VICTORIA UNIVERSITY OF TECHNOLOGY

DEPARTMENT OF COMPUTER AND MATHEMATICAL SCIENCES

Isotopy of 6-Connected Digital Knots

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(43 MATH 7)

October, 1994

(AMS : 57M25)

TECHNICAL REPORT

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ISOTOPY OF 6-CONNECTED DIGITAL KNOTS

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ABSTRACT

Investigation of the topological nature of digitized knot structures is necessary for laying the foundations of a theory for dealing with the processing of such images. Images containing knots arise, for example, in the electron microscopy of DNA and RNA. The classical theory of knots is well developed, and one important result is that two knots are equivalent if their planar representations can be deformed into each other by a sequence of knot moves. We show here that digital knots, suitably defined, can be considered equal if they can be deformed into each other by a sequence of digital moves.

Keywords: digital topology, knots, equivalence, isotopy

1 CLASSICAL BACKGROUND

A *knot* is a continuous embedding $K : S^1 \to \mathbf{R}^3$. In some of the literature, the embedding is required to be differentiable,[2] or just piecewise linear.[4] One of the fundamental problems in knot theory is to determine whether two knots K_1 and K_2 are equivalent, and to give a suitable definition for equivalence. For our purposes, we are interested in *ambient isotopy*. Suppose there is a continuous function $H : [0,1] \times \mathbf{R}^3 \to \mathbf{R}^3$ which satisfies:

1. $H(0,x) = x$,
2. for each $t \in [0,1]$, $H(t,x)$ is a homeomorphism,
3. $H(1,x) = h(x)$, with $h \circ K_1 = K_2$.

Then the function h is an *ambient isotopy*, and the knots K_1 and K_2 are said to be *ambient isotopic.*

It is convenient to refer to knots not by their embedding functions, but by a projection of the image of the knot onto a plane, with the provision that crossings are clearly distinguished. For example, the *trefoil knot* can be described by the projection shown on the left in figure 1 which gives an intuitive understanding of the structure of the knot.

The right hand diagram shows the same knot, slightly deformed so that all lines in the projection are parallel to coordinate axes. This is an example of a *planar isotopy.*

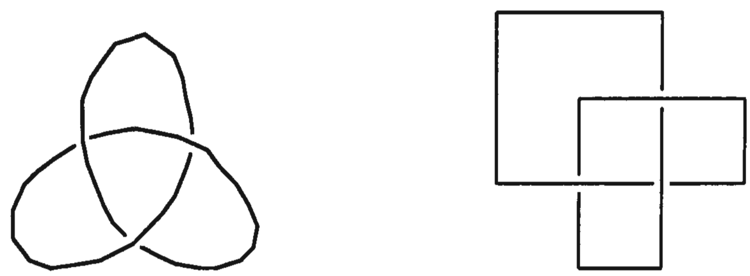
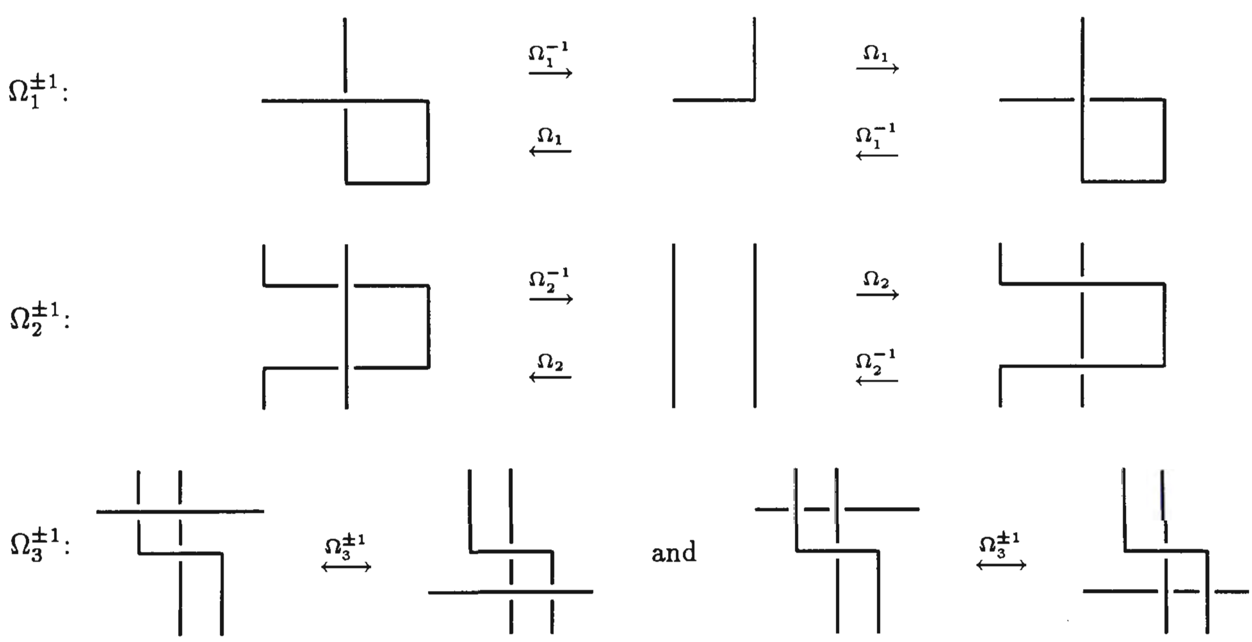


