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RESEARCH

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Norm inequalities of Čebyšev type for power series in Banach algebras

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Abstract

Let $f(\lambda) = \sum_{n=0}^{\infty} \alpha_n \lambda^n$ be a function defined by power series with complex coefficients and convergent on the open disk $D(0, R) \subset \mathbb{C}$, $R > 0$ and $x, y \in \mathcal{B}$, a Banach algebra, with $xy = yx$. In this paper we establish some upper bounds for the norm of the Čebyšev type difference $f(\lambda)f(\lambda xy) - f(\lambda x)f(\lambda y)$, provided that the complex number λ and the vectors $x, y \in \mathcal{B}$ are such that the series in the above expression are convergent. Applications for some fundamental functions such as the exponential function and the resolvent function are provided as well.

MSC: 47A63; 47A99

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1 Introduction

For two Lebesgue integrable functions $f, g : [a, b] \rightarrow \mathbb{R}$, consider the Čebyšev functional:

$$C(f, g) := \frac{1}{b-a} \int_a^b f(t)g(t) dt - \frac{1}{(b-a)^2} \int_a^b f(t) dt \int_a^b g(t) dt. \quad (1.1)$$

In 1935, Grüss [1] showed that

$$|C(f, g)| \leq \frac{1}{4}(M-m)(N-n), \quad (1.2)$$

provided that there exist real numbers m, M, n, N such that

$$m \leq f(t) \leq M \quad \text{and} \quad n \leq g(t) \leq N \quad \text{for a.e. } t \in [a, b]. \quad (1.3)$$

The constant $\frac{1}{4}$ is best possible in (1.1) in the sense that it cannot be replaced by a smaller quantity.

Another, however, less known result, even though it was obtained by Čebyšev in 1882 [2], states that

$$|C(f, g)| \leq \frac{1}{12} \|f'\|_{\infty} \|g'\|_{\infty} (b-a)^2, \quad (1.4)$$

provided that f', g' exist and are continuous on $[a, b]$ and $\|f'\|_{\infty} = \sup_{t \in [a, b]} |f'(t)|$. The constant $\frac{1}{12}$ cannot be improved in the general case.

The Čebyšev inequality (1.4) also holds if $f, g : [a, b] \rightarrow \mathbb{R}$ are assumed to be absolutely continuous and $f', g' \in L_\infty[a, b]$, while $\|f'\|_\infty = \text{ess sup}_{t \in [a, b]} |f'(t)|$.

A mixture between Grüss' result (1.2) and Čebyšev's one (1.4) is the following inequality obtained by Ostrowski in 1970 [3]:

$$|C(f, g)| \leq \frac{1}{8}(b-a)(M-m)\|g'\|_\infty, \tag{1.5}$$

provided that f is *Lebesgue integrable* and satisfies (1.3), while g is absolutely continuous and $g' \in L_\infty[a, b]$. The constant $\frac{1}{8}$ is best possible in (1.5).

The case of *Euclidean norms* of the derivative was considered by Lupaş in [4] in which he proved that

$$|C(f, g)| \leq \frac{1}{\pi^2} \|f'\|_2 \|g'\|_2 (b-a), \tag{1.6}$$

provided that f, g are absolutely continuous and $f', g' \in L_2[a, b]$. The constant $\frac{1}{\pi^2}$ is the best possible.

Recently, Cerone and Dragomir [5] have proved the following results:

$$|C(f, g)| \leq \inf_{\gamma \in \mathbb{R}} \|g - \gamma\|_q \cdot \frac{1}{b-a} \left(\int_a^b \left| f(t) - \frac{1}{b-a} \int_a^b f(s) ds \right|^p dt \right)^{\frac{1}{p}}, \tag{1.7}$$

where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$ or $p = 1$ and $q = \infty$, and

$$|C(f, g)| \leq \inf_{\gamma \in \mathbb{R}} \|g - \gamma\|_1 \cdot \frac{1}{b-a} \text{ess sup}_{t \in [a, b]} \left| f(t) - \frac{1}{b-a} \int_a^b f(s) ds \right|, \tag{1.8}$$

provided that $f \in L_p[a, b]$ and $g \in L_q[a, b]$ ($p > 1, \frac{1}{p} + \frac{1}{q} = 1; p = 1, q = \infty$ or $p = \infty, q = 1$).

Notice that for $q = \infty, p = 1$ in (1.7) we obtain

$$\begin{aligned} |C(f, g)| &\leq \inf_{\gamma \in \mathbb{R}} \|g - \gamma\|_\infty \cdot \frac{1}{b-a} \int_a^b \left| f(t) - \frac{1}{b-a} \int_a^b f(s) ds \right| dt \\ &\leq \|g\|_\infty \cdot \frac{1}{b-a} \int_a^b \left| f(t) - \frac{1}{b-a} \int_a^b f(s) ds \right| dt \end{aligned} \tag{1.9}$$

and if g satisfies (1.3), then

$$\begin{aligned} |C(f, g)| &\leq \inf_{\gamma \in \mathbb{R}} \|g - \gamma\|_\infty \cdot \frac{1}{b-a} \int_a^b \left| f(t) - \frac{1}{b-a} \int_a^b f(s) ds \right| dt \\ &\leq \left\| g - \frac{n+N}{2} \right\|_\infty \cdot \frac{1}{b-a} \int_a^b \left| f(t) - \frac{1}{b-a} \int_a^b f(s) ds \right| dt \\ &\leq \frac{1}{2}(N-n) \cdot \frac{1}{b-a} \int_a^b \left| f(t) - \frac{1}{b-a} \int_a^b f(s) ds \right| dt. \end{aligned} \tag{1.10}$$

The inequality between the first and the last term in (1.10) has been obtained by Cheng and Sun in [6]. However, the sharpness of the constant $\frac{1}{2}$, a generalization for the abstract Lebesgue integral, and the discrete version of it have been obtained in [7].

For other recent results on the Grüss inequality, see [8–22], and the references therein. In order to consider a Čebyšev type functional for functions of vectors in Banach algebras, we need some preliminary definitions and results as follows.

2 Some facts on Banach algebras

Let \mathcal{B} be an algebra. An algebra norm on \mathcal{B} is a map $\|\cdot\| : \mathcal{B} \rightarrow [0, \infty)$ such that $(\mathcal{B}, \|\cdot\|)$ is a normed space, and, further

$$\|ab\| \leq \|a\|\|b\|$$

for any $a, b \in \mathcal{B}$. The normed algebra $(\mathcal{B}, \|\cdot\|)$ is a Banach algebra if $\|\cdot\|$ is a complete norm.

We assume that the Banach algebra is unital, this means that \mathcal{B} has an identity 1 and that $\|1\| = 1$.

Let \mathcal{B} be a unital algebra. An element $a \in \mathcal{B}$ is invertible if there exists an element $b \in \mathcal{B}$ with $ab = ba = 1$. The element b is unique; it is called the inverse of a and written a^{-1} or $\frac{1}{a}$. The set of invertible elements of \mathcal{B} is denoted by $\text{Inv } \mathcal{B}$. If $a, b \in \text{Inv } \mathcal{B}$ then $ab \in \text{Inv } \mathcal{B}$ and $(ab)^{-1} = b^{-1}a^{-1}$.

