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Companions of Hermite-Hadamard Inequality for Convex Functions (II)

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Abstract

Companions of Hermite-Hadamard inequalities for convex functions defined on the positive axis in the case when the integral has either the weight $\frac{1}{t^2}$ or $\frac{1}{t}$, $t > 0$ are given. Applications for special means are provided as well.

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1. Introduction

The following integral inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)\,dt \leq \frac{f(a)+f(b)}{2}, \tag{1.1}$$

which holds for any convex function $f:[a,b]\to\mathbf{R}$, is well known in the literature as the Hermite-Hadamard inequality.

There is an extensive amount of literature devoted to this simple and nice result which has many applications in the Theory of Special Means and in Information Theory for divergence measures, for which we would like to refer the reader to the papers [1] – [61] and the references therein.

Recently we proved the following Hermite-Hadamard type inequality [22]:

Theorem 1. Let $f:[a,b]\subset(0,\infty)\to\mathbf{R}$ be a convex function on $[a,b]$, then we have the inequalities

$$\frac{A\left(\frac{f(a)}{a},\frac{f(b)}{b}\right)}{G^2(a,b)} \geq \frac{1}{b-a}\int_a^b \frac{1}{t^3}f(t)\,dt \geq \frac{f(H(a,b))}{H(a,b)\,G^2(a,b)}, \tag{1.2}$$

where

$$H(p,q) := \frac{2}{\frac{1}{p}+\frac{1}{q}},\ G(p,q) := \sqrt{pq} \text{ and } A(p,q) := \frac{p+q}{2}$$

are the Harmonic, Geometric and Arithmetic means, respectively.

If the function f is concave, then the inequalities (1.2) reverse.

Let us recall the following means :

The *logarithmic mean*:

$$\mathrm{L}=\mathrm{L}(a,b) := \begin{cases} a & \text{if } a=b \\ \dfrac{b-a}{\ln b-\ln a} & \text{if } a\neq b \end{cases} \quad a,b>0;$$

The *identric mean*:

$$\mathrm{I}:=\mathrm{I}(a,b) = \begin{cases} a & \text{if } a=b \\ \dfrac{1}{e}\left(\dfrac{b^b}{a^a}\right)^{\frac{1}{b-a}} & \text{if } a\neq b \end{cases} \quad a,b>0;$$

The *p-logarithmic mean*:

$$\mathrm{L}_p = L_p(a,b) := \begin{cases} \left[\dfrac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}} & \text{if } a \neq b; \\ a & \text{if } a = b \end{cases} \quad \text{where } p \in R\setminus\{-1,0\}$$

and $a, b > 0$.

It is well known that L_p is monotonic nondecreasing over $p \in R$ with $L_{-1} := L$ and $L_0 := I$.

In particular, we have the inequalities

$$H \leq G \leq L \leq I \leq A. \tag{1.3}$$

Utilising Theorem 1, we can state the following proposition [22]:

Proposition 1. For any $0 < a < b$ we have

$$G^2 \geq LH, \tag{1.4}$$

$$0 \leq AL - G^2 \leq \frac{1}{4}(b-a)^2 \frac{AL}{G^2} \tag{1.5}$$

and

$$0 \leq G^2 - HL \leq \frac{1}{4}(b-a)^2 \frac{AL}{G^2}. \tag{1.6}$$

In this paper we establish some companions of Hermite-Hadamard inequalities for convex functions defined on the positive axis in the case when the integral has either the weight $\frac{1}{t^2}$ or $\frac{1}{t}$, $t > 0$. Applications for special means are provided as well.

