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This is the Published version of the following publication

Dragomir, Sever S (2014) Approximating the Riemann-Stieltjes integral via a Chebyshev type functional. Acta et Commentationes Universitatis Tartuensis de Mathematica, 18 (2). 239 - 259. ISSN 1406-2283

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## Approximating the Riemann–Stieltjes integral via a Chebyshev type functional

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ABSTRACT. Some new sharp upper bounds for the absolute value of the error functional  $D(f, u)$  in approximating the Riemann–Stieltjes integral  $\int_a^b f(t) du(t)$  by the quantity  $[u(b) - u(a)] \cdot \frac{1}{b-a} \int_a^b f(t) dt$  are given.

### 1. Introduction

In order to approximate the Riemann–Stieltjes integral  $\int_a^b f(t) du(t)$  by the simpler quantity

$$[u(b) - u(a)] \cdot \frac{1}{b-a} \int_a^b f(t) dt,$$

provided that both integrals exist, Dragomir and Fedotov introduced in [11] the following *error functional of Chebyshev type*

$$D(f; u) = \int_a^b f(t) du(t) - [u(b) - u(a)] \cdot \frac{1}{b-a} \int_a^b f(t) dt,$$

and pointed out the following sharp upper bound for  $|D(f; u)|$ , namely

$$|D(f; u)| \leq \frac{1}{2} L (M - m) (b - a), \quad (1.1)$$

provided the *integrator*  $u : [a, b] \rightarrow \mathbb{R}$  is  $L$ -Lipschitzian on  $[a, b]$ , i.e.,

$$|u(x) - u(y)| \leq L |x - y|$$

for any  $x, y \in [a, b]$  and the *integrand*  $f : [a, b] \rightarrow \mathbb{R}$  is Riemann integrable on  $[a, b]$  and satisfies the boundedness condition

$$-\infty < m \leq f(x) \leq M < \infty \quad \text{for a.e. } x \in [a, b].$$

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Received December 18, 2013.

2010 *Mathematics Subject Classification.* 26D15, 41A55.

*Key words and phrases.* Riemann–Stieltjes integral, quadrature rules, Grüss type inequalities, Chebyshev functional, error bounds.

<http://dx.doi.org/10.12097/ACUTM.2014.18.20>

The multiplicative constant  $\frac{1}{2}$  in (1.1) is the best possible in the sense that it cannot be replaced by a smaller constant.

In the follow-up paper [12], the authors provided a different bound, namely

$$|D(f; u)| \leq \frac{1}{2} K (b - a) \bigvee_a^b(u), \quad (1.2)$$

provided that  $f$  is  $K$ -Lipschitzian and  $u$  is of bounded variation on  $[a, b]$ .

The result (1.2) was improved in [10] for the case of monotonic non-decreasing functions. We have shown in this case that

$$|D(f; u)| \leq \frac{1}{2} K (b - a) [u(b) - u(a) - S(u)] \quad (1.3)$$

$$\left( \leq \frac{1}{2} K (b - a) [u(b) - u(a)] \right),$$

where

$$S(u) := \frac{4}{(b - a)^2} \int_a^b u(t) \left( t - \frac{a + b}{2} \right) dt \geq 0.$$

In (1.3) the constant  $\frac{1}{2}$  is the best possible in both inequalities.

For other sharp bounds on the error functional  $D(f; u)$ , see the recent papers [9], [7] and [14]. For other inequalities for the Riemann–Stieltjes integral, see [1], [2], [3], [4], [6] and [13].

The main aim of this paper is to further investigate the error functional  $D(f; u)$ . Two representations are given. These are applied to obtain some inequalities for  $D(f; u)$  which improve earlier results.

Applications for the classical *Chebyshev functional*  $C(f, g)$ , where

$$C(f, g) := \frac{1}{b - a} \int_a^b f(t) g(t) dt - \frac{1}{b - a} \int_a^b f(t) dt \cdot \frac{1}{b - a} \int_a^b g(t) dt, \quad (1.4)$$

and  $f, g$  are integrable and belonging to different classes of functions, are also provided.

## 2. Representation results

For a function  $g : [a, b] \rightarrow \mathbb{R}$ , consider the *generalised trapezoid error transform*  $\Phi_g : [a, b] \rightarrow \mathbb{R}$  given by

$$\Phi_g(t) := \frac{1}{b - a} [(b - t)g(a) + (t - a)g(b)] - g(t), \quad t \in [a, b]$$

and if  $g$  is Lebesgue integrable, the *Ostrowski transform*, which is the error of approximating the function by its integral mean, defined by

$$\Theta_g(t) := g(t) - \frac{1}{b - a} \int_a^b g(s) ds, \quad t \in [a, b].$$

We also define the kernel  $Q : [a, b]^2 \rightarrow \mathbb{R}$ ,

$$Q(t, s) := \begin{cases} t - b & \text{if } a \leq s \leq t \leq b, \\ t - a & \text{if } a \leq t < s \leq b. \end{cases} \quad (2.1)$$

The following representation result in terms of  $\Theta_g$  and  $Q$  may be stated.