For a unital Banach algebra we also have:

- (i) if $a \in \mathcal{B}$ and $\lim_{n \rightarrow \infty} \|a^n\|^{1/n} < 1$, then $1 - a \in \text{Inv } \mathcal{B}$;
- (ii) $\{a \in \mathcal{B} : \|1 - b\| < 1\} \subset \text{Inv } \mathcal{B}$;
- (iii) $\text{Inv } \mathcal{B}$ is an open subset of \mathcal{B} ;
- (iv) the map $\text{Inv } \mathcal{B} \ni a \mapsto a^{-1} \in \text{Inv } \mathcal{B}$ is continuous.

For simplicity, we denote $\lambda 1$, where $\lambda \in \mathbb{C}$ and 1 is the identity of \mathcal{B} , by λ . The resolvent set of $a \in \mathcal{B}$ is defined by

$$\rho(a) := \{\lambda \in \mathbb{C} : \lambda - a \in \text{Inv } \mathcal{B}\};$$

the spectrum of a is $\sigma(a)$, the complement of $\rho(a)$ in \mathbb{C} , and the resolvent function of a is $R_a : \rho(a) \rightarrow \text{Inv } \mathcal{B}$, $R_a(\lambda) := (\lambda - a)^{-1}$. For each $\lambda, \gamma \in \rho(a)$ we have the identity

$$R_a(\gamma) - R_a(\lambda) = (\lambda - \gamma)R_a(\lambda)R_a(\gamma).$$

We also have $\sigma(a) \subset \{\lambda \in \mathbb{C} : |\lambda| \leq \|a\|\}$. The spectral radius of a is defined as $v(a) = \sup\{|\lambda| : \lambda \in \sigma(a)\}$.

If a, b are commuting elements in \mathcal{B} , i.e. $ab = ba$, then

$$v(ab) \leq v(a)v(b) \quad \text{and} \quad v(a + b) \leq v(a) + v(b).$$

Let \mathcal{B} a unital Banach algebra and $a \in \mathcal{B}$. Then

- (i) the resolvent set $\rho(a)$ is open in \mathbb{C} ;
- (ii) for any bounded linear functionals $\lambda : \mathcal{B} \rightarrow \mathbb{C}$, the function $\lambda \circ R_a$ is analytic on $\rho(a)$;
- (iii) the spectrum $\sigma(a)$ is compact and nonempty in \mathbb{C} ;
- (iv) for each $n \in \mathbb{N}$ and $r > v(a)$, we have

$$a^n = \frac{1}{2\pi i} \int_{|\xi|=r} \xi^n (\xi - a)^{-1} d\xi;$$

- (v) we have $v(a) = \lim_{n \rightarrow \infty} \|a^n\|^{1/n}$.

Let f be an analytic functions on the open disk $D(0, R)$ given by the *power series* $f(\lambda) := \sum_{j=0}^{\infty} \alpha_j \lambda^j$ ($|\lambda| < R$). If $\nu(a) < R$, then the series $\sum_{j=0}^{\infty} \alpha_j a^j$ converges in the Banach algebra \mathcal{B} because $\sum_{j=0}^{\infty} |\alpha_j| \|a^j\| < \infty$, and we can define $f(a)$ to be its sum. Clearly $f(a)$ is well defined and there are many examples of important functions on a Banach algebra \mathcal{B} that can be constructed in this way. For instance, the *exponential map* on \mathcal{B} denoted \exp and defined as

$$\exp a := \sum_{j=0}^{\infty} \frac{1}{j!} a^j \quad \text{for each } a \in \mathcal{B}.$$

If \mathcal{B} is not commutative, then many of the familiar properties of the exponential function from the scalar case do not hold. The following key formula is valid, however, with the additional hypothesis of commutativity for a and b from \mathcal{B} :

$$\exp(a + b) = \exp(a) \exp(b).$$

In a general Banach algebra \mathcal{B} it is difficult to determine the elements in the range of the exponential map $\exp(\mathcal{B})$, *i.e.* the element which have a '*logarithm*'. However, it is easy to see that if a is an element in \mathcal{B} such that $\|1 - a\| < 1$, then a is in $\exp(\mathcal{B})$. That follows from the fact that if we set

$$b = - \sum_{n=1}^{\infty} \frac{1}{n} (1 - a)^n,$$

then the series converges absolutely and, as in the scalar case, substituting this series into the series expansion for $\exp(b)$ yields $\exp(b) = a$.

It is well known that if x and y are commuting, *i.e.* $xy = yx$, then the exponential function satisfies the property

$$\exp(x) \exp(y) = \exp(y) \exp(x) = \exp(x + y).$$

Also, if x is invertible and $a, b \in \mathbb{R}$ with $a < b$ then

$$\int_a^b \exp(tx) dt = x^{-1} [\exp(bx) - \exp(ax)].$$

Moreover, if x and y are commuting and $y - x$ is invertible, then

$$\begin{aligned} \int_0^1 \exp((1-s)x + sy) ds &= \int_0^1 \exp(s(y-x)) \exp(x) ds \\ &= \left(\int_0^1 \exp(s(y-x)) ds \right) \exp(x) \\ &= (y-x)^{-1} [\exp(y-x) - I] \exp(x) \\ &= (y-x)^{-1} [\exp(y) - \exp(x)]. \end{aligned}$$

Let $f(\lambda) = \sum_{n=0}^{\infty} \alpha_n \lambda^n$ be a function defined by power series with complex coefficients and convergent on the open disk $D(0, R) \subset \mathbb{C}$, $R > 0$ and $x, y \in \mathcal{B}$ with $xy = yx$. In this paper

we establish some upper bounds for the norm of the Čebyšev type difference

$$f(\lambda)f(\lambda xy) - f(\lambda x)f(\lambda y) \tag{2.1}$$

provided that the complex number λ and the vectors $x, y \in \mathcal{B}$ are such that the series in (2.1) are convergent. Applications for some fundamental functions such as the exponential function and the resolvent function are provided as well.

Inequalities for functions of operators in Hilbert spaces may be found in [23–26], the recent monographs [27–29], and the references therein.

3 The results

We denote by \mathbb{C} the set of all complex numbers. Let α_n be nonzero complex numbers and let

$$R := \frac{1}{\limsup |\alpha_n|^{\frac{1}{n}}}.$$

Clearly $0 \leq R \leq \infty$, but we consider only the case $0 < R \leq \infty$.

