VICTORIA UNIVERSITY
OF TECHNOLOGY

DEPARTMENT OF MATHEMATICS, COMPUTING AND OPERATIONS RESEARCH

ON SERIES INVOLVING ROOTS OF TRANSCENDENTAL EQUATIONS ARISING FROM INTEGRAL EQUATIONS

Peter Cerone

(12 MATH 1)
JUNE, 1991.

TECHNICAL REPORT

DEPT OF MCOR
FOOTSCRAY INSTITUTE OF TECHNOLOGY
VICTORIA UNIVERSITY OF TECHNOLOGY
BALLARAT ROAD (P O BOX 64), FOOTSCRAY
VICTORIA, AUSTRALIA 3011
TELEPHONE (03) 688-4249/4225
FACSIMILE (03) 687-7632

ON SERIES INVOLVING ROOTS OF TRANSCENDENTAL EQUATIONS ARISING FROM INTEGRAL EQUATIONS

Peter Cerone
(12 MATH 1)
JUNE, 1991.

ON SERIES INVOLVING ROOTS OF TRANSCENDENTAL EQUATIONS ARISING FROM INTEGRAL EQUATIONS

P. CERONE

Victoria University of Technology, Australia, 3011.

Series arising from an integral equation are summed. The series involve inverse powers of roots from the characteristic equation. It is demonstrated how previous similar series obtained from differential -difference equations are particular cases of the present development.

1. INTRODUCTION

Silberstein [3] found the sums of two series arising from the differential - difference equation

$$u'(x) = u(x - \eta).$$

More recently Cerone and Keane [2] generalised the results to obtain the sum of of series $\sum (p_j)^{-k}$ and $\sum (1 + \eta p_j)^{-k}$ where p_j are the roots of $p = e^{-\eta p}$ and summation is over all p_j.

In this paper a method for developing the sum of similar series is derived from the renewal equation describing births in a one - sex population. A number of generalisations and extensions are also examined.

Sums of series of the form

$$\sum \frac{1}{(p_j - \alpha)^n \mu_j} \quad , \quad \alpha \neq p_j \quad , \quad n \in I_+$$

are obtained where the summation is over all the roots p_j of $\phi^*(p) = 1$

$$\text{and } \mu_j = - \left[\frac{d}{dp} \phi^*(p) \right]_{p = p_j} .$$

2. BASIC EQUATION AND RESULTS

Consider the births B(t) at time t from a single ancestor aged x at the zero of time. Thus

$$B(t) = \frac{\phi(x+t)}{l(x)} + \int_0^t B(t-x)\phi(x)\,dx\,, \qquad (1)$$

where $\phi(x)$ is the net maternity function

and $l(x)$ is the probability of surviving to age x of a new born.

Taking the Laplace transform of (1) we obtain after minor manipulation

$$B(t) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{pt}\,\frac{e^{px}\int_x^\infty e^{-pu}\phi(u)\,du}{l(x)\,[1-\phi^*(p)]}\,dp \qquad (2)$$

where $\phi^*(p)$ is the Laplace transform of $\phi(x)$ and γ is chosen in such a manner as to ensure convergence. If we now allow $t \to 0+$ then since the Laplace transform gives the mean value at a discontinuity (Bellman and Cooke [1]), we obtain from (2) that

$$\frac{1}{2}\phi(x+) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{px}\int_x^\infty e^{-pu}\phi(u)du}{1-\phi^*(p)}\,dp\,. \qquad (3)$$

Proceeding in a formal fashion we integrate (3) from t to ∞ to give

$$\frac{1}{2}\int_t^\infty \phi(x)\,dx = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} \frac{\left\{\int_t^\infty \phi(u)du - e^{pt}\int_t^\infty e^{-pu}\phi(u)du\right\}}{p\,[1-\phi^*(p)]}\,dp\,. \qquad (4)$$

Putting t = 0 in equation (4) we obtain

$$\frac{M_0}{2} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{M_0 - \phi^*(p)}{p[1-\phi^*(p)]}\,dp$$

where M_0 is the zeroth moment of ϕ.