Figure 1: A trefoil knot.

Using projections such as these, knots can be determined up to ambient isotopy by means of the *Reidemeister moves*, of which there are three:



Reidemeister[4] has shown that two knots K_1 and K_2 are ambient isotopic if and only if their projections can be deformed into each other by a sequence of Reidemeister moves.

2 DIGITAL DEFINITIONS

In $\mathbb{Z}^3$, a 6-*connected digital knot* (hereafter called a *digital knot*), is a sequence of points $p_0, p_1, \ldots, p_n$ for which p_i and p_j are 6-adjacent[3] if and only if $i \equiv j \pmod{n-1}$.

Given a digital knot, its *continuous image* is the union of all the unit line segments $[p_i, p_j]$ connnecting adjacent points. The continuous image can be obtained by replacing each point p_i with a unit cube C_i centred at p_i and performing a deformation retract on the resulting structure. In order to ensure that this operation gives the required result, we have to ensure that the knot does not brush past itself. Thus if two points p_i and p_j are not

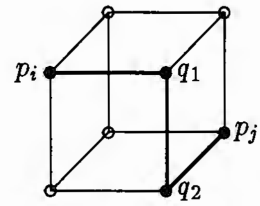


Figure 2: A double corner.

6-adjacent but are 18-adjacent,[3] then we require that there is a point q which is 6-adjacent to both p_i and p_j. That is, two points 18-adjacent but not 6-adjacent must be adjacent to a corner point. Similarly, if two points p_i and p_j are neither 6-adjacent nor 18-adjacent but are 26-adjacent, then we require that there are two points q_1 and q_2 for which p_i, q_1, q_2, p_j forms a 6-adjacent path. In this case p_i and p_j are points on either side of a "double corner" as shown in figure 2.

A digital knot satisfying these further criteria will be called *proper*. We will in general make no distinction between the points of a knot and the structure obtained by replacing each point with a unit cube; but will use the two forms interchangeably, the use being dictated by the context. An example of a proper digital knot is given in figure 3.