**Lemma 1.** *If  $f, u : [a, b] \rightarrow \mathbb{R}$  are bounded functions and such that the Riemann–Stieltjes integral  $\int_a^b f(t) du(t)$  and the Riemann integral  $\int_a^b f(t) dt$  exist, then we have the representation*

$$D(f; u) = \int_a^b \Theta_f(s) du(s) = \frac{1}{b-a} \int_a^b \left( \int_a^b Q(t, s) df(t) \right) du(s). \quad (2.2)$$

*Proof.* We have by the definition of  $Q$  and integrating by parts in the Riemann–Stieltjes integral that

$$\begin{aligned} & \frac{1}{b-a} \int_a^b \left( \int_a^b Q(t, s) df(t) \right) du(s) \\ &= \frac{1}{b-a} \int_a^b \left[ \int_a^s (t-a) df(t) + \int_s^b (t-b) df(t) \right] du(s) \\ &= \frac{1}{b-a} \int_a^b \left[ f(t)(t-a) \Big|_a^s - \int_a^s f(t) dt + (t-b)f(t) \Big|_s^b - \int_s^b f(t) dt \right] du(s) \\ &= \frac{1}{b-a} \int_a^b \left[ f(s)(s-a) - \int_a^s f(t) dt + (b-s)f(s) - \int_s^b f(t) dt \right] du(s) \\ &= \int_a^b \Theta_f(s) du(s), \end{aligned}$$

and the second equality is proved.

The first identity is obvious by the definition of  $D(f; u)$ .  $\square$

The following corollary can be stated about the representation of the Chebyshev functional  $C(f, g)$  defined in (1.4).

**Corollary 1.** *Assume that  $f, g : [a, b] \rightarrow \mathbb{R}$  are Riemann integrable on  $[a, b]$ , then*

$$\begin{aligned} C(f, g) &= \frac{1}{b-a} \int_a^b \Theta_f(s) g(s) ds \\ &= \frac{1}{(b-a)^2} \int_a^b \left( \int_a^b Q(t, s) df(t) \right) g(s) ds. \end{aligned}$$

*Proof.* It is well known (see for instance [5, Theorem 7.33, p. 162] that if  $g$  is Riemann integrable and  $u(t) = \int_a^t g(s) ds$ , then for any Riemann integrable function  $f$  we have that the Riemann–Stieltjes integral  $\int_a^b f(t) du(t)$

exists and  $\int_a^b f(t) du(t) = \int_a^b f(t) g(t) dt$ . Therefore, we have  $D(f; u) = (b-a)C(f, g)$  and

$$\int_a^b \left( \int_a^b Q(t, s) df(t) \right) du(s) = \int_a^b \left( \int_a^b Q(t, s) df(t) \right) g(s) ds.$$

□

The second representation of  $D(f; u)$  is incorporated in

**Lemma 2.** *With the assumptions in Lemma 1, we have*

$$D(f; u) = \int_a^b \Phi_u(t) df(t) = \frac{1}{b-a} \int_a^b \left( \int_a^b Q(t, s) du(s) \right) df(t), \quad (2.3)$$

where  $Q$  is defined by (2.1).

*Proof.* By the Fubini type theorem for the Riemann–Stieltjes integral (see for instance [5, Theorem 7.41, p. 167]) we have that

$$\int_a^b \left( \int_a^b Q(t, s) du(s) \right) df(t) = \int_a^b \left( \int_a^b Q(t, s) df(t) \right) du(s),$$

and the equality between the first and the last term in (2.3) is proved.

Now, observe that

$$\begin{aligned} \int_a^b Q(t, s) du(s) &= \int_a^t (t-b) du(s) + \int_t^b (t-a) du(s) \\ &= (t-b)[u(t) - u(a)] + (t-a)[u(b) - u(t)] \\ &= (b-a)\Phi_u(t), \end{aligned}$$

for any  $t \in [a, b]$ , and then integrating over  $f(t)$ , we deduce the second equality in (2.3). □

**Corollary 2.** *Assume that  $f$  and  $g$  are Riemann integrable on  $[a, b]$ . Then*

$$\begin{aligned} C(f, g) &= \frac{1}{b-a} \int_a^b \tilde{\Phi}_g(t) df(t) \\ &= \frac{1}{(b-a)^2} \int_a^b \left( \int_a^b Q(t, s) g(s) ds \right) df(t), \end{aligned}$$

where

$$\tilde{\Phi}_g(t) = \Phi_{\int_a^b g} (t) = \frac{t-a}{b-a} \int_a^b g(s) ds - \int_a^t g(s) ds, \quad t \in [a, b].$$

### 3. Bounds in the case when $u$ is of bounded variation

The following lemma is of interest in itself.

**Lemma 3.** *If  $p : [a, b] \rightarrow \mathbb{R}$  is continuous on  $[a, b]$  and  $v : [a, b] \rightarrow \mathbb{R}$  is of bounded variation on  $[a, b]$ , then*

$$\begin{aligned} \left| \int_a^b p(t) dv(t) \right| &\leq \int_a^b |p(t)| d\bigvee_a^t(v) \\ &\leq \left[ \bigvee_a^b(v) \right]^{\frac{1}{q}} \left\{ \int_a^b |p(t)|^r d\left[ \bigvee_a^t(v) \right] \right\}^{\frac{1}{r}} \\ &\leq \max_{t \in [a, b]} |p(t)| \bigvee_a^b(v), \end{aligned} \tag{3.1}$$

where  $q > 1$ ,  $1/q + 1/r = 1$ .

*Proof.* Since the Stieltjes integral  $\int_a^b p(t) dv(t)$  exists, for any division  $I_n : a = t_0 < t_1 < \dots < t_{n-1} < t_n = b$  with the norm  $v(I_n) := \max_{i \in \{0, \dots, n-1\}} (t_{i+1} - t_i) \rightarrow 0$  and for any intermediate points  $\xi_i \in [t_i, t_{i+1}]$ ,  $i \in \{0, \dots, n-1\}$ , we have

$$\begin{aligned} \left| \int_a^b p(t) dv(t) \right| &= \left| \lim_{v(I_n) \rightarrow 0} \sum_{i=0}^{n-1} p(\xi_i) [v(t_{i+1}) - v(t_i)] \right| \\ &\leq \lim_{v(I_n) \rightarrow 0} \sum_{i=0}^{n-1} |p(\xi_i)| |v(t_{i+1}) - v(t_i)|. \end{aligned}$$