Denote by

$$D(0, R) = \begin{cases} \{z \in \mathbb{C} : |z| < R\}, & \text{if } R < \infty, \\ \mathbb{C}, & \text{if } R = \infty, \end{cases}$$

consider the functions

$$\lambda \mapsto f(\lambda) : D(0, R) \rightarrow \mathbb{C}, \quad f(\lambda) := \sum_{n=0}^{\infty} \alpha_n \lambda^n$$

and

$$\lambda \mapsto f_A(\lambda) : D(0, R) \rightarrow \mathbb{C}, \quad f_A(\lambda) := \sum_{n=0}^{\infty} |\alpha_n| \lambda^n.$$

Let \mathcal{B} be a unital Banach algebra and 1 its unity. Denote by

$$B(0, R) = \begin{cases} \{x \in \mathcal{B} : \|x\| < R\}, & \text{if } R < \infty, \\ \mathcal{B}, & \text{if } R = \infty. \end{cases}$$

We associate to f the map

$$x \mapsto \tilde{f}(x) : B(0, R) \rightarrow \mathcal{B}, \quad \tilde{f}(x) := \sum_{n=0}^{\infty} \alpha_n x^n.$$

Obviously, \tilde{f} is correctly defined because the series $\sum_{n=0}^{\infty} \alpha_n x^n$ is absolutely convergent, since $\sum_{n=0}^{\infty} \|\alpha_n x^n\| \leq \sum_{n=0}^{\infty} |\alpha_n| \|x\|^n$.

In addition, we assume that $s_2 := \sum_{n=0}^{\infty} n^2 |\alpha_n| < \infty$. Let $s_0 := \sum_{n=0}^{\infty} |\alpha_n| < \infty$ and $s_1 := \sum_{n=0}^{\infty} n |\alpha_n| < \infty$.

With the above assumptions we have the following.

Theorem 1 *Let $\lambda \in \mathbb{C}$ such that $\max\{|\lambda|, |\lambda|^2\} < R < \infty$ and let $x, y \in \mathcal{B}$ with $\|x\|, \|y\| \leq 1$, and $xy = yx$. Then:*

(i) *We have*

$$\begin{aligned} & \| \tilde{f}(\lambda \cdot 1) \tilde{f}(\lambda xy) - \tilde{f}(\lambda x) \tilde{f}(\lambda y) \| \\ & \leq \sqrt{2} \psi \min\{\|x - 1\|, \|y - 1\|\} f_A(|\lambda|^2) \end{aligned} \tag{3.1}$$

where

$$\psi^2 := s_0 s_2 - s_1^2. \tag{3.2}$$

(ii) *We also have*

$$\begin{aligned} & \| \tilde{f}(\lambda \cdot 1) \tilde{f}(\lambda xy) - \tilde{f}(\lambda x) \tilde{f}(\lambda y) \| \\ & \leq \sqrt{2} \min\{\|x - 1\|, \|y - 1\|\} f_A(|\lambda|) \\ & \quad \times \{ f_A(|\lambda|) [|\lambda| f'_A(|\lambda|) + |\lambda|^2 f''_A(|\lambda|)] - [|\lambda| f'_A(|\lambda|)]^2 \}^{1/2}. \end{aligned} \tag{3.3}$$

Proof For $m \geq 1$ and since $xy = yx$ we have

$$\begin{aligned} & \sum_{n=0}^m \sum_{j=0}^m \alpha_n \alpha_j \lambda^n \lambda^j (x^n - x^j) y^n \\ & = \sum_{n=0}^m \sum_{j=0}^m \alpha_n \alpha_j \lambda^n \lambda^j x^n y^n - \sum_{n=0}^m \sum_{j=0}^m \alpha_n \alpha_j \lambda^n \lambda^j x^j y^n \\ & = \sum_{j=0}^m \alpha_j \lambda^j \sum_{n=0}^m \alpha_n \lambda^n x^n y^n - \sum_{j=0}^m \alpha_j \lambda^j x^j \sum_{n=0}^m \alpha_n \lambda^n y^n \\ & = \sum_{j=0}^m \alpha_j \lambda^j \sum_{n=0}^m \alpha_n \lambda^n (xy)^n - \sum_{j=0}^m \alpha_j \lambda^j x^j \sum_{n=0}^m \alpha_n \lambda^n y^n \end{aligned} \tag{3.4}$$

for any $\lambda \in \mathbb{C}$.

Taking the norm in (3.4) we have

$$\begin{aligned} & \left\| \sum_{j=0}^m \alpha_j \lambda^j \sum_{n=0}^m \alpha_n \lambda^n (xy)^n - \sum_{j=0}^m \alpha_j \lambda^j x^j \sum_{n=0}^m \alpha_n \lambda^n y^n \right\| \\ & \leq \sum_{n=0}^m \sum_{j=0}^m |\alpha_n| |\alpha_j| |\lambda|^n |\lambda|^j \| (x^n - x^j) y^n \| \\ & \leq \sum_{n=0}^m \sum_{j=0}^m |\alpha_n| |\alpha_j| |\lambda|^n |\lambda|^j \| x^n - x^j \| \| y^n \| \\ & \leq \sum_{n=0}^m \sum_{j=0}^m |\alpha_n| |\alpha_j| |\lambda|^n |\lambda|^j \| x^n - x^j \| \| y \| \| y \| \end{aligned}$$

$$\begin{aligned} &\leq \sum_{n=0}^m \sum_{j=0}^m |\alpha_n| |\alpha_j| |\lambda|^n |\lambda|^j \|x^n - x^j\| \\ &= 2 \sum_{0 \leq j < n \leq m} |\alpha_n| |\alpha_j| |\lambda|^n |\lambda|^j \|x^n - x^j\|, \end{aligned} \tag{3.5}$$

for any $\lambda \in \mathbb{C}$ and $m \geq 1$.

Observe that

$$\begin{aligned} L &:= \sum_{0 \leq j < n \leq m} |\alpha_n| |\alpha_j| |\lambda|^n |\lambda|^j \|x^n - x^j\| \\ &= \sum_{0 \leq j < n \leq m} |\alpha_n| |\alpha_j| |\lambda|^n |\lambda|^j \left\| \sum_{\ell=j}^{n-1} (x^{\ell+1} - x^\ell) \right\| \\ &= \sum_{0 \leq j < n \leq m} |\alpha_n| |\alpha_j| |\lambda|^n |\lambda|^j \left\| \sum_{\ell=j}^{n-1} x^\ell (x - 1) \right\| \\ &\leq \|x - 1\| \sum_{0 \leq j < n \leq m} |\alpha_n| |\alpha_j| |\lambda|^n |\lambda|^j \sum_{\ell=j}^{n-1} \|x\|^\ell \end{aligned} \tag{3.6}$$

for any $\lambda \in \mathbb{C}$ and $m \geq 1$.