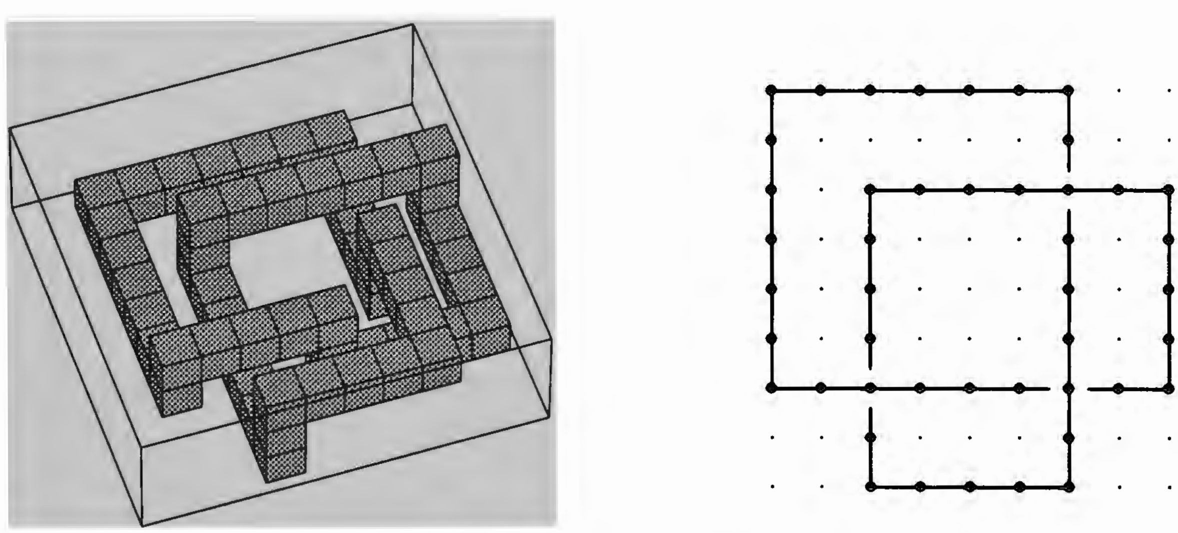


Figure 3: A digital trefoil knot.

2.1 Constructing a digital knot

Given a projection of (the image of) a classical knot K, we can obtain a digital knot by first deforming the projection by planar isotopy so that the result contains just straight lines and right angles.

We then ensure that every crossing has the form as shown in figure 4.

Such a crossing may be represented as:

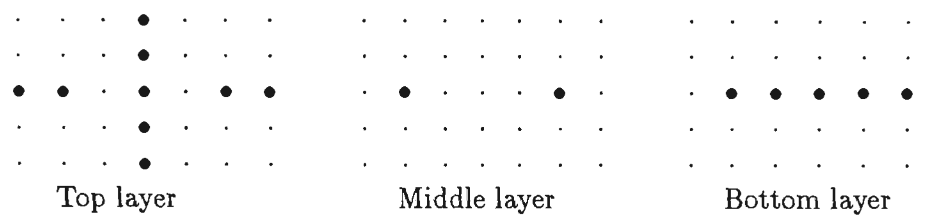


Figure 4: A crossing in a proper digital knot.

			1			
			1			
1	1–3	3	1, 3	3	1–3	1
			1			
			1			

where the number represent the level of the cubes; 1 is a first or top level cube, 2 a middle and 3 a bottom level cube. The terms 1–3 and 1, 3 represent a vertical tower of three cubes and a crossing respectively. We must ensure that all crossings are separated to the extent that no new, unwanted adjacencies occur, and we may have to include some vertical towers to ensure that the knot "joins up" properly. Using this notation, a proper digital trefoil is:

1–3	3	3	3	3	3	3		
1						3		
1		1	1	1	1	1, 3	1	1
1		1				3		1
1		1–3				1–3		1–3
1		3				1		1
1	1	1, 3	1	1–3	3	1, 3	3	3
		3				1		
		3	3	1–3	1	1		

Using the same notation, another proper digital trefoil is:

		1	1	1–5		
		1		5		
3	3	1, 3	3	3, 5	3	3–7
3		1		5		7
3		1–7	7	5, 7	7	7
3				5		
3–5	5	5	5	5		

2.2 Equivalence of digital knots

We can now give a definition as to the equivalence of two digital knots.