2. The Results

We start with the following companion of Hermite-Hadamard inequality:

Theorem 2. Let $f : [a,b] \subset (0,\infty) \to \mathbf{R}$ be a convex function on $[a,b]$, then we have the inequalities

$$\frac{1}{b-a}\left[\frac{L(a,b)-a}{L(a,b)} \cdot \frac{f(a)}{a} + \frac{b-L(a,b)}{L(a,b)} \cdot \frac{f(b)}{b}\right] \tag{2.1}$$

$$\geq \frac{1}{b-a}\int_a^b \frac{1}{t^2} f(t)\,dt \geq \frac{f\left(\frac{G^2(a,b)}{L(a,b)}\right)}{G^2(a,b)}.$$

If the function f is concave, then the inequalities (2.1) reverse.

Proof. Define $x := \frac{1}{b} < \frac{1}{a} := y$ and take $t, s \in (x, y)$.

By the convexity of f on $[a, b]$ we have

$$f\left(\frac{1}{t}\right) - f\left(\frac{1}{s}\right) \geq f'_{+}\left(\frac{1}{s}\right)\left(\frac{1}{t} - \frac{1}{s}\right).$$

Integrating over t on $[x, y]$ and dividing by $y - x$ we get

$$\frac{1}{y-x}\int_x^y f\left(\frac{1}{t}\right) dt - f\left(\frac{1}{s}\right) \geq f'_{+}\left(\frac{1}{s}\right)\left(\frac{1}{y-x}\int_x^y \frac{dt}{t} - \frac{1}{s}\right) \tag{2.2}$$

for any $s \in (x, y)$.

Since

$$\frac{1}{y-x}\int_x^y \frac{dt}{t} = \frac{\ln y - \ln x}{y - x} = \frac{1}{L(x, y)},$$

then from (2.2) we get

$$\frac{1}{y-x}\int_x^y f\left(\frac{1}{t}\right) dt - f\left(\frac{1}{s}\right) \geq f'_{+}\left(\frac{1}{s}\right)\left(\frac{1}{L(x, y)} - \frac{1}{s}\right) \tag{2.3}$$

for any $s \in (x, y)$.

Taking $s = L(x, y)$ in (2.3) we get the following inequality of interest in itself

$$\frac{1}{y-x}\int_x^y f\left(\frac{1}{t}\right) dt \geq f\left(\frac{1}{L(x, y)}\right). \tag{2.4}$$

Changing the variable $u = \frac{1}{t}$ we obtain

$$\frac{1}{y-x}\int_x^y f\left(\frac{1}{t}\right) dt = \frac{1}{y-x}\int_{\frac{1}{y}}^{\frac{1}{x}} \frac{1}{u^2} f(u)\, du$$

$$= \frac{1}{\frac{1}{a} - \frac{1}{b}}\int_a^b \frac{1}{u^2} f(u)\, du = \frac{G^2(a, b)}{b - a}\int_a^b \frac{1}{u^2} f(u)\, du$$

and since

$$\frac{1}{L(x, y)} = \frac{\ln y - \ln x}{y - x} = \frac{\ln \frac{1}{a} - \ln \frac{1}{b}}{\frac{1}{a} - \frac{1}{b}} = \frac{G^2(a, b)}{L(a, b)},$$

then from (2.4) we get the second inequality in (2.1).

We know that for any convex function g on $[m, M]$ we have the inequality

$$\frac{M-v}{M-m}f(m)+\frac{v-m}{M-m}f(M)$$

$$\geq f\left[\frac{(M-v)m+(v-m)M}{M-m}\right]=f(v) \tag{2.5}$$

for any $v\in[m,M]$.

Now, if we write the inequality (2.5) for $m=\frac{1}{y}=a$, $v=\frac{1}{t}$ and $M=\frac{1}{x}=b$ then we get

$$\frac{\frac{1}{x}-\frac{1}{t}}{\frac{1}{x}-\frac{1}{y}}f\left(\frac{1}{y}\right)+\frac{\frac{1}{t}-\frac{1}{y}}{\frac{1}{x}-\frac{1}{y}}f\left(\frac{1}{x}\right)\geq f\left(\frac{1}{t}\right)$$

for any $t\in[x,y]$, which is equivalent to

$$\frac{xy}{y-x}\left[\left(\frac{1}{x}-\frac{1}{t}\right)f\left(\frac{1}{y}\right)+\left(\frac{1}{t}-\frac{1}{y}\right)f\left(\frac{1}{x}\right)\right]\geq f\left(\frac{1}{t}\right) \tag{2.6}$$

for any $t\in[x,y]$.

