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THE NESTED BINARY CES COMPOSITE PRODUCTION FUNCTION:

CRTS with different (but constant) pair-wise elasticities of substitution among three factors

by

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ABSTRACT

The policy debate on global warming has raised the prospect of large taxes on Greenhouse pollutants leading to a very substantial rise in the price of energy. Models in which output is produced according to a technology in which capital (K), labour (L) and energy (E) are substitutable run into the difficulty of how to allow parsimoniously for the higher likely substitutability between K and E than between L and E. Nesting all three factors in a single CES aggregator function is unsatisfactory because of the constancy over pairs of factors of partial substitution elasticities. This paper is a variation on the CES theme. It presents a new composite three-input production function (based on CES and Leontief components) which allows the partial substitution elasticities between capital and labour, capital and energy, and between labour and energy, to differ but to remain individually constant.

JEL classification D2, E1.

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1. Motivation

The policy debate on global warming has raised the prospect of large taxes on Greenhouse pollutants (see, e.g., Oliveira Martins *et al.*, 1993). This is turn could lead to a very substantial rise in the price of energy. The experience of the OPEC oil price shocks suggests that large energy taxes could result in significant substitution away from energy and into capital (and to a lesser extent, labour).

The default setting of models in the ORANI (Dixon, Parmenter, Sutton and Vincent, 1982) family precludes substitution between primary factors and materials (where the latter includes energy and fuels). A simple approach towards rectifying this situation is to treat energy as an honorary primary factor so that it is included among the inputs in a CES aggregator function near the top of the production tree.

Of course this is second best. The first best (but usually infeasible) alternative is to model the production process in great detail (along the lines of activity analysis or the multisectoral CGE approximation thereto). Putting energy as an aggregate input near the top of the production structure obviates the need for the massive amounts of data required by the detailed approach. There is a serious disadvantage with putting energy along with capital and labour into a CES aggregator, though: all three pair-wise substitution elasticities have the same value, a happenstance which can have bizarre consequences. For example, with output held fixed, a large rise in the price of energy will lead to equal percentage rises in the demands for labour and capital. In jocular discourse this is referred to as the *tread-mill* effect of greenhouse abatement.

The problem addressed in this paper is how to nest capital (K), labour (L) and energy (E) within a composite or nested production structure in a way which allows the Allen-Uzawa partial substitution elasticities ν_{KL}, ν_{KE} and ν_{LE} to be chosen at will by the person calibrating the model. It is well known that a production function in which these three elasticities differ cannot be a CES function; i.e., one in which each of the three elasticities is globally constant. Whilst several functional forms allow differing ν_{ij} values, many of them

* Without implicating him in any remaining errors, we wish to thank Keith R. McLaren for critically reading a draft.

probably also involve unacceptably large variations in the ν_{ij} in response to changing relative prices. This is an unwelcome degree of flexibility since neither prior beliefs nor econometric evidence give much guidance on what sorts of variations are acceptable.

In this paper we build on two well known and simple tools, the two-factor CES production function and the Leontief function to obtain a composite (nested) production function with the following properties:

(i) constant returns to scale prevail;

(ii) the three Allen-Uzawa partial substitution elasticities can differ in virtually any desired way;

(iii) even under a doubling or more in the price of energy relative to capital and labour, the variation in each of the partial substitution elasticities is very slight under a suitable calibration of the model. Under another calibration of the model, these three elasticities can be kept absolutely constant, even when relative prices change by a large multiple.

Below we assume that labour is a composite obtained by a CRESH (Hanoch, 1971) or other aggregation over occupations, and that the {K, L, E} aggregator function under focus is nested within a Leontief function (of which the other principal arguments are likely to be material inputs).

2. Structure of the Proposed Composite Production Function

The nested production function proposed here is built from binary CES functions defined on the three factors K (capital), L (labour) and E (energy). Notionally each factor is split up into two parts; each of these then combines with one of the other two factors in a CES nest, with a total of three such nests being formed. This is depicted in Figure 1, while notation is given in Table 1. Lower-case Roman letters indicate the proportional changes in the variables denoted by the corresponding upper-case letters. Thus, for example, k_L signifies the proportional change in K_L , where the latter symbol indicates the amount of capital K assigned to the $\overline{KL}$ nest in Figure 1.

Formally the production function for the output Y of the (K, L, E) factor nest is:

$$Y = \text{Min}\left\{\frac{\overline{KL}}{C_{KL}}, \frac{\overline{KE}}{C_{KE}}, \frac{\overline{LE}}{C_{LE}}\right\}, \qquad (2.1)$$

where

$$\overline{ij} = A_{ij}[\delta_{ij} i_j^{-\rho_{ij}} + (1-\delta_{ij})\, j_i^{-\rho_{ij}}]^{-1/\rho_{ij}} \qquad (2.2)$$

$$(\overline{ij} = \overline{KL}, \overline{KE}, \overline{LE};\ i = K, L;\ j = L, E;\ i \neq j)\ .$$