That is,

$$\frac{M_0}{2} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{M_0 - 1}{p[1-\phi^*(p)]}\,dp + \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{dp}{p} \quad . \tag{5}$$

Evaluation of these integrals using residues and noting that the second integral in (5) gives 1/2 (see Cerone and Keane [2]) gives

$$S_1 = \sum \frac{1}{p_j\,\mu_j} = \frac{1}{2}\,\frac{M_0+1}{M_0-1} \tag{6}$$

where p_j are the roots of $\phi^*(p) = 1$ and are assumed to be simple,

$$\mu_j = -\left[\frac{d\,\phi^*(p)}{dp}\right]_{p=p_j}$$

and the summation is over all the p_j.

It is a straight forward matter to deduce from equation (5) that

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{dp}{p\,[1-\phi^*(p)]} = \frac{1}{2} \tag{7}$$

and so from (4)

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{pt}\int_t^{\infty} e^{-pu}\,\phi(u)du}{p\,[1-\phi^*(p)]}\,dp = 0. \tag{8}$$

Integrating (8) from x to ∞ and putting x = 0 gives

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{M_0 - \phi^*(p)\,dp}{p^2\,[1-\phi^*(p)]} = 0 \tag{9}$$

Now, for the nth moment

$$M_n = \int_0^{\infty} u^n \phi(u)\, du < \infty ,$$

we can develop $\phi^*(p)$ into a Taylor series expansion about $p = 0$ since

$$M_n = (-1)^n \left[\frac{d^n}{dp^n} \phi^*(p) \right]_{p=0} . \qquad (10)$$

Hence we can write

$$\phi^*(p) = M_0 - \frac{M_1}{1!} p + O(p^2) ,$$

and so there is a pole at $p = 0$ in equation (9) with residue $\frac{M_1}{1 - M_0}$.

Further, evaluation of equation (9) gives the sum, S_2 as

$$S_2 = \sum \frac{1}{p_j^2 \mu_j} = \frac{M_1}{(1 - M_0)^2} . \qquad (11)$$

Continuing in this manner we can obtain in a formal fashion a countably infinite number of series of the form

$$S_n = \sum \frac{1}{p_j^n \mu_j} . \qquad (12)$$

For the n^{th} step, with $n \geq 2$ we have

$$\frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\left[M_0 - \frac{M_1}{1!} p + ... + (-1)^{n-2} \frac{M_{n-2}}{(n-2)!} p^{n-2} - \phi^*(p) \right] dp}{p^n [1 - \phi^*(p)]} = 0. \qquad (13)$$

Developing $\phi^*(p)$, in the numerator of equation (13), in a Taylor series expansion about $p = 0$ shows a simple pole at $p = 0$ the contribution from which is given by

$$\frac{(-1)^{n-1}}{(n-1)!} \frac{M_{n-1}}{M_0 - 1} . \qquad (14)$$

Using equation (14) and obtaining the contribution from p_j the roots of $\phi^*(p) = 1$ in equation (13) gives the sum of the series in (12) by

$$S_n = \frac{1}{M_0 - 1} \sum_{k=2}^{n-1} (-1)^{(n-k+1)} \frac{M_{n-k}}{(n-k)!} S_k + (-1)^n \frac{M_{n-1}}{(n-1)!\,(M_0 - 1)^2} . \qquad (15)$$

Equation (15) holds for n = 2, 3, ... The result shown in equation (11) is obtained on putting n = 2 in equation (15) and remembering that the term in the sigma sign is taken to be zero.

At each step of the procedure once an expression has been found for the n^{th} series, it can be shown that

$$\frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{dp}{p^n [1 - \phi^*(p)]} = 0 \qquad \text{for } n = 2, 3, \ldots\ldots \qquad (16)$$

since from (13) and (15) we have

$$S_n = \sum \frac{1}{p_j^n \mu_j} = - \operatorname{Res}^{(n)}_{p=0} \qquad (17)$$

where $\operatorname{Res}^{(n)}_{p=0}$ is the contribution of a pole of order n at p = 0 so that

$$\operatorname{Res}^{(n)}_{p=0} = \frac{1}{(n-1)!} \left[\frac{d^{n-1}}{dp^{n-1}} \left[\frac{1}{1 - \phi^*(p)} \right] \right]_{p=0} . \qquad (18)$$

The above result signifies that each of the coefficient integrals of the moments M in equation (13) is zero.