We have

$$\sum_{\ell=j}^{n-1} \|x\|^\ell \leq (n-j) \max_{\ell \in \{j, \dots, n-1\}} \|x\|^\ell \leq (n-j) \max_{\ell \in \{1, \dots, m-1\}} \|x\|^\ell$$

and then

$$L \leq \|x - 1\| \max_{\ell \in \{1, \dots, m-1\}} \|x\|^\ell \sum_{0 \leq j < n \leq m} |\alpha_n| |\alpha_j| |\lambda|^n |\lambda|^j (n-j). \tag{3.7}$$

From the first inequality in (3.7) and since $\|x\| < 1$ we have

$$\begin{aligned} &\left\| \sum_{j=0}^m \alpha_j \lambda^j \sum_{n=0}^m \alpha_n \lambda^n (xy)^n - \sum_{j=0}^m \alpha_j \lambda^j x^j \sum_{n=0}^m \alpha_n \lambda^n y^n \right\| \\ &\leq 2 \|x - 1\| \sum_{0 \leq j < n \leq m} |\alpha_n| |\alpha_j| |\lambda|^n |\lambda|^j (n-j) \\ &= \|x - 1\| \sum_{n=0}^m \sum_{j=0}^m |\alpha_n| |\alpha_j| |\lambda|^n |\lambda|^j |n-j|. \end{aligned} \tag{3.8}$$

(i) Using the Cauchy-Bunyakovsky-Schwarz inequality for double sums,

$$\sum_{n=0}^m \sum_{j=0}^m p_{n,j} a_{n,j} b_{n,j} \leq \left(\sum_{n=0}^m \sum_{j=0}^m p_{n,j} a_{n,j}^2 \right)^{1/2} \left(\sum_{n=0}^m \sum_{j=0}^m p_{n,j} b_{n,j}^2 \right)^{1/2},$$

where $p_{n,j}, a_{n,j}, b_{n,j} \geq 0$ for $n, j \in \{0, \dots, m\}$, we have

$$\begin{aligned} & \sum_{n=0}^m \sum_{j=0}^m |\alpha_n| |\alpha_j| |\lambda|^n |\lambda|^j |n-j| \\ & \leq \left(\sum_{n=0}^m \sum_{j=0}^m |\alpha_n| |\alpha_j| |\lambda|^{2n} |\lambda|^{2j} \right)^{1/2} \left(\sum_{n=0}^m \sum_{j=0}^m |\alpha_n| |\alpha_j| |n-j|^2 \right)^{1/2} \\ & = \sqrt{2} \left(\sum_{n=0}^m |\alpha_n| |\lambda|^{2n} \right) \left[\sum_{n=0}^m |\alpha_n| \sum_{n=0}^m n^2 |\alpha_n| - \left(\sum_{n=0}^m n |\alpha_n| \right)^2 \right]^{1/2} \end{aligned} \tag{3.9}$$

for any $\lambda \in \mathbb{C}$ and $m \geq 1$.

From (3.8) and (3.9) we get the inequality

$$\begin{aligned} & \left\| \sum_{j=0}^m \alpha_j \lambda^j \sum_{n=0}^m \alpha_n \lambda^n (xy)^n - \sum_{j=0}^m \alpha_j \lambda^j x^j \sum_{n=0}^m \alpha_n \lambda^n y^n \right\| \\ & \leq \sqrt{2} \|x - 1\| \left(\sum_{n=0}^m |\alpha_n| |\lambda|^{2n} \right) \\ & \quad \times \left[\sum_{n=0}^m |\alpha_n| \sum_{n=0}^m n^2 |\alpha_n| - \left(\sum_{n=0}^m n |\alpha_n| \right)^2 \right]^{1/2}. \end{aligned} \tag{3.10}$$

Since the series

$$\sum_{j=0}^{\infty} \alpha_j \lambda^j, \sum_{n=0}^{\infty} \alpha_n \lambda^n (xy)^n, \sum_{j=0}^{\infty} \alpha_j \lambda^j x^j, \sum_{n=0}^{\infty} \alpha_n \lambda^n y^n$$

are convergent in \mathcal{B} , $\sum_{n=0}^{\infty} |\alpha_n| |\lambda|^{2n}$ is convergent and the limit

$$\lim_{m \rightarrow \infty} \left[\sum_{n=0}^m |\alpha_n| \sum_{n=0}^m n^2 |\alpha_n| - \left(\sum_{n=0}^m n |\alpha_n| \right)^2 \right]^{1/2}$$

exists, then by letting $m \rightarrow \infty$ in (3.10) we deduce the desired result in (3.1) for x . Due to the commutativity of x with y , a similar result can be stated for y , and taking the minimum, we deduce the desired result.

(ii) Using the Cauchy-Bunyakovsky-Schwarz inequality for double sums,

$$\sum_{n=0}^m \sum_{j=0}^m p_{n,j} a_{n,j} \leq \left(\sum_{n=0}^m \sum_{j=0}^m p_{n,j} \right)^{1/2} \left(\sum_{n=0}^m \sum_{j=0}^m p_{n,j} a_{n,j}^2 \right)^{1/2}$$

where $p_{n,j}, a_{n,j} \geq 0$ for $n, j \in \{0, \dots, m\}$, we also have

$$\begin{aligned} & \sum_{n=0}^m \sum_{j=0}^m |\alpha_n| |\alpha_j| |\lambda|^n |\lambda|^j |n-j| \\ & \leq \left(\sum_{n=0}^m \sum_{j=0}^m |\alpha_n| |\alpha_j| |\lambda|^n |\lambda|^j \right)^{1/2} \left(\sum_{n=0}^m \sum_{j=0}^m |\alpha_n| |\alpha_j| |\lambda|^n |\lambda|^j |n-j|^2 \right)^{1/2} \end{aligned}$$

$$\begin{aligned}
 &= \sqrt{2} \left(\sum_{n=0}^m |\alpha_n| |\lambda|^n \right) \\
 &\quad \times \left[\sum_{n=0}^m |\alpha_n| |\lambda|^n \sum_{n=0}^m n^2 |\alpha_n| |\lambda|^n - \left(\sum_{n=0}^m n |\alpha_n| |\lambda|^n \right)^2 \right]^{1/2}
 \end{aligned} \tag{3.11}$$

for any $\lambda \in \mathbb{C}$ and $m \geq 1$.

From (3.8) and (3.11) we have

$$\begin{aligned}
 &\left\| \sum_{j=0}^m \alpha_j \lambda^j \sum_{n=0}^m \alpha_n \lambda^n (xy)^n - \sum_{j=0}^m \alpha_j \lambda^j x^j \sum_{n=0}^m \alpha_n \lambda^n y^n \right\| \\
 &\leq \sqrt{2} \|x - 1\| \left(\sum_{n=0}^m |\alpha_n| |\lambda|^n \right) \\
 &\quad \times \left[\sum_{n=0}^m |\alpha_n| |\lambda|^n \sum_{n=0}^m n^2 |\alpha_n| |\lambda|^n - \left(\sum_{n=0}^m n |\alpha_n| |\lambda|^n \right)^2 \right]^{1/2}
 \end{aligned} \tag{3.12}$$

for any $\lambda \in \mathbb{C}$ and $m \geq 1$.