DEFINITION 2.1. *Two digital knots D_1 and D_2 are ambient isotopic if and only if their continuous images are ambient isotopic.*

The main aim of this paper is to show that ambient isotopy can be determined entirely digitally, that is, without recourse to the continuous image. To do this, we define some digital moves.

DEFINITION 2.2. *A* handle move *is performed by taking a line of three points (cubes) and shifting it one unit in a direction perpendicular to itself, and then adding a point (cube) at each end to join the moved line to its old position.*

Figure 5 illustrates this in the plane.

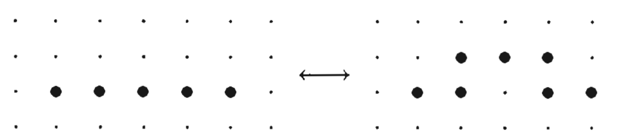


Figure 5: An example of a handle move.

The move can of course be performed in reverse, by "collapsing" a handle down to a straight line. A handle move is *proper* if no cube in the new line is adjacent to any cubes outside the line except for the two added cubes at each end.

DEFINITION 2.3. *A* reflection move *is performed by taking a corner cube, removing it, and attaching to the corner a cube in the only other position which preserves* 6*-connectedness.*

An example of a reflection move is given in figure 6.

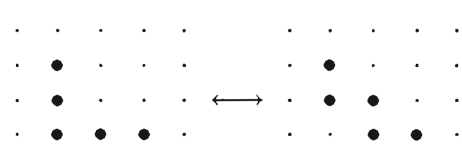
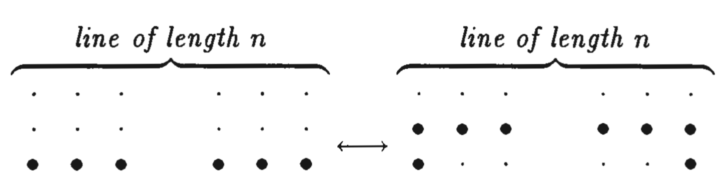


Figure 6: An example of a reflection move.

Again, such a move is *proper* if it introduces no extra adjacencies other than those in the corner itself. We will show in the next section that these are the ony two moves we need. We first show that these moves can be easily extended.

LEMMA 2.4. *Using digital moves a line of any length can be lifted, as shown in the following diagram.*



Proof. By elementary induction; the result is true for lines of length 3, by definition, and if it is true for lines of length k, then it is true for lines of length $k+1$, by making a reflection move at one end. □

LEMMA 2.5. *Using digital moves, two consecutive sides of a rectangle can be moved to the other two sides.*

Proof. Again, by induction; this time double induction on the lengths of the sides of the rectangle. The sides of a rectangle of length 2×2 can be so swapped, by a single reflection move. Suppose the result is true for rectangles of length $k \times l$. Then given $(k+1) \times l$ rectangle, we can produce such a side swap by first swapping the sides of a $k \times l$ rectangle, and then performing a line lift of length l to bring the remaining side out to its required position. Similarly we can swap the sides of a $k \times (l+1)$ rectangle. □

LEMMA 2.6. *Using digital moves, three sides of a rectangle can be flattened down to the fourth side.*

Proof. Using lemma 2.4, we can pull the middle of the three sides down by using an extended handle move. Repeating this as many times as necessary will flatten the rectangle as desired. Of course this can also be done in reverse. □

3 MAIN RESULT

We can now state the main theorem of this paper.

THEOREM 3.1. *Two proper digital knots D_1 and D_2 are ambient isotopic if and only if they can be transformed into each other by a sequence of proper handle moves and proper reflection moves.*

Proof. We will demonstrate this by showing that each Reidemeister move can be performed by a sequence of digital moves. We must therefore make the assumption that the digital knots D_1 and D_2 can be projected onto a plane in such a way that there is no point in the plane which is the image of more than two points of the knot. With this assumption we can then investigate each of the Reidemeister moves.