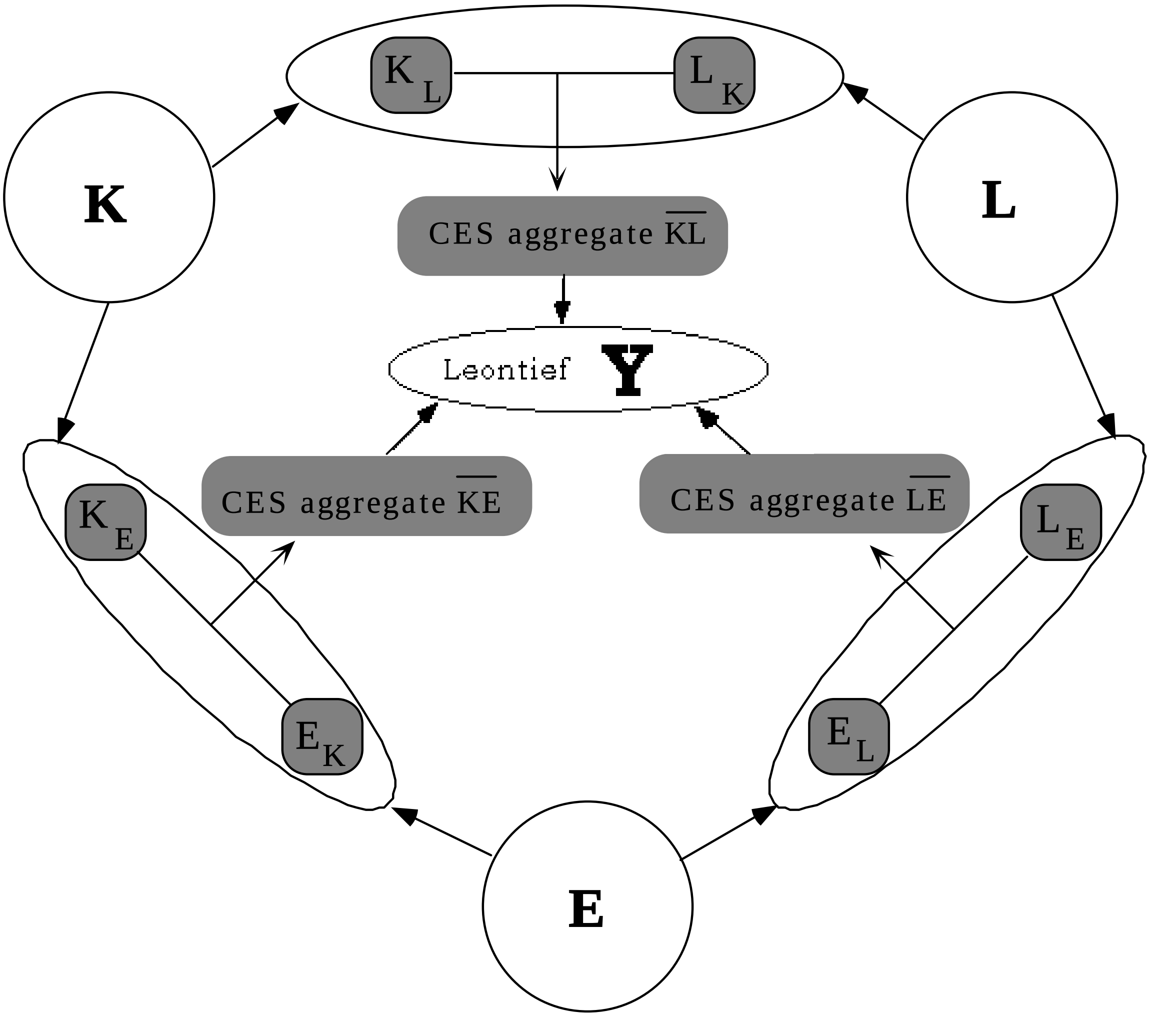

Figure 1 Triad of binary CES functions. $\overline{KL}$, $\overline{KE}$ and $\overline{LE}$ combine in fixed proportions to give the output Y of the {K, L, E} factor nest. The entities shown above against a shaded background are unobservable. Putting i_j = L_E in (2.2) above indicates the quantity of total labour usage L which is assigned to the production of the notional aggregate $\overline{LE}$ (see Figure 1.)

The CES functions (2.2) can be written schematically:

$$\overline{ij} = CES_{ij} (\alpha_{ij}\, i,\ \alpha_{ji}\, j); \qquad (i \neq j) \qquad (2.3)$$

for example,

$$\overline{LE} = CES_{LE} (\alpha_{LE}\, L, \alpha_{EL}\, E)\ . \qquad (2.4)$$

3. First-Order Conditions for Cost Minimization in Differential and in Levels Form

First-order conditions for cost-minimization are now written down; levels solutions are numbered in formats like (3.1L), while equation numbers like (3.1D) are the corresponding linearized (differential, or proportional change) forms — note that the latter refer only to the calibration in which the σ_{ij} (ij = KL, KE, LE) are absolutely constant. With all three factors commanding positive prices, a necessary condition for cost minimization is that fixed proportions among $\overline{KE}$, $\overline{KL}$ and $\overline{LE}$ be maintained; that is, that

$$Y = \left\{\frac{\overline{KL}}{C_{KL}}\right\} = \left\{\frac{\overline{KE}}{C_{KE}}\right\} = \left\{\frac{\overline{LE}}{C_{LE}}\right\}; \tag{3.1L}$$

$$y = (\overline{kl}) = (\overline{ke}) = (\overline{le}); \tag{3.1D}$$

where y and $(\overline{ke})$, etc., are the proportional changes in Y and $\overline{KE}$, etc. It is also necessary that $\overline{KE}$, $\overline{KL}$ and $\overline{LE}$ be produced at minimum cost; hence the following CES factor demand functions apply:

$$k_L = (\overline{kl}) + \sigma_{KL} S_{LK} (p_L - p_K) \tag{3.2D}$$

$$K_L = [\overline{KL}/A_{KL}]\left(\delta_{KL} + (1-\delta_{KL})\left[\frac{\delta_{KL}}{(1-\delta_{KL})}\frac{p_L}{p_K}\right]^{\rho_{KL}\sigma_{KL}}\right)^{1/\rho_{KL}} \tag{3.2L}$$

$$l_K = (\overline{kl}) + \sigma_{KL} S_{KL} (p_K - p_L) \tag{3.3D}$$

$$L_K = [\overline{KL}/A_{KL}]\left((1-\delta_{KL}) + \delta_{KL}\left[\frac{(1-\delta_{KL})}{\delta_{KL}}\frac{p_K}{p_L}\right]^{\rho_{KL}\sigma_{KL}}\right)^{1/\rho_{KL}} \tag{3.3L}$$