Equations (17) and (18) give a different, although equivalent, representation for S_n as equation (15). These expressions hold for n > 1. The sum S_1 is given by equation (6).

We note at this stage that the above procedure would have to be modified if $\phi^*(0) = 1$.

It is further of interest to note that series of the general form

$$\sigma_n(\alpha) = \sum \frac{1}{(p_j - \alpha)^n \mu_j} \tag{19}$$

can be summed by the above development, where $\phi^*(\alpha) \neq 1$. This may be accomplished by multiplying equation (3) by $e^{-\alpha x} x^{n-1}$, $n \geq 1$ before integration. An easier way, is to take n = 1 to obtain a generalisation of (6) as

$$\sigma_1(\alpha) = \sum \frac{1}{(p_j - \alpha)\mu_j} = \frac{1}{2} \frac{L_0(\alpha) + 1}{L_0(\alpha) - 1} = \frac{1}{2} + \frac{1}{L_0(\alpha) - 1} \tag{20}$$

where $L_n(\alpha) = \int_0^\infty e^{-\alpha x} x^n \phi(x)\, dx.$

Formal differentiation of equation (20) with respect to α would give expressions for $\sigma_n(\alpha)$ as

$$\sigma_n(\alpha) = \sum \frac{1}{(p_j - \alpha)^n \mu_j} = \frac{1}{(n-1)!} \frac{d^{n-1}}{d\alpha^{n-1}} \left[\frac{1}{L_0(\alpha) - 1}\right] \tag{21}$$

or alternatively by

$$\sigma_n(\alpha) = \frac{1}{n-1} \sigma'_{n-1}(\alpha) \quad , \quad n = 2, 3, \ldots . \tag{22}$$

It is of interest to note that (21) is similar to (17) and (18) with $\operatorname{Res}^{(n)}_{p=\alpha}$. Further we may note that since $S_n = \sigma_n(0)$ we may obtain the previous results for S_n by using equations (20) and (21) and putting $\alpha = 0$, after the differentiation.

Alternatively corresponding expressions to equations (15) and (16) could be obtained by replacing S_n by $\sigma_n(\alpha)$, M_n by $L_n(\alpha)$ and $\text{Res}^{(n)}_{p=0}$ by $\text{Res}^{(n)}_{p=\alpha}$

3. PARTICULAR RESULTS

To reproduce the results of Silberstein [3] and Cerone and Keane [2] we need to take

$$\phi(x) = H(x - \eta) \tag{23}$$

where H(u) is the Heaviside unit function defined as one for $u > 0$ and zero otherwise.

With $\phi(x)$ as in (23) and using (20) we obtain

$$\sum \frac{1}{1 + \eta p_j} = \frac{1}{2} \tag{24}$$

where the summation is over all the roots p_j of $pe^{\eta p} = 1$ and we have allowed $\alpha \to 0$.

If the M_n are not finite then the results for the sum of the S_n series would need to be modified. This can be done by replacing the M_n by $L_n(\alpha)$ and allowing $\alpha \to 0$.

Thus from equation (11) or (21) we have

$$S_2 = \lim_{\alpha \to 0} \sum \frac{1}{(p_j - \alpha)^2 \mu_j} = \lim_{\alpha \to 0} \frac{L_1(\alpha)}{(1 - L_0(\alpha))^2}$$

and so

$$\sum \frac{1}{p_j(1 + \eta p_j)} = 1 . \tag{25}$$

Generalisations can be obtained by taking other forms of $\phi(x)$ such as

$$\phi(x) = x^n H(x - b) H(c - x) . \tag{26}$$

As a demonstration we will consider

$$\phi(x) = x H(x - 1) . \tag{27}$$

Now, the Laplace Transform of (27) gives

$$\phi^*(p) = e^{-p} \left(\frac{1}{p} + \frac{1}{p^2}\right)$$

and so from (20) we need to take $\alpha \to 0$ since M_0 is not finite, giving,

$$\sum \frac{1 + p_j}{(1 + p_j)^2 + 1} = \frac{1}{2} \lim_{\alpha \to 0} \frac{L_0(\alpha) + 1}{L_0(\alpha) - 1}$$

where $L_0(\alpha) = \phi^*(\alpha)$.