However,

$$|v(t_{i+1}) - v(t_i)| \leq \bigvee_{t_i}^{t_{i+1}}(v) = \bigvee_a^{t_{i+1}}(v) - \bigvee_a^{t_i}(v), \tag{3.2}$$

for any  $i \in \{0, \dots, n-1\}$  and by (3.2) we have

$$\begin{aligned} \left| \int_a^b p(t) dv(t) \right| &\leq \lim_{v(I_n) \rightarrow 0} \sum_{i=0}^{n-1} |p(\xi_i)| \left[ \bigvee_a^{t_{i+1}}(v) - \bigvee_a^{t_i}(v) \right] \\ &= \int_a^b |p(t)| d\left[ \bigvee_a^t(v) \right], \end{aligned}$$

and the last Riemann–Stieltjes integral exists since  $|p|$  is continuous and  $\bigvee_a^t(v)$  is monotonic nondecreasing.

The last part follows from the following Hölder type inequality

$$\left| \int_a^b g(t) dv(t) \right| \leq [v(b) - v(a)]^{\frac{1}{q}} \left[ \int_a^b |g(t)|^r dv(t) \right]^{\frac{1}{r}}, \quad q > 1, \quad \frac{1}{q} + \frac{1}{r} = 1,$$

that holds for any continuous function  $g : [a, b] \rightarrow \mathbb{R}$  and any monotonic nondecreasing function  $v : [a, b] \rightarrow \mathbb{R}$ . The details are omitted.  $\square$

The following result holds.

**Theorem 1.** *Assume that  $f, u : [a, b] \rightarrow \mathbb{R}$  are of bounded variation and such that the Riemann–Stieltjes integral  $\int_a^b f(t) du(t)$  exists. Then*

$$\begin{aligned} |D(f; u)| &\leq \frac{1}{b-a} \left[ \int_a^b \bigvee_a^s(f) (2s-a-b) d \left( \bigvee_a^s(u) \right) \right. \\ &\quad \left. + 2 \int_a^b \left( \bigvee_a^s(u) \cdot \bigvee_a^s(f) \right) ds - \bigvee_a^b(u) \int_a^b \left( \bigvee_a^s(f) \right) ds \right] \\ &\leq \frac{1}{b-a} \int_a^b \bigvee_a^s(f) (2s-a-b) d \left( \bigvee_a^s(u) \right) \\ &\quad + \frac{1}{b-a} \int_a^b \left( \bigvee_a^s(u) \cdot \bigvee_a^s(f) \right) ds \\ &\leq \frac{1}{b-a} \int_a^b \bigvee_a^s(f) (2s-a-b) d \left( \bigvee_a^s(u) \right) \\ &\quad + \bigvee_a^b(u) \cdot \bigvee_a^b(f). \end{aligned} \tag{3.3}$$

*Proof.* Utilising the identity (2.2) and the first inequality in (3.1) we have

$$\begin{aligned} |D(f; u)| &\leq \frac{1}{b-a} \int_a^b \left| \int_a^b Q(t, s) df(t) \right| d \left( \bigvee_a^s(u) \right) \\ &= \frac{1}{b-a} \int_a^b \left| \int_a^s (t-a) df(t) + \int_s^b (t-b) df(t) \right| d \left( \bigvee_a^s(u) \right) \\ &\leq \frac{1}{b-a} \int_a^b \left[ \left| \int_a^s (t-a) df(t) \right| + \left| \int_s^b (t-b) df(t) \right| \right] d \left( \bigvee_a^s(u) \right) \\ &=: I. \end{aligned} \tag{3.4}$$

Since  $f$  is of bounded variation, by the same inequality in (3.1) we have

$$\begin{aligned} \left| \int_a^s (t-a) df(t) \right| &\leq \int_a^s (t-a) d \left( \bigvee_a^t(f) \right) \\ &= \bigvee_a^s(f) \cdot (s-a) - \int_a^s \left( \bigvee_a^t(f) \right) dt \end{aligned}$$

and

$$\begin{aligned} \left| \int_s^b (t-b) df(t) \right| &\leq \int_s^b (t-b) d \bigvee_s^t(f) = \int_s^b \left( \bigvee_s^t(f) \right) dt \\ &= \int_s^b \left[ \bigvee_a^t(f) - \bigvee_a^s(f) \right] dt = \int_s^b \left( \bigvee_a^t(f) \right) dt - (b-s) \bigvee_a^s(f). \end{aligned}$$

This gives that

$$\begin{aligned} I &\leq \frac{1}{b-a} \int_a^b \left[ \bigvee_a^s(f) (s-a) - \int_a^s \left( \bigvee_a^t(f) \right) dt \right. \\ &\quad \left. + \int_s^b \left( \bigvee_a^t(f) \right) dt - (b-s) \bigvee_a^s(f) \right] d \left( \bigvee_a^s(u) \right) \\ &= \frac{1}{b-a} \int_a^b \left[ \bigvee_a^s(f) (2s-a-b) - \int_a^s \left( \bigvee_a^t(f) \right) dt \right. \\ &\quad \left. + \int_s^b \left( \bigvee_a^t(f) \right) dt \right] d \left( \bigvee_a^s(u) \right) \\ &= \frac{1}{b-a} \int_a^b (2s-a-b) \bigvee_a^s(f) d \left( \bigvee_a^s(u) \right) \\ &\quad + \frac{1}{b-a} \int_a^b \left[ \int_a^b \left( \bigvee_a^t(f) \right) dt - 2 \int_a^s \left( \bigvee_a^t(f) \right) dt \right] d \left( \bigvee_a^s(u) \right) \\ &= \frac{1}{b-a} \int_a^b (2s-a-b) \bigvee_a^s(f) d \left( \bigvee_a^s(u) \right) \\ &\quad + \frac{1}{b-a} \int_a^b \left( \bigvee_a^t(f) \right) dt \cdot \bigvee_a^b(u) \\ &\quad - \frac{2}{b-a} \int_a^b \left( \int_a^s \left( \bigvee_a^t(f) \right) dt \right) d \left( \bigvee_a^s(u) \right). \end{aligned} \tag{3.5}$$