$$k_E = (\overline{ke}) + \sigma_{KE} S_{EK} (p_E - p_K) \tag{3.4D}$$

$$K_E = [\overline{KE}/A_{KE}]\left(\delta_{KE} + (1-\delta_{KE})\left[\frac{\delta_{KE}}{(1-\delta_{KE})}\frac{p_E}{p_K}\right]^{\rho_{KE}\sigma_{KE}}\right)^{1/\rho_{KE}} \tag{3.4L}$$

$$e_K = (\overline{ke}) + \sigma_{KE} S_{KE} (p_K - p_E) \tag{3.5D}$$

$$E_K = [\overline{KE}/A_{KE}]\left((1-\delta_{KE}) + \delta_{KE}\left[\frac{(1-\delta_{KE})}{\delta_{KE}}\frac{p_K}{p_E}\right]^{\rho_{KE}\sigma_{KE}}\right)^{1/\rho_{KE}} \tag{3.5L}$$

$$l_E = (\overline{le}) + \sigma_{LE} S_{EL} (p_E - p_L) \tag{3.6D}$$

$$L_E = [\overline{LE}/A_{LE}]\left((1-\delta_{LE}) + \delta_{LE}\left[\frac{(1-\delta_{LE})}{\delta_{LE}}\frac{p_E}{p_L}\right]^{\rho_{LE}\sigma_{LE}}\right)^{1/\rho_{LE}} \quad (3.6L)$$

$$e_L = (\overline{le}\ ) + \sigma_{LE}S_{LE}\,(p_L - p_E) \quad (3.7D)$$

$$E_L = [\overline{LE}/A_{LE}]\left((1-\delta_{LE}) + \delta_{LE}\left[\frac{(1-\delta_{LE})}{\delta_{LE}}\frac{p_L}{p_E}\right]^{\rho_{LE}\sigma_{LE}}\right) \ . \quad (3.7L)$$

The levels demands for K, L and E are found by addition of the relevant components above:

$$K = K_L + K_E \ ; \quad (3.8L)$$

$$E = E_K + E_L \ . \quad (3.10L)$$

The differential forms of (3.8L)—(3.10L) are:

$$\alpha_{KL}k_L + \alpha_{KE}k_E = k \ , \quad (3.8D)$$

$$\alpha_{LK}l_K + \alpha_{LE}l_E = l \ , \quad (3.9D)$$

$$\alpha_{EK}e_K + \alpha_{EL}e_L = e \ , \quad (3.10D)$$

Using (3.1) and the above identities, we eliminate some of the unobservables (namely, k_L, k_E, l_K, l_E, e_K, and e_L), and obtain the differential forms of the factor demand functions:

$$k = y + \alpha_{KL}\sigma_{KL}S_{LK}\,(p_L - p_K) + \alpha_{KE}\sigma_{KE}S_{EK}\,(p_E - p_K) \ ; \quad (3.11D)$$

$$l = y + \alpha_{LK}\sigma_{KL}S_{KL}\,(p_K - p_L) + \alpha_{LE}\sigma_{LE}S_{EL}\,(p_E - p_L) \ ; \quad (3.12D)$$

$$e = y + \alpha_{EK}\sigma_{KE}S_{KE}\,(p_K - p_E) + \alpha_{EL}\sigma_{LE}S_{LE}\,(p_L - p_E) \ . \quad (3.13D)$$

The corresponding factor demand functions in the levels are obtained by making the appropriate substitutions from (3.2L)—(3.7L) into (3.8L)—(3.10L).

Allen-Uzawa pair-wise substitution elasticities can be defined at two different levels: between each pair of factors making up the binary CES nests

Table 1

Notation

$\overline{LK} \equiv \overline{KL}$:	a notional CES aggregate of capital and labour
$\overline{EK} \equiv \overline{KE}$:	a notional CES aggregate of capital and energy
$\overline{EL} \equiv \overline{LE}$:	a notional CES aggregate of labour and energy
K_L :	the amount of capital assigned to the $\overline{KL}$ nest
L_K :	the amount of labour assigned to the $\overline{KL}$ nest
K_E :	the amount of capital assigned to the $\overline{KE}$ nest
E_K :	the amount of energy assigned to the $\overline{KE}$ nest
E_L :	the amount of energy assigned to the $\overline{LE}$ nest
L_E :	the amount of labour assigned to the LE nest
K :	the total use of capital — $K \equiv K_L + K_E$
L :	the total use of labour — $L \equiv L_K + L_E$
E :	the total use of energy — $E \equiv E_K + E_L$
P_j (j=K,L,E) :	the price of factor j
α_{ij} (i,j=K,L,E; i≠j) :	the proportion of factor i assigned to nest $\overline{ij}$; for example, $\alpha_{KL} = K_L/K$ is the proportion of capital assigned to nest $\overline{KL}$. ***Note that these are variables, not parameters.*** Also note: $\Sigma_{j\neq i}\alpha_{ij} \equiv 1$.
σ_{ij} :	the (micro) elasticity of substitution between i and j in the production of $\overline{ij}$ ($\overline{ij} = \overline{KL}, \overline{KE}, \overline{LE}$)
A_{ij} :	multiplicative parameter of the CES production function for $\overline{ij}$
δ_{ij} :	distribution parameter for factor i in the CES production function for $\overline{ij}$ ($\overline{ij} = \overline{KL}, \overline{KE}, \overline{LE}$). Thus $\overline{ij} = A_{ij}[\delta_{ij} i_j^{-\rho_{ij}} + (1-\delta_{ij}) j_i^{-\rho_{ij}}]^{-1/\rho_{ij}}$ where $\rho_{ij} = 1/\sigma_{ij} - 1$ ($\overline{ij} = \overline{KL}, \overline{KE}, \overline{LE}$; i = K, L; j = L, E; i ≠ j)
ν_{ij} :	the overall (macro) elasticity of substitution of factor i for factor j (i, j = K, L, E)
S_{ij} :	the share of factor i in the value of $\overline{ij}$: $S_{LK} = P_L L_K / (P_L L_K + P_K K_L)$ $= P_L \alpha_{LK} L / (P_L \alpha_{LK} L + P_K \alpha_{KL} K)$
C_{ij} :	Outer-nest input-output coefficient for $\overline{ij}$
$\mu_{ij} = A_{ij}/C_{ij}$:	Leontief-CES combined parameter (ij = KL, KE, LE)