Integrating (2.6) over t on $[x,y]$ and dividing by $y-x$ we get

$$\frac{xy}{y-x}\left[\left(\frac{1}{x}-\frac{1}{L(x,y)}\right)f\left(\frac{1}{y}\right)+\left(\frac{1}{L(x,y)}-\frac{1}{y}\right)f\left(\frac{1}{x}\right)\right] \tag{2.7}$$

$$\geq\frac{1}{y-x}\int_x^y f\left(\frac{1}{t}\right)dt,$$

which is an inequality of interest in itself.

Writing the inequality (2.7) in terms of a and b we obtain

$$\frac{1}{b-a}\left[\left(b-\frac{G^2(a,b)}{L(a,b)}\right)f(a)+\left(\frac{G^2(a,b)}{L(a,b)}-a\right)f(b)\right] \tag{2.8}$$

$$\geq\frac{G^2(a,b)}{b-a}\int_a^b\frac{1}{u^2}f(u)\,du.$$

Since

$$\left(b-\frac{G^2(a,b)}{L(a,b)}\right)f(a)=ba\left(1-\frac{a}{L(a,b)}\right)\frac{f(a)}{a}$$

$$=G^2(a,b)\cdot\frac{L(a,b)-a}{L(a,b)}\cdot\frac{f(a)}{a}$$

and

$$\left(\frac{G^2(a,b)}{L(a,b)}-a\right)f(b)=ab\left(\frac{b}{L(a,b)}-1\right)\frac{f(b)}{b}$$
$$=G^2(a,b)\cdot\frac{b-L(a,b)}{L(a,b)}\cdot\frac{f(b)}{b},$$

then by (2.8) we get

$$\frac{1}{b-a}\left[G^2(a,b)\cdot\frac{L(a,b)-a}{L(a,b)}\cdot\frac{f(a)}{a}+G^2(a,b)\cdot\frac{b-L(a,b)}{L(a,b)}\cdot\frac{f(b)}{b}\right]$$
$$\geq\frac{G^2(a,b)}{b-a}\int_a^b\frac{1}{u^2}f(u)\,du.$$

Finally, dividing by $G^2(a,b)$ we get the first inequality in (2.1). □

We have the following result as well:

Theorem 3. Let $f:[a,b]\subset(0,\infty)\to\mathbf{R}$ be a convex function on $[a,b]$, then we have the inequalities

$$\frac{1}{b-a}\left[\frac{b-L(a,b)}{L(a,b)}f(a)+\frac{L(a,b)-a}{L(a,b)}f(b)\right]\tag{2.9}$$
$$\geq\frac{1}{b-a}\int_a^b\frac{1}{t}f(t)\,dt\geq\frac{f(L(a,b))}{L(a,b)}.$$

If the function f is concave, then the inequalities (2.9) reverse.

Proof. By the convexity of f we have

$$f(t)-f(s)\geq f'_+(s)(t-s)$$

for any $t,s\in(a,b)$.

If we multiply this inequality by $\frac{1}{t}$ and integrate over t on $[a,b]$ we get by division with $b-a$

$$\frac{1}{b-a}\int_a^b\frac{f(t)}{t}dt-f(s)\frac{1}{b-a}\int_a^b\frac{1}{t}dt\geq f'_+(s)\left(1-s\frac{1}{b-a}\int_a^b\frac{1}{t}dt\right)\tag{2.10}$$

and since

$$\frac{1}{b-a}\int_a^b \frac{1}{t}dt = \frac{1}{L(a,b)}$$

then we get from (2.10) the inequality

$$\frac{1}{b-a}\int_a^b \frac{f(t)}{t}dt - \frac{f(s)}{L(a,b)} \geq f'_+(s)\left(1-\frac{s}{L(a,b)}\right) \tag{2.11}$$

for any $s \in (a,b)$, which is an inequality of interest in itself.