If we denote $f(u) := \sum_{n=0}^{\infty} \alpha_n u^n$, then for $|u| < R$ we have

$$\sum_{n=0}^{\infty} n \alpha_n u^n = u f'(u)$$

and

$$\sum_{n=0}^{\infty} n^2 \alpha_n u^n = u (u g'(u))'.$$

However

$$u (u g'(u))' = u g'(u) + u^2 g''(u)$$

and then

$$\sum_{n=0}^{\infty} n^2 \alpha_n u^n = u g'(u) + u^2 g''(u).$$

Therefore

$$\sum_{n=0}^{\infty} n^2 |\alpha_n| |\lambda|^n = |\lambda| f'_A(|\lambda|) + |\lambda|^2 f''_A(|\lambda|)$$

and

$$\sum_{n=0}^m n |\alpha_n| |\lambda|^n = |\lambda| f'(|\lambda|)$$

for $|\lambda| < R$.

Since all the series whose partial sums are involved in (3.12) are convergent, then by letting $m \rightarrow \infty$ in (3.12) we deduce the desired inequality (3.3) for x . Due to the commutativity of x with y , a similar result can be stated for y , and taking the minimum, we deduce the desired result. \square

Remark 1 If $R = \infty$, Theorem 1 holds true. Moreover, in this case the restrictions $\|x\|, \|y\| \leq 1$ need no longer be imposed.

Remark 2 We observe that if the power series $f(\lambda) = \sum_{n=0}^{\infty} \alpha_n \lambda^n$ has the radius of convergence $R > 1$, then

$$\sum_{n=0}^{\infty} |\alpha_n| = f_A(1), \quad \sum_{n=0}^{\infty} n^2 |\alpha_n| = f'_A(1) + f''_A(1)$$

and

$$\sum_{n=0}^{\infty} n |\alpha_n| = f'_A(1).$$

In this case ψ is finite and

$$\begin{aligned} \psi &= \lim_{m \rightarrow \infty} \left[\sum_{n=0}^m |\alpha_n| \sum_{n=0}^m n^2 |\alpha_n| - \left(\sum_{n=0}^m n |\alpha_n| \right)^2 \right]^{1/2} \\ &= \{f_A(1)[f'_A(1) + f''_A(1)] - [f'_A(1)]^2\}^{1/2}. \end{aligned}$$

Therefore, if $\lambda \in \mathbb{C}$ with $|\lambda|, |\lambda|^2, |\lambda|\|x\|, |\lambda|\|y\| < R$, then from (3.1) we have

$$\begin{aligned} &\|\tilde{f}(\lambda \cdot 1)\tilde{f}(\lambda xy) - \tilde{f}(\lambda x)\tilde{f}(\lambda y)\| \\ &\leq \sqrt{2}\{f_A(1)[f'_A(1) + f''_A(1)] - [f'_A(1)]^2\}^{1/2} \\ &\quad \times \min\{\|x - 1\|, \|y - 1\|\} f_A(|\lambda|^2). \end{aligned} \tag{3.13}$$

Corollary 1 Under the assumptions of Theorem 1 we have the inequalities

$$\|\tilde{f}(\lambda \cdot 1)\tilde{f}(\lambda x^2) - \tilde{f}^2(\lambda x)\| \leq \sqrt{2}\psi \|x - 1\| f_A(|\lambda|^2) \tag{3.14}$$

provided $\lambda \in \mathbb{C}$ with $|\lambda|, |\lambda|^2, |\lambda|\|x\| < R$, and

$$\begin{aligned} &\|\tilde{f}(\lambda \cdot 1)\tilde{f}(\lambda x^2) - \tilde{f}^2(\lambda x)\| \\ &\leq \sqrt{2}\|x - 1\| f_A(|\lambda|) \\ &\quad \times \{f_A(|\lambda|)[|\lambda|f'_A(|\lambda|) + |\lambda|^2 f''_A(|\lambda|)] - [|\lambda|f'_A(|\lambda|)]^2\}^{1/2} \end{aligned} \tag{3.15}$$

provided $\lambda \in \mathbb{C}$ with $|\lambda|, |\lambda|\|x\| < R$.

Theorem 2 Let $f(\lambda) = \sum_{n=0}^{\infty} \alpha_n \lambda^n$ be a function defined by power series with complex coefficients and convergent on the open disk $D(0, R) \subset \mathbb{C}$, $R > 0$, and $x, y \in \mathcal{B}$ with $xy = yx$ and $\|x\|, \|y\| < 1$.

If $\lambda \in \mathbb{C}$ with $|\lambda|, |\lambda|\|x\|, |\lambda|\|y\| < R$, then

$$\| \tilde{f}(\lambda \cdot 1) \tilde{f}(\lambda xy) - \tilde{f}(\lambda x) \tilde{f}(\lambda y) \| \leq \min \left\{ \frac{\|x-1\|}{1-\|x\|}, \frac{\|y-1\|}{1-\|y\|} \right\} [f_{A^2}^2(|\lambda|) - f_{A^2}(|\lambda|^2)], \quad (3.16)$$

where

$$f_{A^2}(\lambda) := \sum_{n=0}^{\infty} |\alpha_n|^2 \lambda^n \quad (3.17)$$

has the radius of convergence R^2 .

Proof As pointed out in (3.6), we have

$$\begin{aligned} L &\leq \|x-1\| \sum_{0 \leq j < n \leq m} |\alpha_n| |\alpha_j| |\lambda|^n |\lambda|^j \sum_{\ell=j}^{n-1} \|x\|^\ell \\ &\leq \|x-1\| \sum_{\ell=0}^{m-1} \|x\|^\ell \sum_{0 \leq j < n \leq m} |\alpha_n| |\alpha_j| |\lambda|^n |\lambda|^j \end{aligned} \quad (3.18)$$

for any $\lambda \in \mathbb{C}$ and $m \geq 1$.

Denote

$$K_m := \sum_{0 \leq j < n \leq m} |\alpha_n| |\alpha_j| |\lambda|^n |\lambda|^j.$$

We obviously have

$$\begin{aligned} K_m &= \frac{1}{2} \left(\sum_{n,j=0}^m |\alpha_n| |\alpha_j| |\lambda|^n |\lambda|^j - \sum_{n=0}^m |\alpha_n|^2 |\lambda|^{2n} \right) \\ &= \frac{1}{2} \left[\left(\sum_{n=0}^m |\alpha_n| |\lambda|^n \right)^2 - \sum_{n=0}^m |\alpha_n|^2 |\lambda|^{2n} \right]. \end{aligned}$$

From (3.8) and (3.18) we get the inequality

$$\begin{aligned} &\left\| \sum_{j=0}^m \alpha_j \lambda^j \sum_{n=0}^m \alpha_n \lambda^n (xy)^n - \sum_{j=0}^m \alpha_j \lambda^j x^j \sum_{n=0}^m \alpha_n \lambda^n y^n \right\| \\ &\leq \|x-1\| \sum_{\ell=0}^{m-1} \|x\|^\ell \\ &\quad \times \left[\left(\sum_{n=0}^m |\alpha_n| |\lambda|^n \right)^2 - \sum_{n=0}^m |\alpha_n|^2 |\lambda|^{2n} \right], \end{aligned} \quad (3.19)$$

for any $\lambda \in \mathbb{C}$ and $m \geq 1$.