$\overline{KL}$, $\overline{KE}$ and $\overline{LE}$; and among the three aggregate factors K, L and E. Assuming price taking behaviour (as we have done above in deriving the factor demands), the partial substitution elasticities among the aggregate factors may be defined as:

$$\nu_{ij} = \frac{\partial \ln(i)}{\partial \ln P_j} \Big/ \left\{ \frac{jP_j}{P_L L + P_K K + P_E E} \right\} \Bigg]_{\text{fixed } Y,\, P_i,\, i \neq j} ,$$

$$(i,j = K, L, E;\ i \neq j)$$

From (3.11D)—(3.13D) it follows that the connection between the two sorts of partial substitution elasticities is:

$$\nu_{KL} = \alpha_{LK}\alpha_{KL}\sigma_{KL}\,(P_L L + P_K K + P_E E)/\,(P_L\alpha_{LK}L + P_K\alpha_{KL}K)\,, \tag{3.15}$$

$$\nu_{KE} = \alpha_{EK}\alpha_{KE}\sigma_{KE}\,(P_L L + P_K K + P_E E)/\,(P_E\alpha_{EK}E + P_K\alpha_{KE}K), \tag{3.16}$$

$$\nu_{LE} = \alpha_{EL}\alpha_{LE}\sigma_{LE}\,(P_L L + P_K K + P_E E)/\,(P_E\alpha_{EL}E + P_L\alpha_{LE}L)\,, \tag{3.17}$$

where the σ_{ij} (ij = KL, KE, LE) are the partial substitution elasticities applying to the binary nests $\overline{KL}$, $\overline{KE}$ and $\overline{LE}$. Note that the allocation shares (α_{LK}, α_{EK} etc.) are variables, rather than parameters.

Equations (3.15)—(3.17) reveal two possibilities for calibration of the model: either the micro elasticities (the σs) can be treated as parameters, implying variation in the macro elasticities (the νs); alternatively (and this requires careful interpretation), the macro elasticities can be treated as parameters, in which case the micro elasticities must be free to vary.

The choice between these alternatives depends on how one regards the binary nests. If they are taken literally as the preferred technological specification, then the constancy of the σs will be chosen. This option will guarantee global regularity.

If one regards the available empirical evidence as being conveniently summarized by the νs, however, one will not feel squeamish about allowing the endogenization of the σs as variables while keeping the νs constant at their initial values. In this case the binary nests are regarded as a convenient device for preserving the regularity of the production system (which they will do, provided no σ is driven to a negative value[1]); that is, the story about the latent variables K_L, L_E, etc., has only an 'as if' interpretation.

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[1] With each micro production function (those for $\overline{KL}$, $\overline{KE}$ and $\overline{LE}$) having just two inputs, each cross substitution elasticity must be non-negative to ensure regularity. A more general statement of the required curvature conditions is given in Allen (1938), p.505.

Can a σ_{ij} be driven to a negative value by a valid choice of the exogenous variables (Y, P_K, P_L and P_E)? We will show that provided all the ν_{ij}s have been set to positive numbers, the answer is "No!". We will do this by *reductio ad absurdum*:

> Suppose some particular σ_{ij} were negative. Even so, the Leontief production functions (2.3) and the CES demand functions (3.2L)—(3.7L) guarantee the positivity of $\overline{KL}$, $\overline{KE}$, $\overline{LE}$, K_L, K_E, L_K, L_E, E_K and E_L (and hence of every α_{ij}), as well as of aggregate K, L and E. Now solve the relevant equation among (3.15)—(3.17) for the σ_{ij} that was assumed to be negative. Provided the corresponding ν_{ij} has been set to a positive value, the solution so obtained is positive, contradicting the initial supposition. Hence with every ν_{ij} positive, no σ_{ij} can be driven to a negative value by *any* set of relative prices, and the global regularity of the underlying micro system is preserved.

4. Illustrative Hypothetical Partial Equilibrium Simulation Experiment

A simulation experiment was conducted with the aim of illustrating the stability of the pair-wise macro substitution elasticities for the nested production function described above, and to show that the model can be calibrated in such a way that variations in the factor mix under a large increase in the price of energy seem plausible.

The experiment was conducted over 252 sub-intervals with the price of energy doubling over the entire interval. This allows results to be plotted as a function of the ratio of the price of energy to the prices of the other factors.

The wage and rental rates P_L and P_K were set exogenously at one throughout the simulations. The initial value of the price of energy P_E was 1. The initial values of K, L, E and Y were set at 3, 6, 1 and 10, which would reflect a stylized economy in which capital accounted for one third of value added, and in which the cost of energy was around one tenth of total value added. The initial (arbitrarily selected) values of the macro Allen-Uzawa substitution elasticities were:

$$\nu_{KL} = 1.28 \; ; \quad \nu_{KE} = 0.5 \; ; \quad \nu_{LE} = 0\,.1.$$

The value of ν_{KL} is the default long-run value of capital-labour substituion elasticities in the ORANI model, while the remaining νs are an educated guess.