Move $\Omega_1^{\pm 1}$: The projection of the continuous image of a loop will be a Jordan curve in the plane. We note here that the points themselves may not from a digital Jordan curve in the sense of Rosenfeld[5]; an example of this is given in the left most diagram in figure 7, for which the projetion of the loop is a square of four points. We define a point (X, Y, Z) on the loop as follows: X is the largest x value taken by all points on the loop, Y is the largest y value taken by all points whose x coordinate is X, and Z is the largest z value taken by all points whose x and y coordinates are X and Y respectively. We shall refer to the point (X, Y, Z) as the *outer point*. Let n be the number of points on the loop, excluding the points at the crossing. Following Stout,[6] we will refer to the ordered quadruple (n, X, Y, Z) as the *rank* of the loop. Since ranks can be ordered lexicographically, we can proceed by strong induction.

Given the properness of the knot, the smallest possible loop is that shown in the left most diagram in figure 7, for which $n = 5$.

The sequence of diagrams in figure 7 show how such a loop can be transformed, by digital moves, into a non-loop, for which the projection onto the (x, y) plane is just a single point.

The result is thus true for all loops with length 5. Note that by lemma 2.4 this is also true for all loops with the same projection, no matter what their height.

We consider the different configurations at the outer point (X, Y, Z), and show that at each configuration a digital move (or series of such moves) can be made which reduces the rank. Diagram 8 shows all possible configurations (viewed from above), where • represents a point on the knot, · represents a point not on the knot, and ∘ a point which could be either. Similarly, a solid line indicates part of the projection of the knot, and the dashed lines possible extensions of the projection.

Case (i). Suppose that all the ∘ points are not part of the knot projection. Then a corner move at the outer

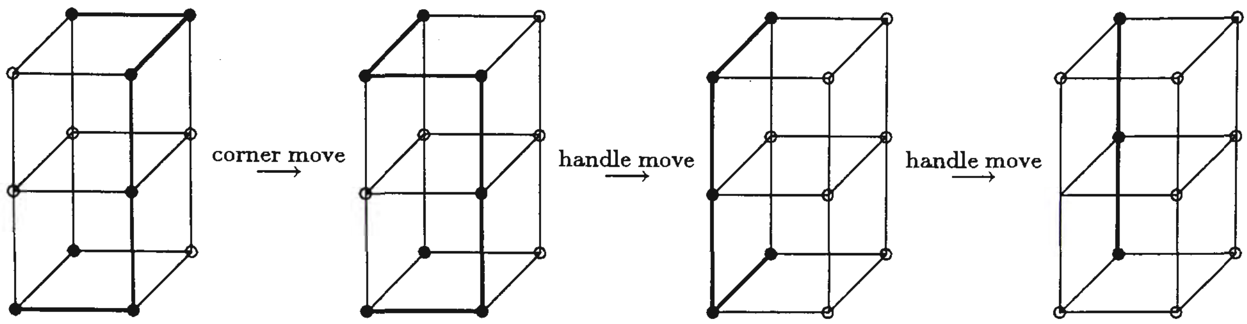


Figure 7: Digital moves transforming a small loop.

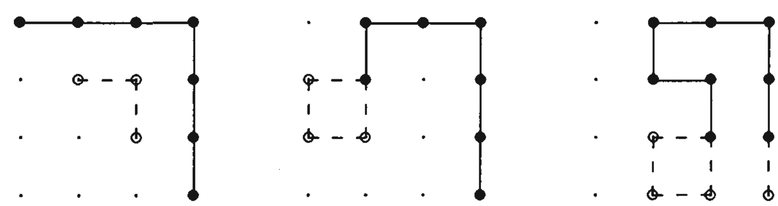


Figure 8: Possible configurations at the outer point.