By taking $s = L(a,b)$ in (2.11) we get the second inequality in (2.5).

From the inequality (2.5) we get

$$\frac{b-t}{b-a}f(a) + \frac{t-a}{b-a}f(b) \geq f(t) \tag{2.12}$$

for any $t \in [a,b]$.

If we multiply this inequality by $\frac{1}{t}$ and integrate over t on $[a,b]$ we get by division with $b-a$

$$\frac{b\frac{1}{b-a}\int_a^b \frac{1}{t}dt - 1}{b-a}f(a) + \frac{1-a\frac{1}{b-a}\int_a^b \frac{1}{t}dt}{b-a}f(b) \geq \frac{1}{b-a}\int_a^b \frac{f(t)}{t}dt,$$

which is equivalent to

$$\frac{\frac{b}{L(a,b)}-1}{b-a}f(a) + \frac{1-\frac{a}{L(a,b)}}{b-a}f(b) \geq \frac{1}{b-a}\int_a^b \frac{f(t)}{t}dt,$$

which proves the first inequality as well. □

3. Applications for Special Means

We have:

Proposition 2. For any $0 < a < b$ we have

$$b^{\frac{b-L(a,b)}{b-a}} a^{\frac{L(a,b)-a}{b-a}} \geq G(a,b). \tag{3.1}$$

Proof. Using the inequality (2.1) for the convex function $f : [a,b] \subset (0,\infty) \to \mathbf{R}$, $f(t) = t\ln t$ we get

$$\frac{1}{b-a}\left[\frac{L(a,b)-a}{L(a,b)}\cdot\ln a+\frac{b-L(a,b)}{L(a,b)}\cdot\ln b\right] \tag{3.2}$$
$$\geq\frac{1}{b-a}\int_a^b\frac{1}{t}\ln(t)\,dt\geq\frac{\frac{G^2(a,b)}{L(a,b)}\ln\left(\frac{G^2(a,b)}{L(a,b)}\right)}{G^2(a,b)}.$$

Observe that

$$\frac{1}{b-a}\int_a^b\frac{1}{t}\ln(t)\,dt=\frac{1}{2}\frac{(\ln b)^2-(\ln a)^2}{b-a} \tag{3.3}$$
$$=\frac{1}{2}\frac{[\ln b-\ln a][\ln b+\ln a]}{b-a}=\frac{\ln G(a,b)}{L(a,b)}$$

and

$$\frac{1}{b-a}\left[\frac{L(a,b)-a}{L(a,b)}\cdot\ln a+\frac{b-L(a,b)}{L(a,b)}\cdot\ln b\right]$$
$$=\ln\left[b^{\frac{b-L(a,b)}{(b-a)L(a,b)}}a^{\frac{L(a,b)-a}{(b-a)L(a,b)}}\right], \tag{3.4}$$

then by (3.2) we get

$$\ln\left[b^{\frac{b-L(a,b)}{(b-a)L(a,b)}}a^{\frac{L(a,b)-a}{(b-a)L(a,b)}}\right]\geq\ln\left[G(a,b)\right]^{\frac{1}{L(a,b)}}\geq\ln\left(\frac{G^2(a,b)}{L(a,b)}\right)^{\frac{1}{L(a,b)}},$$

which is equivalent to

$$b^{\frac{b-L(a,b)}{(b-a)L(a,b)}}a^{\frac{L(a,b)-a}{(b-a)L(a,b)}}\geq\left[G(a,b)\right]^{\frac{1}{L(a,b)}}\geq\left(\frac{G^2(a,b)}{L(a,b)}\right)^{\frac{1}{L(a,b)}}. \tag{3.5}$$

