeros of $g_1(z)$ with restriction (B3).

Theorem 2

The characteristic function

$$q_j(z) = z^{a+1} - bz^a - be^{2\pi ij/k} \tag{B4}$$

has 'a' zeros on the contour $C: |z| \leq \left|\frac{ab}{a+1}\right|$ for each $j = 0,1,2,3,....(k-1)$ with restriction (B3).

The study of the zeros of (B4), (B2) and (B1) is important in the area of queueing theory, and several papers have been devoted to this study, see for example, Chaudhry, Harris and Marchal [3] and Zhao [16]. Their studies have concentrated, amongst other things, on robustness of methods for locating zeros inside a unit circle. In this paper the location of dominant zeros of (B1), (B2) and (B4) is of prime importance.

Proof

The restriction (B3) comes from (1.20) which is required for the convergence of the infinite series in equation (1.19).

Let $A(z) = -bz^a$. Then $A(z)$ has 'a' zeros in the contour C and

$$|A(z)| \leq b\left(\frac{ab}{a+1}\right)^a.$$

Now

$$|q_j(z) - A(z)| = |z^{a+1} - be^{2\pi ij/k}|$$

$$\leq \left(\frac{ab}{a+1}\right)^{a+1} + b$$

$$= b\left(1 + b^{a}\left(\frac{a}{a+1}\right)^{a+1}\right).$$

By Rouche's theorem, see Takagi [15], it is required that

$$|q_j(z) - A(z)| \leq |A(z)|$$

hence

$$\left(1 + \left(\frac{a}{a+1}\right)\left(\frac{ab}{a+1}\right)^{a}\right) \leq \left(\frac{ab}{a+1}\right)^{a}$$

and

$$1 \leq \left(\frac{ab}{a+1}\right)^{a}\left(\frac{1}{a+1}\right)$$

which is satisfied since (B3) applies.

Hence the characteristic equation (B4) has 'a' zeros in the contour C and one zero with modulus bigger than $\left|\frac{ab}{a+1}\right|$.

Now from (B4), letting $j = 0$ gives the characteristic function (B1). Theorem 1 now follows since at least one zero of (B1) must be real, it is evident that $\xi_0 > b$ for $b > 0$ and $|\xi_0| > \left|\frac{ab}{a+1}\right|$ for $b < 0$.

Note, that restriction (B3) is imperative for theorem 2 to apply. If, for example $a=1, b=\frac{1}{2}, k=1$ which indicates that (B3) is not satisfied then $q_0(z)=z^2-\frac{z}{2}-\frac{1}{2}$ gives the two zeros as $z=\left\{-\frac{1}{2},1\right\}$, neither of which are in the contour $C: |z| \leq \frac{1}{4}$.

Theorem 3

The characteristic function $g_1(z)$ has 'ak' zeros in the contour $C: |z| \leq \left|\frac{ab}{a+1}\right|$ with restriction (B3).

Proof

Let $B(z)=\left(q_j(z)\right)^k=\left(z^{a+1}-bz^a-be^{2\pi ij/k}\right)^k$. Utilizing theorem 2, $B(z)$ has therefore 'ak' zeros in the contour C and therefore the remaining 'k' zeros have modulus bigger than $\left|\frac{ab}{a+1}\right|$.

In the contour C,

$$|B(z)|=\left|\left(z^{a+1}-bz^a-be^{2\pi ij/k}\right)^k\right|$$

$$\leq\left\{\left(\frac{ab}{a+1}\right)^{a+1}+b\left(\frac{ab}{a+1}\right)^a+b\right\}^k$$

$$=b^k\left\{\left(\frac{ab}{a+1}\right)^a\left(\frac{2a+1}{a+1}\right)+1\right\}^k \qquad \text{(B5)}$$

Now,

$$|g_1(z)-B(z)|=\left|(G(z))^k-b^k-\left(z^{a+1}-bz^a-be^{2\pi ij/k}\right)^k\right|$$

for every $j = 0,1,2,\ldots,(k-1)$, and $G(z) = z^a(z-b)$.