However, integrating by parts in the Riemann–Stieltjes integral we have

$$\begin{aligned} & \int_a^b \left( \int_a^s \left( \bigvee_a^t(f) \right) dt \right) d \left( \bigvee_a^s(u) \right) \\ &= \int_a^s \left( \bigvee_a^t(f) \right) dt \cdot \bigvee_a^s(u) \Big|_a^b - \int_a^b \bigvee_a^s(u) \cdot \bigvee_a^s(f) ds \\ &= \bigvee_a^b(u) \cdot \int_a^b \left( \bigvee_a^t(f) \right) dt - \int_a^b \bigvee_a^s(u) \cdot \bigvee_a^s(f) ds. \end{aligned}$$

Inserting this value in the expression of  $I$  from (3.5) we deduce the first inequality in (3.3).

The other inequalities are obvious.  $\square$

The following result may be stated as well.

**Theorem 2.** *If  $u : [a, b] \rightarrow \mathbb{R}$  is of bounded variation and  $f : [a, b] \rightarrow \mathbb{R}$  is  $L$ -Lipschitzian, then*

$$\begin{aligned} |D(f; u)| &\leq L \left[ \frac{1}{2} (b-a) \bigvee_a^b(u) \right. \\ &\quad \left. - \frac{2}{b-a} \int_a^b \left( \bigvee_a^s(u) \right) \left( s - \frac{a+b}{2} \right) ds \right] \quad (3.6) \\ &\leq \frac{1}{2} L (b-a) \bigvee_a^b(u). \end{aligned}$$

The constant  $\frac{1}{2}$  is sharp in both inequalities.

*Proof.* It is well known that if  $p : [\alpha, \beta] \rightarrow \mathbb{R}$  is  $L$ -Lipschitzian and  $v : [\alpha, \beta] \rightarrow \mathbb{R}$  is Riemann integrable, then the Riemann–Stieltjes integral  $\int_\alpha^\beta p(s) dv(s)$  exists and  $\left| \int_\alpha^\beta p(s) dv(s) \right| \leq L \int_\alpha^\beta |p(s)| ds$ . Utilising this property, we then have

$$\begin{aligned} \left| \int_a^s (t-a) df(t) \right| &\leq L \int_a^s (t-a) dt = \frac{L}{2} (s-a)^2, \\ \left| \int_s^b (t-b) df(t) \right| &\leq L \int_s^b (b-t) dt = \frac{L}{2} (b-s)^2. \end{aligned}$$

Therefore, by relation (3.4) we have

$$I \leq \frac{L}{2(b-a)} \int_a^b \left[ (b-s)^2 + (s-a)^2 \right] d \left( \bigvee_a^s(u) \right)$$

$$\begin{aligned}
 &= \frac{L}{2(b-a)} \left[ \left[ (b-s)^2 + (s-a)^2 \right] \bigvee_a^s(u) \Big|_a^b - 2 \int_a^b \bigvee_a^s(u) (2s-a-b) ds \right] \\
 &= \frac{L}{2(b-a)} \left[ (b-a)^2 \bigvee_a^b(u) - 4 \int_a^b \bigvee_a^s(u) \left( s - \frac{a+b}{2} \right) ds \right]
 \end{aligned}$$

and the first inequality in (3.6) is proved.

To prove the last part, we use the Chebyshev inequality which states that for two nondecreasing functions  $g$  and  $h$ ,

$$\frac{1}{b-a} \int_a^b g(s) h(s) ds \geq \frac{1}{b-a} \int_a^b g(s) ds \cdot \frac{1}{b-a} \int_a^b h(s) ds.$$

Then

$$\begin{aligned}
 &\frac{1}{b-a} \int_a^b \bigvee_a^s(u) \left( s - \frac{a+b}{2} \right) ds \\
 &\geq \frac{1}{b-a} \int_a^b \left( \bigvee_a^s(u) \right) ds \cdot \frac{1}{b-a} \int_a^b \left( s - \frac{a+b}{2} \right) ds
 \end{aligned}$$

and since  $\int_a^b \left( s - \frac{a+b}{2} \right) ds = 0$ , the inequality is proved.

For the sharpness of the constant, we consider the functions  $f(t) = t - \frac{a+b}{2}$ ,  $t \in [a, b]$ , and  $u : [a, b] \rightarrow \mathbb{R}$  defined by

$$u(t) := \begin{cases} 1 & \text{if } t = a, \\ 0 & \text{if } t \in (a, b), \\ 1 & \text{if } t = b. \end{cases}$$

Then  $f$  is Lipschitzian with  $L = 1$  and  $u$  is of bounded variation on  $[a, b]$ .

We have  $\bigvee_a^s(u) = 1$ ,  $s \in (a, b)$ , and  $\bigvee_a^b(u) = 2$ . Also,

$$\begin{aligned}
 D(f; u) &= \int_a^b f(t) du(t) - \frac{u(b) - u(a)}{b-a} \int_a^b f(t) dt \\
 &= \int_a^b f(t) du(t) \\
 &= f(t) u(t) \Big|_a^b - \int_a^b u(t) df(t) = b - a
 \end{aligned}$$

and

$$\int_a^b \left( \bigvee_a^s(u) \right) \left( s - \frac{a+b}{2} \right) ds = \int_a^b \left( s - \frac{a+b}{2} \right) ds = 0.$$

Replacing the values in (3.6) we get in all sides the same quantity  $b - a$ . This shows that the constant  $\frac{1}{2}$  is the best possible in both inequalities.  $\square$

**Remark 1.** The inequality between the first and last term in (3.6) was firstly discovered by Dragomir and Fedotov in [12] where they also showed the sharpness of the constant  $\frac{1}{2}$ .