Since all the series whose partial sums are involved in (3.19) are convergent, then by letting $m \rightarrow \infty$ in (3.19) we deduce the desired inequality (3.16) for x . Due to the commutativity of x with y , a similar result can be stated for y , and taking the minimum, we deduce the desired result. \square

Remark 3 Since the power series $f_{A^2}(\lambda) := \sum_{n=0}^{\infty} |\alpha_n|^2 \lambda^n$ is not easy to compute, we can provide some bounds for the quantity

$$D_f(|\lambda|) := f_A^2(|\lambda|) - f_{A^2}(|\lambda|^2),$$

where $|\lambda| < R$, as follows.

If $|\lambda| < 1$ and $a_{\ell_\infty} := \sup_{n \in \mathbb{N}} \{a_n\} < \infty$, then

$$\begin{aligned} K_m &\leq a_{\ell_\infty}^2 \sum_{0 \leq j < n \leq m} |\lambda|^n |\lambda|^j \\ &= \frac{1}{2} a_{\ell_\infty}^2 \left[\left(\sum_{n=0}^m |\lambda|^n \right)^2 - \sum_{n=0}^m |\lambda|^{2n} \right] \end{aligned}$$

and by taking $m \rightarrow \infty$ in this inequality we get

$$D_f(|\lambda|) \leq \frac{1}{2} a_{\ell_\infty}^2 \left[\left(\frac{1}{1-|\lambda|} \right)^2 - \frac{1}{1-|\lambda|^2} \right] \tag{3.20}$$

for $|\lambda| < 1$.

If $|\lambda| < 1$ and

$$a_{\ell_1} := \lim_{m \rightarrow \infty} \left[\left(\sum_{n=0}^m |\alpha_n| \right)^2 - \sum_{n=0}^m |\alpha_n|^2 \right] < \infty$$

then

$$\begin{aligned} K_m &\leq \max_{n \in \{0, \dots, m\}} |\lambda|^{2n} \sum_{0 \leq j < n \leq m} |\alpha_n| |\alpha_j| \\ &\leq \frac{1}{2} \left[\left(\sum_{n=0}^m |\alpha_n| \right)^2 - \sum_{n=0}^m |\alpha_n|^2 \right] \end{aligned}$$

and by taking $m \rightarrow \infty$ in this inequality we get

$$D_f(|\lambda|) \leq \frac{1}{2} a_{\ell_1} \tag{3.21}$$

for $|\lambda| < 1$.

If the series $\sum_{n=0}^{\infty} |\alpha_n|$ and $\sum_{n=0}^{\infty} |\alpha_n|^2$ are convergent, then

$$D_f(|\lambda|) \leq \frac{1}{2} \left[\left(\sum_{n=0}^{\infty} |\alpha_n| \right)^2 - \sum_{n=0}^{\infty} |\alpha_n|^2 \right] \tag{3.22}$$

for $|\lambda| < 1$.

If $|\lambda| < 1, p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, and

$$a_{\ell_q} := \lim_{m \rightarrow \infty} \left[\left(\sum_{n=0}^m |\alpha_n|^q \right)^2 - \sum_{n=0}^m |\alpha_n|^{2q} \right] < \infty$$

then by Hölder's inequality we have

$$\begin{aligned} K_m &\leq \left(\sum_{0 \leq j < n \leq m} |\alpha_n|^q |\alpha_j|^q \right)^{1/q} \left(\sum_{0 \leq j < n \leq m} |\lambda|^{pn} |\lambda|^{pj} \right)^{1/p} \\ &\leq \left\{ \frac{1}{2} \left[\left(\sum_{n=0}^m |\alpha_n|^q \right)^2 - \sum_{n=0}^m |\alpha_n|^{2q} \right] \right\}^{1/q} \\ &\quad \times \left\{ \frac{1}{2} \left[\left(\sum_{n=0}^m |\lambda|^{pn} \right)^2 - \sum_{n=0}^m |\lambda|^{2pn} \right] \right\}^{1/p} \end{aligned}$$

and by taking $m \rightarrow \infty$ in this inequality we get

$$D_f(|\lambda|) \leq \frac{1}{2} a_{\ell_q}^{1/q} \left[\left(\frac{1}{1 - |\lambda|^p} \right)^2 - \frac{1}{1 - |\lambda|^{2p}} \right]^{1/p} \tag{3.23}$$

for $|\lambda| < 1$.

If the series $\sum_{n=0}^{\infty} |\alpha_n|^q$ and $\sum_{n=0}^{\infty} |\alpha_n|^{2q}$ are convergent, then

$$D_f(|\lambda|) \leq \frac{1}{2} \left[\left(\sum_{n=0}^{\infty} |\alpha_n|^q \right)^2 - \sum_{n=0}^{\infty} |\alpha_n|^{2q} \right]^{1/p} \left[\left(\frac{1}{1 - |\lambda|^p} \right)^2 - \frac{1}{1 - |\lambda|^{2p}} \right]^{1/p} \tag{3.24}$$

for $|\lambda| < 1$.

The following result also holds.

Theorem 3 Let $f(\lambda) = \sum_{n=0}^{\infty} \alpha_n \lambda^n$ be a function defined by power series with complex coefficients and convergent on the open disk $D(0, R) \subset \mathbb{C}, R > 0$, and $x, y \in \mathcal{B}$ with $xy = yx$ and $\|x\|, \|y\| < 1$.

If $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $\lambda \in \mathbb{C}$ with $|\lambda|, |\lambda|^p, |\lambda| \|x\|, |\lambda| \|y\| < R$, then

$$\begin{aligned} &\| \tilde{f}(\lambda \cdot 1) \tilde{f}(\lambda xy) - \tilde{f}(\lambda x) \tilde{f}(\lambda y) \| \\ &\leq \frac{1}{2} \min \left\{ \frac{\|x - 1\|}{(1 - \|x\|^p)^{1/p}}, \frac{\|y - 1\|}{(1 - \|y\|^p)^{1/p}} \right\} \\ &\quad \times \varphi^{1/q} [f_A^2(|\lambda|^p) - f_{A^2}(|\lambda|^{2p})]^{1/p}, \end{aligned} \tag{3.25}$$

where

$$\varphi := \lim_{m \rightarrow \infty} \sum_{n,j=0}^m |\alpha_n| |\alpha_j| |n - j| \tag{3.26}$$

is assumed to exist and be finite.

Proof Using Hölder’s inequality for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and (3.6), we have

$$\begin{aligned}
 L &\leq \|x - 1\| \sum_{0 \leq j < n \leq m} |\alpha_n| |\alpha_j| |\lambda|^n |\lambda|^j (n - j)^{1/q} \left(\sum_{\ell=j}^{n-1} \|x\|^{\ell p} \right)^{1/p} \\
 &\leq \|x - 1\| \left(\sum_{\ell=0}^{m-1} \|x\|^{\ell p} \right)^{1/p} \sum_{0 \leq j < n \leq m} |\alpha_n| |\alpha_j| |\lambda|^n |\lambda|^j (n - j)^{1/q}
 \end{aligned} \tag{3.27}$$

for any $\lambda \in \mathbb{C}$ and $m \geq 1$.