Taking the power $L(a,b)$ in (3.5) we obtain

$$b^{\frac{b-L(a,b)}{b-a}}a^{\frac{L(a,b)-a}{b-a}}\geq G(a,b)\geq\frac{G^2(a,b)}{L(a,b)}. \tag{3.6}$$

The second inequality in (3.6) is obvious, so we drop it. □

The following result also holds:

Proposition 3. For any $0<a<b$ we have

$$\frac{G^2(a,b)}{b-a}\left[\frac{L(a,b)-a}{L(a,b)}\cdot\frac{\ln\left(\frac{1}{a}\right)}{a}+\frac{b-L(a,b)}{L(a,b)}\cdot\frac{\ln\left(\frac{1}{b}\right)}{b}\right]$$

$$\geq \ln I\left(\frac{1}{b},\frac{1}{a}\right). \tag{3.7}$$

Proof. If we write the inequality (2.1) for the concave function $f : [a,b]\subset(0,\infty)\to\mathbf{R}$, $f(t)=\ln t$ we have

$$\frac{1}{b-a}\left[\frac{L(a,b)-a}{L(a,b)}\cdot\frac{\ln(a)}{a}+\frac{b-L(a,b)}{L(a,b)}\cdot\frac{\ln(b)}{b}\right] \tag{3.8}$$

$$\leq\frac{1}{b-a}\int_a^b\frac{1}{t^2}\ln(t)\,dt\leq\frac{\ln\left(\frac{G^2(a,b)}{L(a,b)}\right)}{G^2(a,b)}.$$

Integrating by parts, we have

$$\begin{aligned}\frac{1}{b-a}\int_a^b\frac{1}{t^2}\ln(t)\,dt&=-\frac{1}{b-a}\int_a^b\ln(t)\,d\left(\frac{1}{t}\right)\\
&=-\frac{1}{b-a}\left[\frac{1}{t}\ln(t)\Big|_a^b-\int_a^b\frac{1}{t^2}dt\right]\\
&=-\frac{1}{b-a}\left[\frac{1}{b}\ln(b)-\frac{1}{a}\ln(a)+\frac{1}{t}\Big|_a^b\right]\\
&=-\frac{1}{b-a}\left[\frac{1}{b}\ln(b)-\frac{1}{a}\ln(a)+\frac{1}{b}-\frac{1}{a}\right]\\
&=-\frac{1}{b-a}\left[\frac{1}{b}\ln(b)-\frac{1}{a}\ln(a)+\frac{1}{b}-\frac{1}{a}\right]\\
&=\frac{1}{b-a}\left[\frac{1}{b}\ln\left(\frac{1}{b}\right)-\frac{1}{a}\ln\left(\frac{1}{a}\right)-\frac{1}{b}+\frac{1}{a}\right]\\
&=\frac{1}{b-a}\left[\frac{1}{b}\ln\left(\frac{1}{b}\right)-\frac{1}{a}\ln\left(\frac{1}{a}\right)\right]+\frac{1}{ba}.\end{aligned}$$

Observe that

$$K = -\frac{1}{ba}\left[\frac{1}{\frac{1}{b}-\frac{1}{a}}\left[\frac{1}{b}\ln\left(\frac{1}{b}\right)-\frac{1}{a}\ln\left(\frac{1}{a}\right)\right]-1\right] = -\frac{\ln I\left(\frac{1}{b},\frac{1}{a}\right)}{G^2(a,b)},$$

then by (3.8) we get

$$\frac{1}{b-a}\left[\frac{L(a,b)-a}{L(a,b)}\cdot\frac{\ln(a)}{a}+\frac{b-L(a,b)}{L(a,b)}\cdot\frac{\ln(b)}{b}\right]$$
$$\leq -\frac{\ln I\left(\frac{1}{b},\frac{1}{a}\right)}{G^2(a,b)} \leq \frac{\ln\left(\frac{G^2(a,b)}{L(a,b)}\right)}{G^2(a,b)}.$$