The following result may be stated as well.

**Theorem 3.** Assume that  $u : [a, b] \rightarrow \mathbb{R}$  is of bounded variation and  $f : [a, b] \rightarrow \mathbb{R}$  is monotonic nondecreasing and such that the Riemann–Stieltjes integral  $\int_a^b f(t) du(t)$  exists. Then,

$$|D(f; u)| \leq \frac{1}{b-a} \left[ \int_a^b (2s-a-b) f(s) d \left( \bigvee_a^s(u) \right) + 2 \int_a^b \left( \bigvee_a^s(u) \right) f(s) ds - \int_a^b f(s) ds \cdot \bigvee_a^b(u) \right]. \quad (3.7)$$

*Proof.* It is well known that if the Stieltjes integrals  $\int_\alpha^\beta p(t) dv(t)$  and  $\int_\alpha^\beta |p(t)| dv(t)$  exist and  $v$  is monotonic nondecreasing on  $[\alpha, \beta]$ , then

$$\left| \int_\alpha^\beta p(t) dv(t) \right| \leq \int_\alpha^\beta |p(t)| dv(t).$$

Utilising this property we then have

$$\left| \int_a^s (t-a) df(t) \right| \leq \int_a^s (t-a) df(t) = (s-a) f(s) - \int_a^s f(t) dt$$

and

$$\left| \int_s^b (t-b) df(t) \right| \leq \int_s^b (t-b) df(t) = \int_s^b f(t) dt - (b-s) f(s)$$

for any  $s \in [a, b]$ .

Utilising relation (3.4), we obtain

$$\begin{aligned} I &\leq \frac{1}{b-a} \left[ \int_a^b \left\{ (s-a) f(s) - \int_a^s f(t) dt + \int_s^b f(t) dt - (b-s) f(s) \right\} d \left( \bigvee_a^s(u) \right) \right] \\ &= \frac{1}{b-a} \left[ \int_a^b (2s-a-b) f(s) d \left( \bigvee_a^s(u) \right) + \int_a^b \left( \int_s^b f(t) dt \right) d \left( \bigvee_a^s(u) \right) - \int_a^b \left( \int_a^s f(t) dt \right) d \left( \bigvee_a^s(u) \right) \right] =: J. \end{aligned} \quad (3.8)$$

However, integrating by parts in the Riemann–Stieltjes integral, we have

$$\begin{aligned} & \int_a^b \left( \int_s^b f(t) dt \right) d \left( \bigvee_a^s(u) \right) \\ &= \left( \int_s^b f(t) dt \right) \cdot \bigvee_a^s(u) \Big|_a^b - \int_a^b \left( \bigvee_a^s(u) \right) d \left( \int_s^b f(t) dt \right) \\ &= \int_a^b \left( \bigvee_a^s(u) \right) f(s) ds \end{aligned}$$

and

$$\begin{aligned} & \int_a^b \left( \int_a^s f(t) dt \right) d \left( \bigvee_a^s(u) \right) \\ &= \left( \int_a^s f(t) dt \right) \cdot \bigvee_a^s(u) \Big|_a^b - \int_a^b \left( \bigvee_a^s(u) \right) d \left( \int_a^s f(t) dt \right) \\ &= \int_a^b f(t) dt \cdot \bigvee_a^b(u) - \int_a^b \left( \bigvee_a^s(u) \right) f(s) ds. \end{aligned}$$

Therefore,

$$\begin{aligned} J &= \frac{1}{b-a} \left[ \int_a^b (2s-a-b) f(s) d \left( \bigvee_a^s(u) \right) \right. \\ &\quad \left. + \int_a^b \left( \bigvee_a^s(u) \right) f(s) ds - \int_a^b f(t) dt \cdot \bigvee_a^b(u) + \int_a^b \left( \bigvee_a^s(u) \right) f(s) ds \right] \\ &= \frac{1}{b-a} \left[ \int_a^b (2s-a-b) f(s) d \left( \bigvee_a^s(u) \right) \right. \\ &\quad \left. + 2 \int_a^b \left( \bigvee_a^s(u) \right) f(s) ds - \int_a^b f(s) ds \cdot \bigvee_a^b(u) \right]. \end{aligned}$$

This together with inequalities (3.4) and (3.8) produces the desired result (3.7). □

#### 4. Bounds in the case when $f$ is of bounded variation

We can state the following result as well.

**Theorem 4.** *Assume that  $f : [a, b] \rightarrow \mathbb{R}$  is of bounded variation on  $[a, b]$ . If  $u : [a, b] \rightarrow \mathbb{R}$  is continuous and such that there exists constants  $L_a, L_b > 0$  and  $\alpha, \beta > 0$  with the properties*

$$|u(t) - u(a)| \leq L_a (t - a)^\alpha, \quad |u(t) - u(b)| \leq L_b (b - t)^\beta \quad (4.1)$$

for any  $t \in [a, b]$ , then

$$\begin{aligned}
 |D(f; u)| &\leq \frac{1}{b-a} L_a \left[ \int_a^b \left( \bigvee_a^t(f) \right) (t-a)^\alpha dt \right. \\
 &\quad \left. - \alpha \int_a^b \left( \bigvee_a^t(f) \right) (b-t) (t-a)^{\alpha-1} dt \right] \\
 &\quad + \frac{1}{b-a} L_b \left[ \int_a^b \left( \bigvee_a^t(f) \right) (t-a) (b-t)^{\beta-1} dt \right. \\
 &\quad \left. - \int_a^b \left( \bigvee_a^t(f) \right) (b-t)^\beta dt \right].
 \end{aligned} \tag{4.2}$$