The α_{ij}s were set initially at the following arbitrary values:

$$\alpha_{KL} = \; 0.75 \; ; \quad \alpha_{KE} = \; 0.25 \; ;$$

$$\alpha_{LK} = \; 0.90 \; ; \quad \alpha_{LE} = \; 0.10 \; ;$$

$$\alpha_{EK} = 0\,.75 \; ; \quad \alpha_{EL} = 0.25 \; .$$

The constant substitution elasticities within the binary CES nests in the base case are determined from (3.15) — (3.17). The resulting values, which are held fixed throughout the initial simulation, are:

$$\sigma_{KL} = 1.45, \quad \sigma_{KE} = 0.40, \quad \sigma_{LE} = 0.34 \ .$$

In each sub-interval, factor demands and the values of the α_{ij}s were calculated in *MS Excel 7* using (3.2L)–(3.7L). The simulation was repeated treating the macro substitution elasticities (ν_{KL}, ν_{KE} and ν_{LE}) as parameters which were assigned the values set out above.

Some of the simulation results are shown in Figures 2a—2f and in Table 2. Results are given under both choices of parameters:

- micro partial substitution elasticities, σ_{KL}, σ_{KE} and σ_{LE} , held constant;
- macro partial substitution elasticities, ν_{KL}, ν_{KE} and ν_{LE} , held constant;

To enable the charts to be read more easily, each is repeated three times: once when the maximum energy price rise envisaged is a doubling; once when it is a quintupling; and once when a hundred-fold increase is contemplated.

Relative to a non-nested 3-factor CES function, the production function proposed in this paper is relatively heavily endowed with parameters. Whereas the 3-factor CES function has four parameters (one substitution elasticity, one parameter to convert from units of input into units of output, and two independent distribution parameters), the function (2.1)–(2.2) has nine parameters, as follows:

Parameter	*Number*
$\mu_{ij} = A_{ij} \ / C_{ij}$	3
σ_{ij}	3
δ_{ij}	3
total	9

The ρ_{ij} s are related to the σ_{ij} s in the usual CES way ($\rho_{ij} = 1/\sigma_{ij} - 1$); the δ_{ij}s and data on prices can be used to work out the α_{ij} s (or *vice-versa*). To do this we note[2] that the share of factor i in the value of $\overline{ij}$ implied by cost minimization is

$$S_{ij} = \frac{\delta_{ij}^{\ \sigma_{ij}} P_i^{\ \rho_{ij}\ \sigma_{ij}}}{\delta_{ij}^{\ \sigma_{ij}} P_i^{\ \rho_{ij}\ \sigma_{ij}} + (1-\delta_{ij})^{\sigma_{ij}} P_j^{\ \rho_{ij}\ \sigma_{ij}}} \qquad (4.1)$$

(i,j = K, L, E; i≠j) .

[2] See, e.g., Dixon *et al.* (1980), p. 298.