If we multiply this inequality by $-G^2(a,b)<0$ we get

$$\frac{G^2(a,b)}{b-a}\left[\frac{L(a,b)-a}{L(a,b)}\cdot\frac{\ln\left(\frac{1}{a}\right)}{a}+\frac{b-L(a,b)}{L(a,b)}\cdot\frac{\ln\left(\frac{1}{b}\right)}{b}\right]$$
$$\geq \ln I\left(\frac{1}{b},\frac{1}{a}\right) \geq \ln\left(\frac{L(a,b)}{G^2(a,b)}\right)$$

and the inequality (3.7) is proved.

We notice that the second inequality is obvious, since

$$I\left(\frac{1}{b},\frac{1}{a}\right) \geq L\left(\frac{1}{b},\frac{1}{a}\right) = \frac{\frac{1}{b}-\frac{1}{a}}{\ln\frac{1}{b}-\ln\frac{1}{a}}$$
$$= \frac{a-b}{\ln a-\ln b}\cdot\frac{1}{ab} = \frac{L(a,b)}{G^2(a,b)},$$

so we drop it. □

We have:

Proposition 3. For any $0<a<b$ and $p\in(-\infty,0)\cup(1,\infty)\setminus\{2\}$ we have

$$\frac{1}{b-a}\left[\frac{L(a,b)-a}{L(a,b)}\cdot a^{p-1}+\frac{b-L(a,b)}{L(a,b)}\cdot b^{p-1}\right] \tag{3.9}$$
$$\geq L_{p-1}^{p-1}(a,b) \geq \frac{G^{2p-2}(a,b)}{L^p(a,b)}.$$

Proof. Consider the function $f : [a, b] \to (0, \infty)$, $f(t) = t^p$ with $p \in (-\infty, 0) \cup (1, \infty) \setminus \{2\}$, then f is convex on $[a, b]$ and if we apply the inequality (2.1), we get

$$\frac{1}{b-a}\left[\frac{L(a,b)-a}{L(a,b)} \cdot a^{p-1} + \frac{b-L(a,b)}{L(a,b)} \cdot b^{p-1}\right] \tag{3.10}$$

$$\geq \frac{1}{b-a}\int_a^b t^{p-2}dt \geq \frac{G^{2p-2}(a,b)}{L^p(a,b)}.$$

Since

$$\frac{1}{b-a}\int_a^b t^{p-2}dt = L_{p-1}^{p-1}(a,b),$$

then by (3.11) we get the desired result (3.9). □

Utilising Theorem 3 we can get the following inequalities for means:

Proposition 5. For any $0 < a < b$ we have

$$a^{a\frac{b-L(a,b)}{(b-a)L(a,b)}} b^{b\frac{L(a,b)-a}{(b-a)L(a,b)}} \geq I(a,b). \tag{3.11}$$

Proof. Using the inequality (2.9) for the convex function $f : [a, b] \subset (0, \infty) \to \mathbf{R}$, $f(t) = t \ln t$ we get

$$\frac{1}{b-a}\left[\frac{b-L(a,b)}{L(a,b)} a \ln a + \frac{L(a,b)-a}{L(a,b)} b \ln b\right]$$

$$\geq \frac{1}{b-a}\int_a^b \ln t dt \geq \ln(L(a,b)) \tag{3.12}$$

and since

$$\frac{1}{b-a}\int_a^b \ln t dt = \ln I(a,b)$$

and

$$\frac{b-L(a,b)}{(b-a)L(a,b)} a \ln a + \frac{L(a,b)-a}{(b-a)L(a,b)} b \ln b$$

$$= \ln\left[a^{\frac{b-L(a,b)}{(b-a)L(a,b)}a} b^{\frac{L(a,b)-a}{(b-a)L(a,b)}b}\right],$$

then by (3.12) we get

$$\ln\left[a^{\frac{b-L(a,b)}{(b-a)L(a,b)}a} b^{\frac{L(a,b)-a}{(b-a)L(a,b)}b}\right] \geq \ln I(a,b) \geq \ln(L(a,b)).$$