*Proof.* Utilising the identity (2.3) and the first inequality in (3.1), we have successively,

$$\begin{aligned}
 |D(f; u)| &\leq \frac{1}{b-a} \int_a^b \left| \int_a^b Q(t, s) du(s) \right| d \left( \bigvee_a^t(f) \right) \\
 &= \frac{1}{b-a} \int_a^b \left| \int_a^t Q(t, s) du(s) + \int_t^b Q(t, s) du(s) \right| d \left( \bigvee_a^t(f) \right) \\
 &\leq \frac{1}{b-a} \int_a^b \left[ \left| \int_a^t Q(t, s) du(s) \right| + \left| \int_t^b Q(t, s) du(s) \right| \right] d \left( \bigvee_a^t(f) \right) \\
 &= \frac{1}{b-a} \int_a^b [(b-t)|u(t) - u(a)| + (t-a)|u(b) - u(t)|] d \left( \bigvee_a^t(f) \right) \\
 &=: P.
 \end{aligned}$$

Now, on making use of condition (4.1), we can state that

$$\begin{aligned}
 P &\leq \frac{1}{b-a} \int_a^b \left[ L_a (b-t) (t-a)^\alpha + L_b (t-a) (b-t)^\beta \right] d \left( \bigvee_a^t(f) \right) \\
 &= \frac{1}{b-a} \left[ L_a \int_a^b (b-t) (t-a)^\alpha d \left( \bigvee_a^t(f) \right) \right. \\
 &\quad \left. + L_b \int_a^b (t-a) (b-t)^\beta d \left( \bigvee_a^t(f) \right) \right].
 \end{aligned} \tag{4.3}$$

However,

$$\begin{aligned}
 & \int_a^b (b-t)(t-a)^\alpha d\left(\bigvee_a^t(f)\right) \\
 &= (b-t)(t-a)^\alpha \bigvee_a^t(f) \Big|_a^b - \int_a^b \left(\bigvee_a^t(f)\right) d[(b-t)(t-a)^\alpha] \\
 &= - \int_a^b \left(\bigvee_a^t(f)\right) \left[-(t-a)^\alpha + \alpha(b-t)(t-a)^{\alpha-1}\right] dt \\
 &= \int_a^b \left(\bigvee_a^t(f)\right) (t-a)^\alpha dt - \alpha \int_a^b \left(\bigvee_a^t(f)\right) (b-t)(t-a)^{\alpha-1} dt
 \end{aligned}$$

and

$$\begin{aligned}
 & \int_a^b (t-a)(b-t)^\beta d\left(\bigvee_a^t(f)\right) \\
 &= (t-a)(b-t)^\beta \bigvee_a^t(f) \Big|_a^b - \int_a^b \left(\bigvee_a^t(f)\right) d[(t-a)(b-t)^\beta] \\
 &= - \int_a^b \left(\bigvee_a^t(f)\right) \left[(b-t)^\beta - \beta(t-a)(b-t)^{\beta-1}\right] dt \\
 &= \beta \int_a^b \left(\bigvee_a^t(f)\right) (t-a)(b-t)^{\beta-1} dt - \int_a^b \left(\bigvee_a^t(f)\right) (b-t)^\beta dt,
 \end{aligned}$$

and from (4.3) we deduce the desired inequality (4.2).  $\square$

**Corollary 3.** *If  $f$  is as in Theorem 4 and  $u$  is of  $r$ -H-Hölder type, i.e.,*

$$|u(t) - u(s)| \leq H |u - t|^r \quad \text{for any } t, s \in [a, b],$$

where  $H > 0$  and  $r \in (0, 1)$  are given, then

$$\begin{aligned}
 |D(f; u)| \leq & \frac{1}{b-a} H \int_a^b \left(\bigvee_a^t(f)\right) \left\{ (t-a)^r - (b-t)^r \right. \\
 & \left. + r(b-t)^{r-1}(t-a)^{r-1} \left[ (t-a)^{1-r} - (b-t)^{1-r} \right] \right\} dt.
 \end{aligned}$$

**Remark 2.** If  $r = \frac{1}{2}$  in Corollary 3, then we obtain the inequality

$$|D(f; u)| \leq \frac{1}{b-a} H \int_a^b \left( \bigvee_a^t(f) \right) (\sqrt{t-a} - \sqrt{b-t}) \times \left( 1 + \frac{1}{2\sqrt{(b-t)(t-a)}} \right) dt.$$

The following particular result may be useful for applications.

**Corollary 4.** *If  $f$  is as in Theorem 4 and  $u : [a, b] \rightarrow \mathbb{R}$  is Lipschitzian with the constant  $K > 0$ , then*

$$|D(f; u)| \leq \frac{4}{b-a} \cdot K \int_a^b \left( t - \frac{a+b}{2} \right) \cdot \bigvee_a^t(f) dt \leq \begin{cases} K(b-a) \bigvee_a^b(f); \\ \frac{2(b-a)^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}} K \left( \int_a^b \left[ \bigvee_a^t(f) \right]^p dt \right)^{\frac{1}{p}}, \quad p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ 2K \int_a^b \left( \bigvee_a^t(f) \right) dt. \end{cases} \quad (4.4)$$

The multiplication constant 4 is the best possible.

*Proof.* The first inequality follows by Theorem 4 on choosing  $L_a = L_b = K$  and  $\alpha = \beta = 1$ .