Applying Hölder’s inequality once more we have

$$\begin{aligned}
 &\sum_{0 \leq j < n \leq m} |\alpha_n| |\alpha_j| |\lambda|^n |\lambda|^j (n - j)^{1/q} \\
 &\leq \left(\sum_{0 \leq j < n \leq m} |\alpha_n| |\alpha_j| |\lambda|^n (n - j) \right)^{1/q} \left(\sum_{0 \leq j < n \leq m} |\alpha_n| |\alpha_j| |\lambda|^{2n} |\lambda|^{2j} \right)^{1/p} \\
 &= \left(\frac{1}{2} \sum_{n,j=0}^m |\alpha_n| |\alpha_j| |n - j| \right)^{1/q} \\
 &\quad \times \left(\frac{1}{2} \left[\left(\sum_{n=0}^m |\alpha_n| |\lambda|^{np} \right)^2 - \sum_{n=0}^m |\alpha_n|^2 |\lambda|^{2np} \right] \right)^{1/p} \\
 &= \frac{1}{2} \left(\sum_{n,j=0}^m |\alpha_n| |\alpha_j| |n - j| \right)^{1/q} \\
 &\quad \times \left[\left(\sum_{n=0}^m |\alpha_n| |\lambda|^{np} \right)^2 - \sum_{n=0}^m |\alpha_n|^2 |\lambda|^{2np} \right]^{1/p}
 \end{aligned} \tag{3.28}$$

for any $\lambda \in \mathbb{C}$ and $m \geq 1$.

From (3.8) and (3.28) we get the inequality

$$\begin{aligned}
 &\left\| \sum_{j=0}^m \alpha_j \lambda^j \sum_{n=0}^m \alpha_n \lambda^n (xy)^n - \sum_{j=0}^m \alpha_j \lambda^j x^j \sum_{n=0}^m \alpha_n \lambda^n y^n \right\| \\
 &\leq \frac{1}{2} \|x - 1\| \left(\sum_{\ell=0}^{m-1} \|x\|^{\ell p} \right)^{1/p} \left(\sum_{n,j=0}^m |\alpha_n| |\alpha_j| |n - j| \right)^{1/q} \\
 &\quad \times \left[\left(\sum_{n=0}^m |\alpha_n| |\lambda|^{np} \right)^2 - \sum_{n=0}^m |\alpha_n|^2 |\lambda|^{2np} \right]^{1/p},
 \end{aligned} \tag{3.29}$$

for any $\lambda \in \mathbb{C}$ and $m \geq 1$.

Since all the series whose partial sums are involved in (3.29) are convergent, then by letting $m \rightarrow \infty$ in (3.29) we deduce the desired inequality (3.25) for x . Due to the commutativity of x with y , a similar result can be stated for y , and taking the minimum, we deduce the desired result. \square

Remark 4 Observe that

$$[f_A^2(|\lambda|^p) - f_{A^2}(|\lambda|^{2p})]^{1/p} = D_f^{1/p}(|\lambda|^p)$$

and then further bounds for the inequality (3.25) may be provided by the use of Remark 3. However the details are not mentioned here.

We can obtain a simpler upper bound for φ as follows.

Using the Cauchy-Bunyakovsky-Schwarz inequality for double sums

$$\sum_{n=0}^m \sum_{j=0}^m p_{i,j} a_{i,j} \leq \left(\sum_{n=0}^m \sum_{j=0}^m p_{i,j} \right)^{1/2} \left(\sum_{n=0}^m \sum_{j=0}^m p_{i,j} a_{i,j}^2 \right)^{1/2},$$

where $p_{i,j}, a_{i,j} \geq 0$ for $i, j \in \{0, \dots, m\}$, we have

$$\begin{aligned} \sum_{n,j=0}^m |\alpha_n| |\alpha_j| |n-j| &\leq \left(\sum_{n,j=0}^m |\alpha_n| |\alpha_j| \right)^{1/2} \left(\sum_{n,j=0}^m |\alpha_n| |\alpha_j| |n-j|^2 \right)^{1/2} \\ &= \sqrt{2} \sum_{n=0}^m |\alpha_n| \left[\sum_{n=0}^m |\alpha_n| \sum_{n=0}^m n^2 |\alpha_n| - \left(\sum_{n=0}^m n |\alpha_n| \right)^2 \right]^{1/2} \end{aligned} \tag{3.30}$$

for $m \geq 1$.

If the series $\sum_{n=0}^\infty |\alpha_n|$ is finite and ψ , defined by (3.2), is finite, then from (3.30) we have

$$\varphi \leq \sqrt{2} \sum_{n=0}^\infty |\alpha_n| \psi. \tag{3.31}$$

We observe that, if the power series $f(\lambda) = \sum_{n=0}^\infty \alpha_n \lambda^n$ has the radius of convergence $R > 1$, then ψ is finite and

$$\psi = \{f_A(1)[f_A'(1) + f_A''(1)] - [f_A'(1)]^2\}^{1/2}.$$

We have from (3.31) the inequality

$$\varphi \leq \sqrt{2} f_A(1) \{f_A(1)[f_A'(1) + f_A''(1)] - [f_A'(1)]^2\}^{1/2}. \tag{3.32}$$

4 Some examples

As some natural examples that are useful for applications, we can point out that, if

$$\begin{aligned} f(\lambda) &= \sum_{n=1}^\infty \frac{(-1)^n}{n} \lambda^n = \ln \frac{1}{1+\lambda}, \quad \lambda \in D(0,1); \\ g(\lambda) &= \sum_{n=0}^\infty \frac{(-1)^n}{(2n)!} \lambda^{2n} = \cos \lambda, \quad \lambda \in \mathbb{C}; \\ h(\lambda) &= \sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)!} \lambda^{2n+1} = \sin \lambda, \quad \lambda \in \mathbb{C}; \\ l(\lambda) &= \sum_{n=0}^\infty (-1)^n \lambda^n = \frac{1}{1+\lambda}, \quad \lambda \in D(0,1), \end{aligned} \tag{4.1}$$

then the corresponding functions constructed by the use of the absolute values of the coefficients are

$$\begin{aligned}
 f_A(\lambda) &= \sum_{n=1}^{\infty} \frac{1}{n} \lambda^n = \ln \frac{1}{1-\lambda}, \quad \lambda \in D(0, 1); \\
 g_A(\lambda) &= \sum_{n=0}^{\infty} \frac{1}{(2n)!} \lambda^{2n} = \cosh \lambda, \quad \lambda \in \mathbb{C}; \\
 h_A(\lambda) &= \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \lambda^{2n+1} = \sinh \lambda, \quad \lambda \in \mathbb{C}; \\
 l_A(\lambda) &= \sum_{n=0}^{\infty} \lambda^n = \frac{1}{1-\lambda}, \quad \lambda \in D(0, 1).
 \end{aligned} \tag{4.2}$$