The second inequality is obvious and we drop it. □

Proposition 6. For any $0 < a < b$ we have

$$G(a,b) \geq a^{\frac{b-L(a,b)}{b-a}} b^{\frac{L(a,b)-a}{b-a}}. \tag{3.13}$$

Proof. Using the inequality (2.9) for the concave function $f : [a,b] \subset (0,\infty) \to \mathbf{R}$, $f(t) = \ln t$ we get

$$\frac{1}{b-a}\left[\frac{b-L(a,b)}{L(a,b)}\ln(a) + \frac{L(a,b)-a}{L(a,b)}\ln(b)\right] \leq \frac{1}{b-a}\int_a^b \frac{1}{t}\ln t dt \leq \frac{\ln(L(a,b))}{L(a,b)}. \tag{3.14}$$

However

$$\frac{1}{b-a}\int_a^b \frac{1}{t}\ln t dt = \frac{\ln G(a,b)}{L(a,b)}$$

and since

$$\frac{1}{b-a}\left[\frac{b-L(a,b)}{L(a,b)}\ln(a) + \frac{L(a,b)-a}{L(a,b)}\ln(b)\right] = \ln\left[a^{\frac{b-L(a,b)}{(b-a)L(a,b)}} b^{\frac{L(a,b)-a}{(b-a)L(a,b)}}\right],$$

then by (3.12) we get

$$\ln\left[a^{\frac{b-L(a,b)}{(b-a)L(a,b)}} b^{\frac{L(a,b)-a}{(b-a)L(a,b)}}\right] \leq \ln G(a,b)^{\frac{1}{L(a,b)}} \leq \ln\left(L(a,b)^{\frac{1}{L(a,b)}}\right),$$

i.e.

$$a^{\frac{b-L(a,b)}{(b-a)L(a,b)}} b^{\frac{L(a,b)-a}{(b-a)L(a,b)}} \leq G(a,b)^{\frac{1}{L(a,b)}} \leq L(a,b)^{\frac{1}{L(a,b)}}.$$

Taking the power $L(a,b) > 0$ we get

$$a^{\frac{b-L(a,b)}{b-a}} b^{\frac{L(a,b)-a}{b-a}} \leq G(a,b) \leq L(a,b).$$

The second inequality is obvious and we drop it. □

Finally, we have:

Proposition 7. For any $0 < a < b$ and $p \in (-\infty, 0) \cup (1, \infty)$ we have

$$\frac{1}{b-a}\left[\frac{L(a,b)-a}{L(a,b)}\cdot a^{p-1}+\frac{b-L(a,b)}{L(a,b)}\cdot b^{p-1}\right] \tag{3.15}$$
$$\geq L_{p-1}^{p-1}(a,b)\geq L^{p-1}(a,b).$$

Proof. Consider the function $f:[a,b]\to(0,\infty)$, $f(t)=t^p$ with $p\in(-\infty,0)\cup(1,\infty)$, then f is convex on $[a,b]$ and if we apply the inequality (2.9), we get

$$\frac{1}{b-a}\left[\frac{b-L(a,b)}{L(a,b)}a^p+\frac{L(a,b)-a}{L(a,b)}b^p\right] \tag{3.16}$$
$$\geq \frac{1}{b-a}\int_a^b t^{p-1}dt \geq \frac{(L(a,b))^p}{L(a,b)}=(L(a,b))^{p-1}.$$

Since

$$\frac{1}{b-a}\int_a^b t^{p-1}dt = L_p^p(a,b),$$

then by (3.16) we get the desired result (3.15). □

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