Now, on utilising Hölder’s inequality, we have

$$\int_a^b \left( t - \frac{a+b}{2} \right) \cdot \left( \bigvee_a^t(f) \right) dt \leq \begin{cases} \sup_{t \in [a,b]} \left( \bigvee_a^t(f) \right) \int_a^b \left| t - \frac{a+b}{2} \right| dt; \\ \left( \int_a^b \left[ \bigvee_a^t(f) \right]^p dt \right)^{\frac{1}{p}} \left( \int_a^b \left| t - \frac{a+b}{2} \right|^q dt \right)^{\frac{1}{q}}, \quad p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \int_a^b \left( \bigvee_a^t(f) \right) dt \sup_{t \in [a,b]} \left| t - \frac{a+b}{2} \right|. \end{cases} \quad (4.5)$$

However,  $\sup_{t \in [a,b]} \left| t - \frac{a+b}{2} \right| = \frac{b-a}{2}$  and

$$\begin{aligned} \int_a^b \left| t - \frac{a+b}{2} \right|^q dt &= 2 \int_{\frac{a+b}{2}}^b \left( t - \frac{a+b}{2} \right)^q dt \\ &= \frac{(b-a)^{q+1}}{2^q (q+1)}, \quad q \geq 1, \end{aligned}$$

and by (4.5) we deduce

$$\begin{aligned} \int_a^b \left( t - \frac{a+b}{2} \right) \cdot \bigvee_a^t(f) dt &\leq \begin{cases} \frac{(b-a)^2}{4} \bigvee_a^t(f); \\ \frac{(b-a)^{1+\frac{1}{q}}}{2^{(q+1)\frac{1}{q}}} \left( \int_a^b \left[ \bigvee_a^t(f) \right]^p dt \right)^{\frac{1}{p}}, \quad p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{b-a}{2} \int_a^b \left( \bigvee_a^t(f) \right) dt, \end{cases} \end{aligned}$$

and the second part is proved.

To prove the sharpness of the constant 4 in the first inequality in (4.4) assume that there exists  $A > 0$  such that

$$|D(f; u)| \leq \frac{A}{b-a} \cdot K \int_a^b \left( t - \frac{a+b}{2} \right) \cdot \bigvee_a^t(f) dt, \tag{4.6}$$

provided that  $f$  is of bounded variation and  $u$  is  $K$ -Lipschitzian.

Let  $f : [a, b] \rightarrow \mathbb{R}$ ,

$$f(t) = \begin{cases} 0 & \text{if } t \in [a, \frac{a+b}{2}], \\ k & \text{if } t \in (\frac{a+b}{2}, b], \end{cases}$$

with  $k > 0$ . Then

$$\bigvee_a^t(f) = \begin{cases} 0 & \text{if } t \in [a, \frac{a+b}{2}], \\ k & \text{if } t \in (\frac{a+b}{2}, b]. \end{cases}$$

Also, we have

$$\begin{aligned} \int_a^b \left( t - \frac{a+b}{2} \right) \cdot \bigvee_a^t(f) dt &= \int_{\frac{a+b}{2}}^b \left( t - \frac{a+b}{2} \right) k dt \\ &= \frac{k(b-a)^2}{8}. \end{aligned}$$



Consider  $u : [a, b] \rightarrow \mathbb{R}$ ,  $u(t) = \left|t - \frac{a+b}{2}\right|$ . Then  $u$  is  $K$ -Lipschitzian with  $K = 1$ . Also,

$$\begin{aligned} D(f; u) &= \int_a^b f(t) du(t) - \frac{u(b) - u(a)}{b - a} \int_a^b f(t) dt \\ &= k \int_{\frac{a+b}{2}}^b du(t) = k \left[ u(b) - u\left(\frac{a+b}{2}\right) \right] \\ &= \frac{(b-a)k}{2}. \end{aligned}$$

Substituting these values into (4.6) produces the inequality

$$\frac{(b-a)k}{2} \leq \frac{A}{b-a} \cdot \frac{k(b-a)^2}{8},$$

which implies that  $A \geq 4$ . □

## 5. Inequalities for $(l, L)$ -Lipschitzian functions

The following simple lemma holds.

**Lemma 4.** *Let  $u : [a, b] \rightarrow \mathbb{R}$  and  $l, L \in \mathbb{R}$  with  $L > l$ . The following statements are equivalent:*

- (i) *The function  $u - \frac{l+L}{2} \cdot e$ , where  $e(t) = t$ ,  $t \in [a, b]$  is  $\frac{1}{2}(L-l)$ -Lipschitzian;*
- (ii) *We have the inequalities*

$$l \leq \frac{u(t) - u(s)}{t - s} \leq L \quad \text{for each } t, s \in [a, b], t \neq s;$$

- (iii) *We have the inequalities*

$$l(t-s) \leq u(t) - u(s) \leq L(t-s) \quad \text{for each } t, s \in [a, b] \quad \text{with } t > s.$$

The proof is obvious and we omit the details.

**Definition 1** (see also [14]). The function  $u : [a, b] \rightarrow \mathbb{R}$  which satisfies one of the equivalent conditions (i)–(iii) from Lemma 4 is said to be  $(l, L)$ -Lipschitzian on  $[a, b]$ . If  $L > 0$  and  $l = -L$ , then  $(-L, L)$ -Lipschitzian means  $L$ -Lipschitzian in the classical sense.

The following result can be stated.

**Theorem 5.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be a function of bounded variation and  $u : [a, b] \rightarrow \mathbb{R}$  an  $(l, L)$ -Lipschitzian function. Then

$$\begin{aligned}
 |D(f; u)| &\leq \frac{2}{b-a} (L-l) \int_a^b \left(t - \frac{a+b}{2}\right) \cdot \bigvee_a^t(f) dt \\
 &\leq \begin{cases} \frac{1}{2} (L-l) (b-a) \bigvee_a^b(f); \\ \frac{(b-a)^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}} (L-l) \left(\int_a^b \left[\bigvee_a^t(f)\right]^p dt\right)^{\frac{1}{p}}, & p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ (L-l) \int_a^b \left(\bigvee_a^t(f)\right) dt. \end{cases}
 \end{aligned}$$

The constant 2 in the first inequality is sharp.