point, shifting (X, Y, Z) to $(X-1, Y-1, Z)$, will maintain properness, as well as reducing the rank. Suppose however that the top right $\circ$ point $(X-1, Y-1, z)$ was part of the projection. Its vertical distance must be at least two from the outer point, otherwise the knot would not be proper. The other possibility is that the top right $\circ$ point $(X-1, Y-1, z)$ is the crossing point itself, in which case, by properness of the knot, we will have a small loop similar to the example shown on the left in figure 7, for which corner moves and handle moves can be made. If the point is not the crossing, the projection will change to include a point which, although not strictly a crossing point, will be the image of two points on the knot. To avoid this, we may first make a corner move at this point. A corner move on the outer point will then maintain properness, correct projection, and will reduce the rank. We must thus ensure that we can make a corner move at the point $(X-1, Y-1, z)$, and that this move will not affect the properness of the knot, or produce any forbidden aspects of the projection. There are two cases to consider:

1. A corner move will not produce any double points in the projection. In this case the corner move will maintain properness, and we are done.
2. A corner move will produce a double point at on the projection. In which case we have a further "inner point" with (x, y) coordinates $(X-2, Y-2)$. It is clear that we may consider the number of such "inner points"; if there are none, we are done; if there are more than one we just perform corner moves on each one starting at the inner most point. If the inner most point is the crossing point, then as we have seen, moves can be made which will deal with the other inner points.

Case (ii). A handle move along the top row will reduce the rank, and maintain properness.

Case (iii). We will have part of a loop as shown in figure 9, and it is readily seen that a sequence of handle and corner moves will both reduce the rank, and produce a loop which is proper, and whose projection contains no double points.

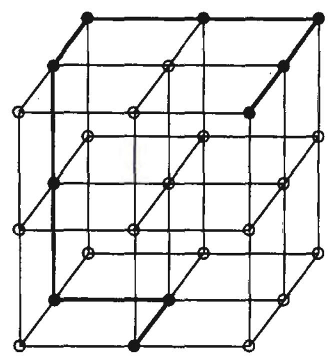


Figure 9: Part of a loop.

Moves $\Omega_{2,3}^{\pm 1}$: We treat these moves together; in each case the move can be effected by taking the upper strand of the knot and moving it over the lower strand or strands. We must ensure that the upper strand is high enough so as not to bump into any of the lower strand(s) which may be sticking up. Our first step is to demonstrate how a given knot segment can be lifted, using digital moves, to any desired level. More formally, given a segment $p_0 = (x_0, y_0, z_0), p_1, \ldots, p_n = (x_n, y_n, z_n)$, we show how digital moves can transform it to $p_0' = (x_0, y_0, z'), p_1', \ldots, p_n' = (x_n, y_n, z')$, where $z' \geq \max\{z_0, z_1, \ldots, z_n\}$. We will do this by induction on the number of corners between p_0 and p_n in the projection of the segment.

Suppose that the projection has no corners; that is, it is a straight line. We then have either the x or y coordinates constant along the line; suppose it to be x. We then have effectively a two dimensional situation, with the knot segment lying entirely in a plane parallel to the (y, z) plane. We now perform an induction argument on the number of corners in this planar segment. If none, we can lift up the line by lemma 2.6 to any height we require. Similarly, if there is only one corner, we can do the same thing. Suppose that there are two or more corners. Then we choose any two consecutive corners. As shown in figure 10, the corners may turn in the same direction, as in (i), or in different directions, as in (ii).

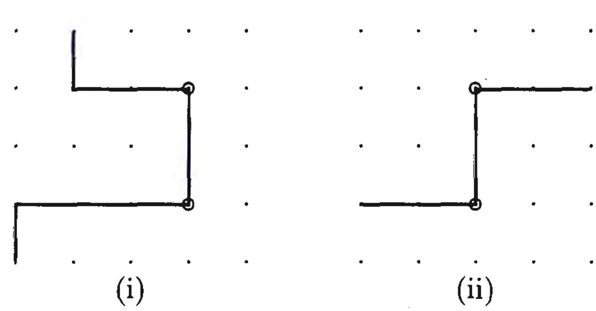


Figure 10: Different consecutive corners.