Hence,

$$\sum \frac{1 + p_j}{(1 + p_j)^2 + 1} = \frac{1}{2} , \tag{28}$$

where the summation is over p_j the roots of $p^2 e^p = p + 1$. We note that both results (24) and (28) could have been obtained from equation (6) by allowing $M_0 \to \infty$. This cannot be done in situations involving other moments since then the rate at which $M_n \to \infty$ matters. In such cases we would need the explicit expression for $L_n(\alpha)$ so that the limit as $\alpha \to 0$ could be taken.

As a further example consider

$$\phi(x) = H(\gamma - x) \quad , \quad \gamma \neq 1 \tag{29}$$

so that

$$\phi^*(p) = \frac{1 - e^{-\gamma p}}{p}$$

and so

$$M_n = \frac{\gamma^{n+1}}{n+1}$$

The restriction on γ is made so that $\phi^*(0) \neq 1$.

Since the M_n are finite, the expressions obtained for S_n can be used directly to give from (6) and (15),

$$S_1 = \sum \frac{1}{p_j \mu_j} = \sum \frac{1}{\gamma p_j + 1 - \gamma} = \frac{1}{2} \cdot \frac{\gamma + 1}{\gamma - 1} \qquad (30)$$

and

$$S_n = \frac{1}{\gamma - 1} \sum_{k=2}^{n-1} \frac{(-\gamma)^{(n-k+1)}}{(n-k+1)!} S_k + \frac{(-\gamma)^n}{n!} \cdot \frac{1}{(\gamma - 1)^2}, \quad n = 2, 3, 4, \ldots \qquad (31)$$

with

$$S_n = \sum \frac{1}{p_j^{n-1} (\gamma p_j + 1 - \gamma)} \qquad (32)$$

and summation is over all p_j the roots of $p = 1 - e^{-\gamma p}$, $\gamma \neq 1$. Further, breaking (32) into partial fractions would produce sums of series of the form

$$\sum \frac{1}{p_j^k} .$$

In particular using (30), (31) and (32) with n = 2 gives

$$\sum \frac{1}{p_j} = - \frac{\gamma}{2(1-\gamma)} .$$

Taking $\phi(x)$ to be represented by a histogram would give a generalisation of the results obtained from $\phi(x)$ given by equation (29).

Thus if,

$$\phi(x) = \sum_{r=0}^{R-1} \alpha_r H(x - b_r) H(b_{r+1} - x) \qquad (33)$$

then

$$\phi^*(p) = \sum_{r=0}^{R} \gamma_r \frac{e^{-pb_r}}{p} \tag{34}$$

where

$$\gamma_r = \begin{cases} \alpha_r & , \quad r = 0 \\ \alpha_r - \alpha_{r-1} & , \quad 0 < r < R \\ -\alpha_{r-1} & , \quad r = R \ . \end{cases}$$

Now, using equations (10) and (33) gives

$$M_n = \frac{-1}{n+1} \sum_{r=0}^{R} \gamma_r b_r^{n+1} \ .$$

Assuming $M_0 = \phi^*(0) = -\sum_{r=0}^{N} \gamma_r b_r \neq 1$

then

$$S_1 = \sum \frac{1}{p_j \mu_j} = \sum \frac{1}{1 + \sum_{r=0}^{R} \gamma_r b_r e^{-p_j b_r}} = \frac{1}{2} \frac{M_0 + 1}{M_0 - 1}$$

and the sum for S_n is given by

$$S_n = \frac{1}{M_0 - 1} \sum_{k=2}^{n-1} \frac{(-1)^{n-k+1}}{(n-k)!} M_{n-k} S_k + \frac{(-1)^n}{(n-1)!} \frac{M_{n-1}}{M_0 - 1)^2}$$

where

$$S_n = \sum \frac{1}{p_j^{n-1} [1 + \sum_{r=0}^{R} \gamma_r b_r e^{-p_j b_r}]} \ .$$

Taking $R = 1$, $b_0 = 0$, $b_1 = \gamma$ will reproduce the results obtained previously

for $\phi(x) = H(\gamma - x)$.