Further more let $c_j = be^{2\pi ij/k}$, such that

$$\left|(G(z))^k - b^k - (G(z)-c_j)^k\right| = \left|-b^k - \sum_{r=1}^{k}\binom{k}{r}(-1)^r c_j^r G^{k-r}(z)\right|$$

$$\leq b^k + \sum_{r=1}^{k}\binom{k}{r} b^r \left[\left(\frac{ab}{a+1}\right)^a \left(\frac{ab}{a+1}+b\right)\right]^{k-r}$$

$$= b^k\left[1+\sum_{r=1}^{k}\binom{k}{r}\left\{\left(\frac{ab}{a+1}\right)^a\left(\frac{2a+1}{a+1}\right)\right\}^{k-r}\right]$$

$$= b^k\left[1+(1+M)^k - M^k\right] \tag{B6}$$

where $M = \left(\frac{ab}{a+1}\right)^a\left(\frac{2a+1}{a+1}\right) > 0$.

By Rouche's theorem, it is required that

$|g_1(z) - B(z)| \leq |B(z)|$ and upon using (B5) and (B6) we have that

$$b^k\left[1+(1+M)^k - M^k\right] \leq b^k\left[(1+M)^k\right]$$

$$1 \leq \left\{\left(\frac{ab}{a+1}\right)^a\left(\frac{2a+1}{a+1}\right)\right\}^k$$

which is satisfied by virtue of restriction (B3). Therefore the characteristic function (B2) has 'ak' zeros in the contoúr $C:|z| \leq \left|\frac{ab}{a+1}\right|$ and 'k' zeros with modulus bigger than $\left|\frac{ab}{a+1}\right|$.

Consider as, an example $a=3, b=10, k=6$ such that restriction (B3) is satisfied and $C:|z|\leq 7½$. The zeros of $q_j(z)$ are listed below, showing that one dominant zero appears from each of the $q_j(z)$, for $j=0,12,3,4,5$.

$q_0(z)$	10.0100	-0.9696	0.4798- 0.8944i	0.4798+0.8944i
$q_1(z)$	10.0051+0.0086i	-0.9157- 0.3231i	0.7697- 0.6786i	0.1412+0.9933i
$q_2(z)$	9.9951+0.0087i	-0.7589- 0.6127i	0.9669- 0.3668i	-0.2031+0.9708i
$q_3(z)$	9.9900	1.0372	-0.5136+ 0.8375i	-0.5136-0.8375i
$q_4(z)$	9.9951-0.0087i	-0.7589+0.6127i	0.9669+ 0.3668i	-0.2031-0.9708i
$q_5(z)$	10.0051-0.0086i	-0.9157+0.3231i	0.7697+0.6786i	0.1412-0.9932i

The dominant zeros of $q_j(z)$are listed in the first column and all have modulus bigger than 7½. These dominant zeros are exactly the same k dominant zeros of (B2).

It appears that the zeros, $\alpha_j(a,b)$ of equation (B4) can be related for $b>0$ and $b<0$. It may be shown that the following relationships hold:

(i) For **all** values of k and a **even**

$$\alpha_j(a,b)=-\alpha_j(a,-b)$$

(ii) For k **odd** and a **odd**

$$\alpha_j(a,b)\neq\alpha_j(a,-b)$$

and,

(iii) For k **even** and a **odd**

$$\alpha_j(a,b)=\begin{cases}-\alpha_{j+\frac{k}{2}}(a,-b) & ;\ \text{for } j<\frac{k}{2}\\ -\alpha_0\ (a,-b) & ;\ \text{for } j=\frac{k}{2}\\ -\alpha_{j-\frac{k}{2}}(a,-b) & ;\ \text{for } j>\frac{k}{2}\end{cases}$$

where $j=0,1,2,\ldots\ldots,(k-1)$.

Conclusion

A Z transformation technique has been described whereby renewal processes were positively exploited to allow for the summation of series in closed form. The closed form representation of the infinite series depend on the dominant zeros of an associated characteristic function. The method may be easily extended to handle multiple delays and more general transformed functions of the type,

$$F_1(z) = \frac{z^{akR+1}}{(z-b)^m\left((z^a(z-b))^k - b^k\right)^R}$$

and

$$F_2(z) = \frac{z^{aR+1}}{(z-c)^m\left(z^{a+1} - bz^a - b\right)^R}$$

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