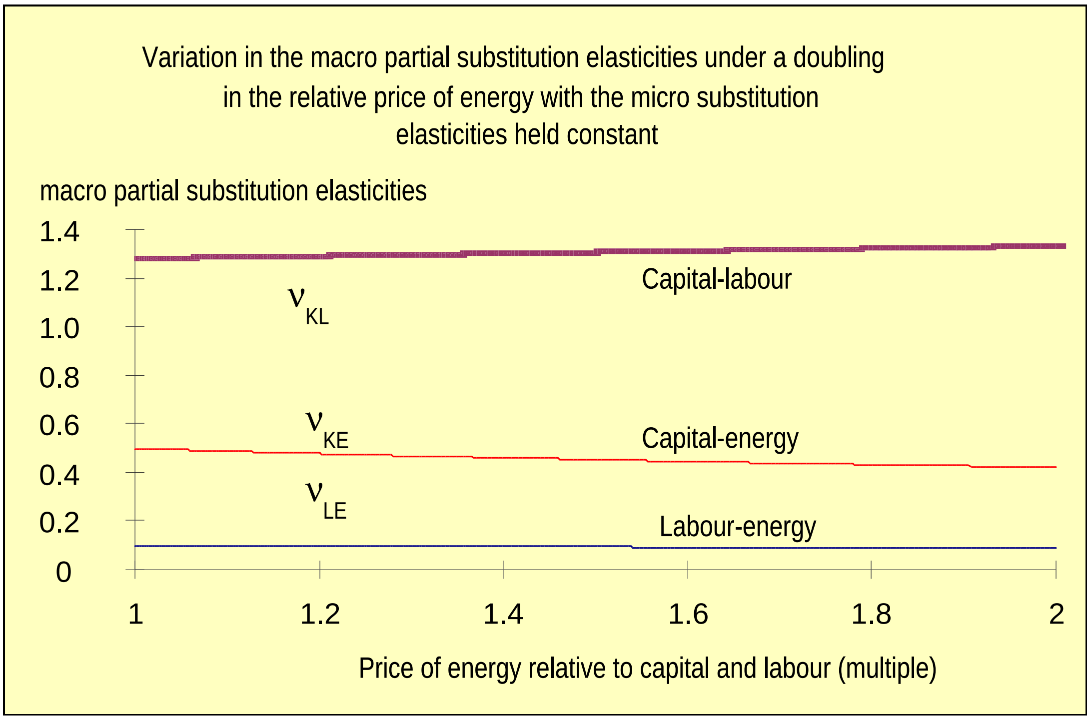


Figure 2a

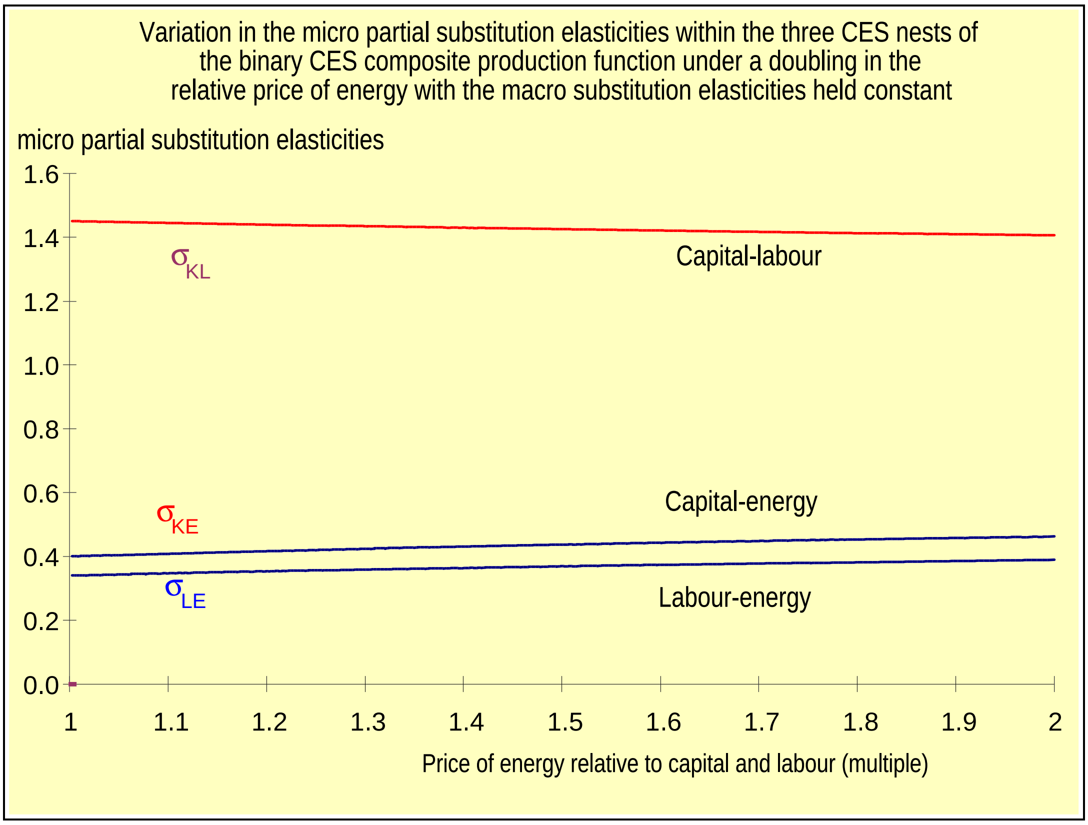


Figure 2b

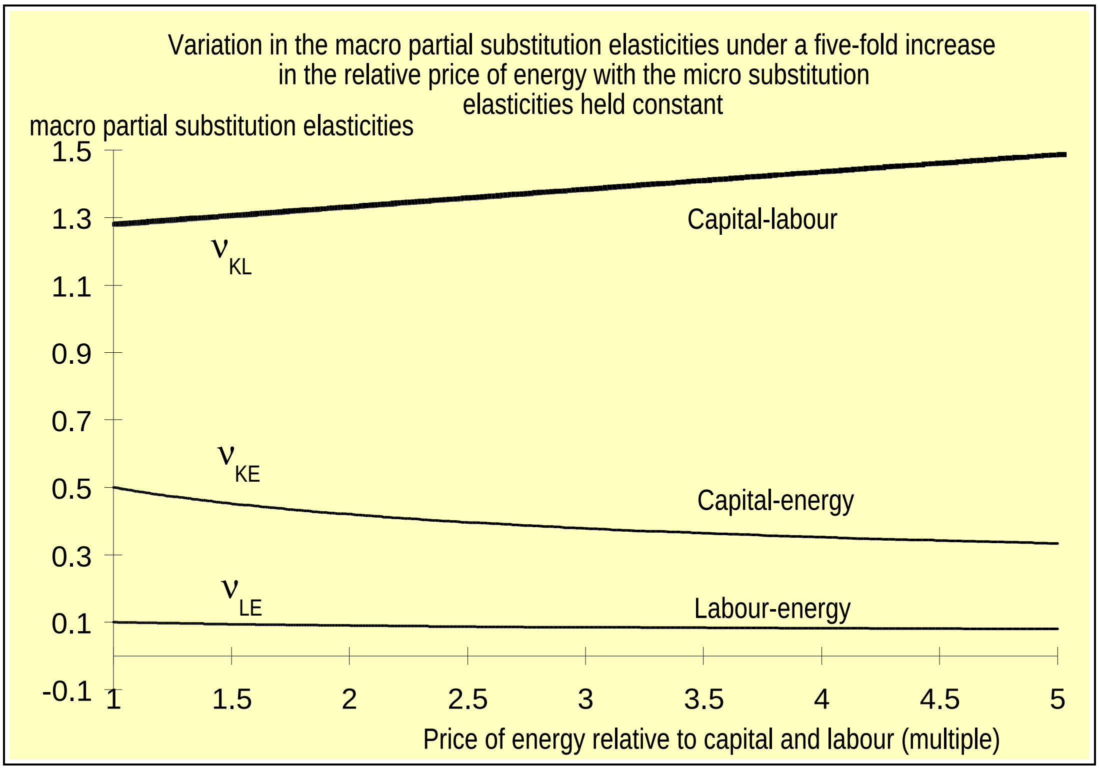
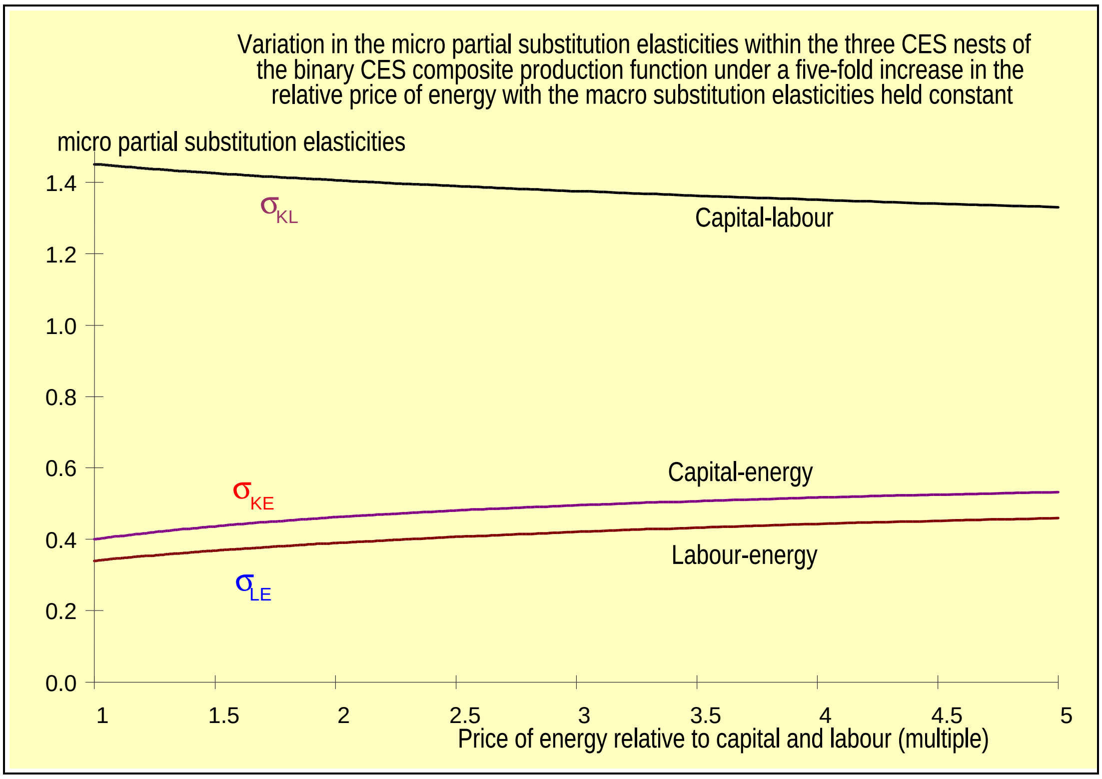