4. SOME SIMPLE DERIVATIONS OF THE RESULTS OF SECTION 2

Consider the integral equation

$$B(t) = F(t) + \int_0^t B(t - x)\, \phi(x)\, dx \tag{35}$$

with $F(t) = e^{\alpha t}$ then we may readily obtain, using Laplace Transform techniques, that

$$B(t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{pt}\, dp}{(p - \alpha)\, [1 - \phi^*(p)]} \, . \tag{36}$$

That is, evaluating (36) using the theory of residues gives

$$B(t) = \frac{e^{\alpha t}}{1 - \phi^*(\alpha)} + \sum \frac{e^{p_j t}}{(p_j - \alpha)\, \mu_j} \tag{37}$$

where we are assuming that the roots of $\phi^*(p) = 1$ are simple and that $\phi^*(\alpha) \neq 1$.

Evaluation of (36) and (37) at $t = 0$ gives, since the Laplace transform gives the mean value at a discontinuity,

$$\frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{dp}{(p - \alpha)\, [1 - \phi^*(p)]} = \frac{F(0+)}{2} = \frac{1}{2} \tag{38}$$

and

$$\sigma_1(\alpha) = \frac{1}{2} - \frac{1}{1 - \phi^*(\alpha)} \, . \tag{39}$$

Differentiation with respect to α gives from (38)

$$\frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} \frac{dp}{(p-\alpha)^n\,[1-\phi^*(p)]} = 0 \qquad n = 2, 3, ,,, \qquad (40)$$

and from equation (39), $\sigma_n(\alpha)$ as given by equation (21).

Equation (21) could be obtained directly from evaluating equation (40) to give

$$\sigma_n(\alpha) = \sum \frac{1}{(p_j - \alpha)^n \mu_j} = - \operatorname{Res}^{(n)}_{p=\alpha} \,, \quad n = 2, 3, \ldots \qquad (41)$$

Further, equations (40) and (41) can be obtained from (35) by taking

$$F(t) = t^{n-1} e^{\alpha t}$$

and noting $F(0+) = \begin{cases} 0 \;, & n > 1 \\ 1 \;, & n = 1 \end{cases}$.

An alternate way to derive the sums of the series would be to take F in equation (35) as

$$e^{\alpha t} f(x+t) \quad .$$

Thus, with the integral equation

$$b(t) = e^{\alpha t} f(x+t) + \int_0^t b(t-x)\,\phi(x)\,dx \qquad (42)$$

and assuming f to have a Taylor series expansion about t = 0 we would obtain upon using equation (33),

$$\frac{1}{2} f(x) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} \sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{(p-\alpha)^{n+1}\,[1-\phi^*(p)]}\,dp \quad . \qquad (43)$$

Hence using residues we get from equation (43),

$$\frac{1}{2} f(x) = \sum_{n=0}^{\infty} f^{(n)}(x) \left[\sum \frac{1}{(p_j - \alpha)^{n+1} \mu_j} + \mathrm{Res}_{p=\alpha}^{n+1} \right] \tag{44}$$

where $\mathrm{Res}_{p=\alpha}^{n+1}$ is the residue at $p = \alpha$ from a pole of order $n+1$ of the integrand in (43).

Equating coefficients of $f^{(n)}(x)$, since $f(x)$ is an arbitrary function, we obtain from equation (44), $\sigma_n(\alpha)$ as given by equations (39) and (41).

5. REFERENCES

[1] R. Bellman and K.L. Cooke. Differential Difference Equations. Academic Press, N.Y., (1963).

[2] P. Cerone and A. Keane. Series of Roots of a Transcendental Equation. J. & Proc., Royal Soc. of N.S.W. (1977), Vol. 110, pp 29 - 31.

[3] L. Silberstein. On a Hystero - Differential Equation Arising in a Probability Problem. Phil. Mag. (1940) Ser 7, 29, pp. 75 - 84.