Other important examples of functions as power series representations with nonnegative coefficients are

$$\begin{aligned}
 \exp(\lambda) &= \sum_{n=0}^{\infty} \frac{1}{n!} \lambda^n, \quad \lambda \in \mathbb{C}, \\
 \frac{1}{2} \ln \left(\frac{1+\lambda}{1-\lambda} \right) &= \sum_{n=1}^{\infty} \frac{1}{2n-1} \lambda^{2n-1}, \quad \lambda \in D(0, 1); \\
 \sin^{-1}(\lambda) &= \sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{1}{2})}{\sqrt{\pi}(2n+1)n!} \lambda^{2n+1}, \quad \lambda \in D(0, 1); \\
 \tanh^{-1}(\lambda) &= \sum_{n=1}^{\infty} \frac{1}{2n-1} \lambda^{2n-1}, \quad \lambda \in D(0, 1); \\
 {}_2F_1(\alpha, \beta, \gamma, \lambda) &= \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)\Gamma(n+\beta)\Gamma(\gamma)}{n!\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\gamma)} \lambda^n, \quad \alpha, \beta, \gamma > 0, \lambda \in D(0, 1);
 \end{aligned} \tag{4.3}$$

where Γ is the *Gamma function*.

If we apply the inequality (3.13) to the exponential function, then we have

$$\|\exp[\lambda(1+xy)] - \exp[\lambda(x+y)]\| \leq \sqrt{2}e \min\{\|x-1\|, \|y-1\|\} \exp(|\lambda|^2) \tag{4.4}$$

for any $x, y \in \mathcal{B}$ with $xy = yx$, $\|x\|, \|y\| < 1$, and $\lambda \in \mathbb{C}$.

If we take $y = -x$ in (4.4), then we get

$$\|\exp[\lambda(1-x^2)] - 1\| \leq \sqrt{2}e \min\{\|x-1\|, \|x+1\|\} \exp(|\lambda|^2) \tag{4.5}$$

for any $x \in \mathcal{B}$ with $\|x\| < 1$ and $\lambda \in \mathbb{C}$.

If we apply the inequality (3.3) for the exponential functions we also have

$$\begin{aligned}
 &\|\exp[\lambda(1+xy)] - \exp[\lambda(x+y)]\| \\
 &\leq \sqrt{2} \min\{\|x-1\|, \|y-1\|\} |\lambda|^{1/2} \exp(2|\lambda|),
 \end{aligned} \tag{4.6}$$

for any $x, y \in \mathcal{B}$ with $xy = yx$, $\|x\|, \|y\| < 1$, and $\lambda \in \mathbb{C}$.

If we take $y = -x$ in (4.6), then we get

$$\|\exp[\lambda(1 - x^2)] - 1\| \leq \sqrt{2} \min\{\|x - 1\|, \|x + 1\|\} |\lambda|^{1/2} \exp(2|\lambda|). \tag{4.7}$$

Now, consider the function $f(\lambda) := \sum_{n=0}^{\infty} \lambda^n = \frac{1}{1-\lambda}$, $\lambda \in D(0, 1)$. If we apply the inequality (3.3) for this function, then we get the result

$$\begin{aligned} & \|(1 - \lambda)^{-1}(1 - \lambda xy)^{-1} - (1 - \lambda x)^{-1}(1 - \lambda y)^{-1}\| \\ & \leq \sqrt{2} \min\{\|x - 1\|, \|y - 1\|\} |\lambda|^{1/2} (1 - |\lambda|)^{-3} \end{aligned} \tag{4.8}$$

for any $x, y \in \mathcal{B}$ with $xy = yx$, $\|x\|, \|y\| < 1$, and $\lambda \in \mathbb{C}$ with $|\lambda| < 1$.

We have in particular the inequalities

$$\|(1 - \lambda)^{-1}(1 - \lambda x^2)^{-1} - (1 - \lambda x)^{-2}\| \leq \sqrt{2} \|x - 1\| |\lambda|^{1/2} (1 - |\lambda|)^{-3} \tag{4.9}$$

and

$$\begin{aligned} & \|(1 - \lambda)^{-1}(1 + \lambda x^2)^{-1} - (1 - \lambda^2 x^2)^{-1}\| \\ & \leq \sqrt{2} \min\{\|x - 1\|, \|x + 1\|\} |\lambda|^{1/2} (1 - |\lambda|)^{-3} \end{aligned} \tag{4.10}$$

for any $x \in \mathcal{B}$ with $\|x\| < 1$ and $\lambda \in \mathbb{C}$ with $|\lambda| < 1$.

Now, if we take $\lambda = \frac{1}{\gamma}$ with $|\gamma| > 1$ then we get from (4.8) the inequality

$$\begin{aligned} & \|\gamma^2(\gamma - 1)^{-1}(\gamma - xy)^{-1} - \gamma^2(\gamma - x)^{-1}(\gamma - y)^{-1}\| \\ & \leq \sqrt{2} \min\{\|x - 1\|, \|y - 1\|\} |\gamma|^{-1/2} (|\gamma| - 1)^{-3} |\gamma|^3, \end{aligned}$$

which is equivalent with

$$\begin{aligned} & \|(\gamma - 1)^{-1}(\gamma - xy)^{-1} - (\gamma - x)^{-1}(\gamma - y)^{-1}\| \\ & \leq \sqrt{2} \min\{\|x - 1\|, \|y - 1\|\} |\gamma|^{1/2} (|\gamma| - 1)^{-3} \end{aligned}$$

for any $x, y \in \mathcal{B}$ with $xy = yx$, $\|x\|, \|y\| < 1$, and $\gamma \in \mathbb{C}$ with $|\gamma| > 1$.

If we use the *resolvent function* notation, then we have the following inequality:

$$\begin{aligned} & \|(\gamma - 1)^{-1}R_{xy}(\gamma) - R_x(\gamma)R_y(\gamma)\| \\ & \leq \sqrt{2} \min\{\|x - 1\|, \|y - 1\|\} |\gamma|^{1/2} (|\gamma| - 1)^{-3} \end{aligned} \tag{4.11}$$

for any $x, y \in \mathcal{B}$ with $xy = yx$, $\|x\|, \|y\| < 1$, and $\gamma \in \mathbb{C}$ with $|\gamma| > 1$.

In particular, we have

$$\|(\gamma - 1)^{-1}R_{x^2}(\gamma) - R_x^2(\gamma)\| \leq \sqrt{2} \|x - 1\| |\gamma|^{1/2} (|\gamma| - 1)^{-3} \tag{4.12}$$

for any $x \in \mathcal{B}$ with $\|x\| < 1$ and $\gamma \in \mathbb{C}$ with $|\gamma| > 1$.

Remark 5 Similar inequalities may be stated for the other power series mentioned at the beginning of this paragraph. However, the details are not presented here.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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