Figure 2c



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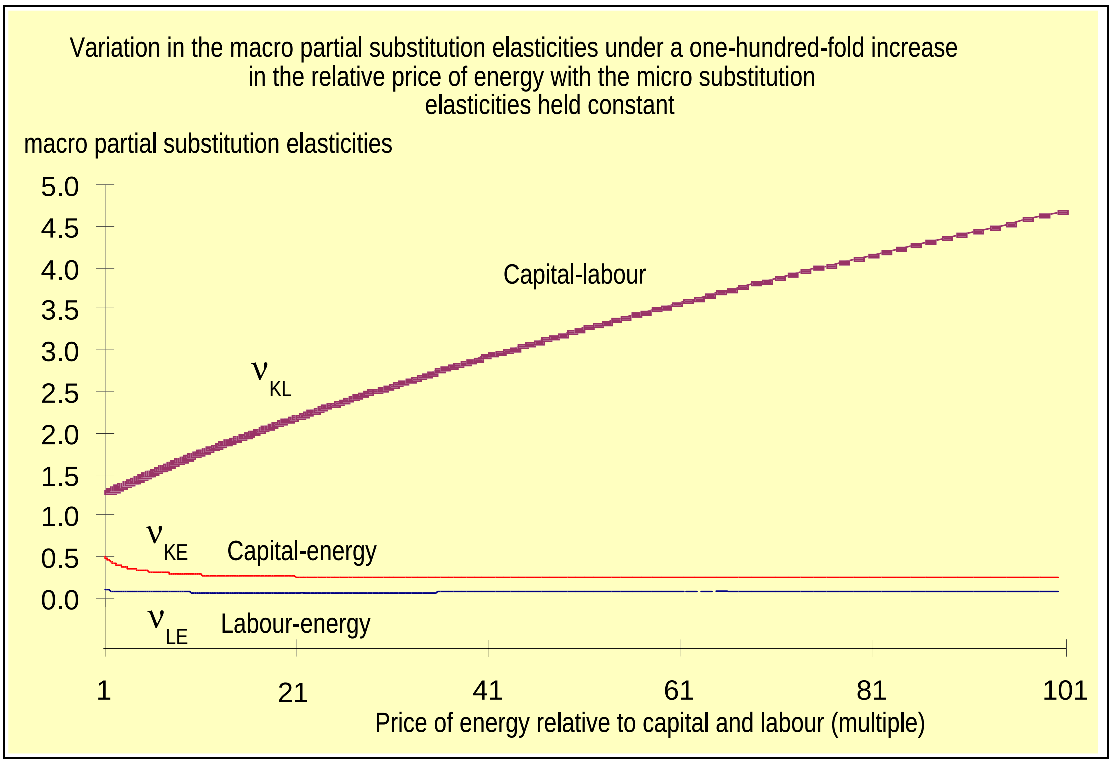