In case (i), suppose that there are corners "outside" the two circled (as shown in the diagram). In this case, a combination of moves will pull the segment between the corners down to the level of the closest corner, and thus reduce the number of corners. If there are no outside corners, then one (or perhaps both) of the end points p_0 and p_n will be the points at the end of the outside segments. In which case the inner segment can be moved to the level of either the end point or a consecutive corner (if there is one); whichever is closest to the inner segment. Again the number of corners is reduced.

In case (ii) we have a similar situation, but instead of moving a segment, we change two sides of a rectangle to the two opposite sides by lemma 2.5, and again the number of corners can be seen to be reduced.

This deals with the case of the projection having no corners.

Suppose now that the projection has some number of corners. We may consider the segment being composed of a number of sub-segments; each of which has a projection without corners, as shown in figure 11. We can apply the previous result and, by a combination of digital moves, build up each subsegment to any required height. That this can be done is an immediate consequence of lemmas 2.5 and 2.6

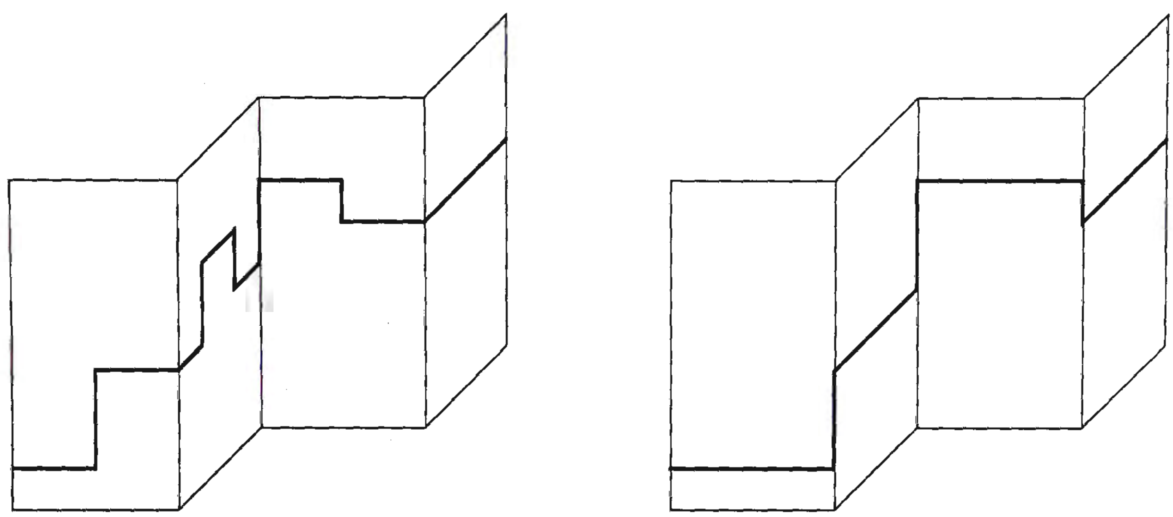


Figure 11: A segment whose projection has corners, and its transformation.

This needs to be done in stages, so as to maintain adjacencies. At each stage, we transform each subsegment in two sides of a rectangle, the end points of which are the end points of the the subsegment. We choose which two sides of the rectangle to use so as not to use either of the sides of the previous rectangle. The right hand diagram in figure 11 illustrates the result. We can now build up each rectangle to any desired height.

Having pulled our segment up to the height we desire, we now have to pull it across the lower segment or crossing, depending on which of the Reidemeister moves we are performing. But this is simply the previous result—moving a segment, this time within a different plane.

This completes the discussion of the Reidemeister moves, and the proof. □

4 CONCLUSION

We have seen that knots can be defined in digital space, and can be classified on their points alone. In particular, if a scanning electron microscope image of a molecule exhibits a knotted structure, we can classify the structure by the method given.

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