Figure 2e

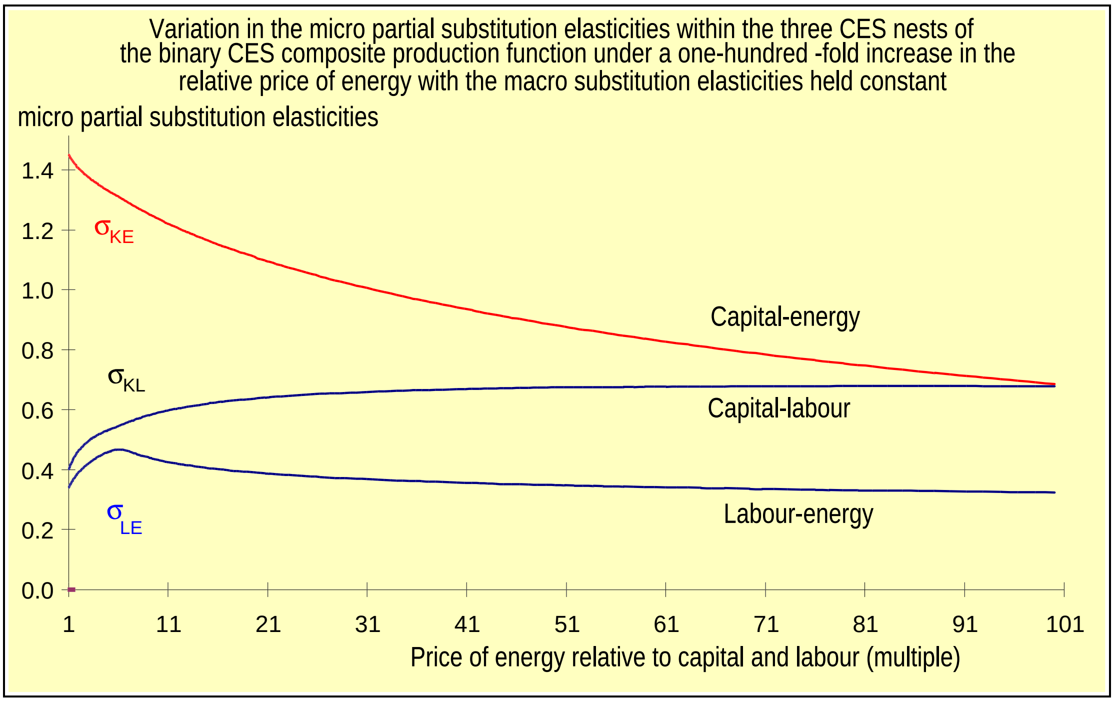

Figure 2f

Table 2

Simulation Results

	micro substitution elasticities (σ_{ij}) held constant				macro substitution elasticities (ν_{ij}) held constant		
	P_E as a multple of P_L or P_K						
	1	2	5	100	2	5	100
L	$\overline{6}$	6.05	6.15	7.01	6.03	6.09	6.21
K	$\overline{3}$	3.12	3.36	5.33	3.19	3.62	9.24
E	$\overline{1}$	0.88	0.77	0.63	0.83	0.64	0.42
Y	$\overline{10}$	10	10	10	10	10	10
ν_{KL}	$\overline{1.28}$	1.33	1.49	4.66	$\overline{1.28}$	$\overline{1.28}$	$\overline{1.28}$
ν_{KE}	$\overline{.5}$	0.42	0.33	0.26	$\overline{.5}$	$\overline{.5}$	$\overline{.5}$
ν_{LE}	$\overline{.1}$	0.09	0.08	0.08	$\overline{.1}$	$\overline{.1}$	$\overline{.1}$
σ_{KL}	$\overline{1.45}$	$\overline{1.45}$	$\overline{1.45}$	$\overline{1.45}$	1.45	1.45	1.45
σ_{KE}	$\overline{0.40}$	$\overline{0.40}$	$\overline{0.40}$	$\overline{0.40}$	0.40	0.40	0.40
σ_{LE}	$\overline{0.34}$	$\overline{0.34}$	$\overline{0.34}$	$\overline{0.34}$	0.34	0.34	0.34
α_{LK}	$\overline{.9}$	0.89	0.88	0.77	0.89	0.87	0.75
α_{LE}	$\overline{.1}$	0.11	0.12	0.23	0.11	0.13	0.25
α_{KL}	$\overline{.75}$	0.72	0.67	0.42	0.72	0.66	0.34
α_{KE}	$\overline{.25}$	0.28	0.33	0.58	0.28	0.34	0.66
α_{EL}	$\overline{.25}$	0.24	0.24	0.22	0.22	0.18	0.36
α_{EK}	$\overline{.75}$	0.76	0.76	0.78	0.78	0.82	0.64
P_E	$\overline{1}$	2	5	100	2	5	100

Note: $\overline{\quad}$ indicates initial setting. The prices of energy and the rental rate on capital are held fixed at unity.

By definition, however, we also have that

$$S_{ij} = \frac{P_i \alpha_{ij} i}{\{ P_i \alpha_{ij} i + P_j \alpha_{ji} j\}} \quad . \qquad (4.2)$$

(i,j = K, L, E; i≠j)

Hence at any given set of factor prices we can infer the values of the δ_{ij}s from the α_{ij}s or *vice-versa*. For example, if the α_{ij}s are given (as in the starting point for the hypothetical simulation above), we would start by computing S_{ij}. We would then calculate the δ_{ij}s as

$$\delta_{ij} = \left\{ 1 + \left(\frac{P_i}{P_j}\right)^{\rho_{ij}} \left[\frac{1 - S_{ij}}{S_{ij}}\right]^{1/\sigma_{ij}} \right\}^{-1} \quad . \qquad (4.3)$$

The μ_{ij}s can then be calculated using the given α_{ij}s and the bench-mark data on Y, K, L, E, P_K, P_L and P_E. If values are 'known' either for the σ_{ij}s or for the ν_{ij}s at the bench-mark data setting, then equations (3.15) through (3.17) can be used to recover the unknown substitution elasticities (ν_{ij}s or σ_{ij}s as the case may be).

In our initial calibration above, we started with α_{ij}s and ν_{ij}s ; equally, we could have started with δ_{ij}s and σ_{ij}s.

5. Concluding Remarks

The salient lack of flexibility of the multi-factor CES production function is the constancy over pairs of factors of its partial substitution elasticities. Above we have shown how CES and Leontief functions can be used to build a composite 3-factor production function in which all three Allen-Uzawa partial substitution elasticities can differ from one another while individually remaining constant. The composite production function is globally regular, and is a suitable vehicle to encapsulate prior intuition and/or empirical evidence on the ease of substitution that applies between the members of different pairs of factors. The above exercise was motivated by the need to allow higher substitutability between energy and capital than between labour and energy, but the tool is generic.

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