*Proof.* Observe that

$$\begin{aligned}
 &D\left(f; u - \frac{l+L}{2} \cdot e\right) \\
 &= \int_a^b \left(f(t) - \frac{1}{b-a} \int_a^b f(s) ds\right) d\left[u(t) - \frac{l+L}{2} \cdot t\right] \\
 &= \int_a^b \left(f(t) - \frac{1}{b-a} \int_a^b f(s) ds\right) du(t) \\
 &\quad - \frac{l+L}{2} \int_a^b \left(f(t) - \frac{1}{b-a} \int_a^b f(s) ds\right) dt \\
 &= D(f; u).
 \end{aligned}$$

Now, applying Corollary 4 for the function  $u - \frac{l+L}{2}e$ , which is  $\frac{1}{2}(L-l)$ -Lipschitzian, we get

$$\begin{aligned}
 \left|D\left(f; u - \frac{l+L}{2}e\right)\right| &\leq \frac{4}{b-a} \cdot \frac{1}{2} (L-l) \int_a^b \left(t - \frac{a+b}{2}\right) \cdot \bigvee_a^t(f) dt \\
 &= \frac{2}{b-a} (L-l) \int_a^b \left(t - \frac{a+b}{2}\right) \cdot \bigvee_a^t(f) dt
 \end{aligned}$$

and the theorem is proved. □

The second result may be stated as follows.

**Theorem 6.** Let  $u : [a, b] \rightarrow \mathbb{R}$  be a function of bounded variation. If  $f : [a, b] \rightarrow \mathbb{R}$  is  $(\phi, \Phi)$ -Lipschitzian with  $\Phi > \phi$ , then

$$\begin{aligned} & \left| D(f; u) - \frac{\phi + \Phi}{2} \left[ \frac{u(b) + u(a)}{2} (b - a) - \int_a^b u(t) dt \right] \right| \\ & \leq \frac{1}{2} (\Phi - \phi) \cdot \left[ \frac{1}{2} (b - a) \bigvee_a^b(u) \right. \\ & \quad \left. - \frac{2}{b - a} \int_a^b \left( \bigvee_a^s(u) \right) \left( s - \frac{a + b}{2} \right) ds \right] \\ & \leq \frac{1}{4} \cdot (\Phi - \phi) (b - a) \bigvee_a^b(u). \end{aligned} \tag{5.1}$$

The constant  $\frac{1}{2}$  in front of  $(\Phi - \phi)$  and  $\frac{1}{4}$  are the best possible.

*Proof.* Observe that

$$\begin{aligned} & D\left(f - \frac{\phi + \Phi}{2} \cdot e; u\right) \\ & = \int_a^b \left[ f(t) - \frac{\phi + \Phi}{2} \cdot t - \frac{1}{b - a} \int_a^b \left( f(s) - \frac{\phi + \Phi}{2} \cdot s \right) ds \right] du(t) \\ & = \int_a^b \left[ f(t) - \frac{1}{b - a} \int_a^b f(s) ds - \left( \frac{\phi + \Phi}{2} t - \frac{1}{b - a} \int_a^b \frac{\phi + \Phi}{2} \cdot s ds \right) \right] du(t) \\ & = D(f; u) - \frac{\phi + \Phi}{2} \int_a^b \left( t - \frac{1}{b - a} \int_a^b s ds \right) du(t) \\ & = D(f; u) - \frac{\phi + \Phi}{2} \int_a^b \left( t - \frac{a + b}{2} \right) du(t). \end{aligned}$$

Integrating by parts in the Riemann–Stieltjes integral we have

$$\int_a^b \left( t - \frac{a + b}{2} \right) du(t) = \frac{u(b) + u(a)}{2} (b - a) - \int_a^b u(t) dt.$$

Then

$$D\left(f - \frac{\phi + \Phi}{2} e; u\right) = D(f; u) - \frac{\phi + \Phi}{2} \left[ \frac{u(b) + u(a)}{2} (b - a) - \int_a^b u(t) dt \right].$$

Now, on applying Theorem 2 for the function  $f - \frac{\phi + \Phi}{2} e$  which is  $\frac{1}{2}(L - l)$ -Lipschitzian, we deduce the desired result (5.1).  $\square$

### 6. Applications for the Chebyshev functional

If we choose  $u(t) := \int_a^t g(\tau) d\tau$ ,  $t \in [a, b]$ , where  $g : [a, b] \rightarrow \mathbb{R}$  is Lebesgue integrable on  $[a, b]$ , then we have the equality

$$C(f; g) = \frac{1}{b-a} D(f; u).$$

Also,  $u$  is of bounded variation on any subinterval  $[a, s]$ ,  $s \in [a, b]$ , and if  $g$  is continuous on  $[a, b]$ , then

$$\bigvee_a^s(u) = \int_a^s |g(\tau)| d\tau, \quad s \in [a, b].$$

If  $f$  is of bounded variation on  $[a, b]$ , then on utilising the inequality (3.3) we have

$$\begin{aligned} |C(f; g)| &\leq \frac{1}{(b-a)^2} \left[ \int_a^b (2s-a-b) |g(s)| \bigvee_a^s(f) ds \right. \\ &\quad \left. + 2 \int_a^b \left( \int_a^s |g(\tau)| d\tau \right) \bigvee_a^s(f) ds - \int_a^b |g(\tau)| d\tau \cdot \int_a^b \left( \bigvee_a^s(f) \right) ds \right] \\ &\leq \frac{1}{(b-a)^2} \int_a^b (2s-a-b) |g(s)| \bigvee_a^s(f) ds \\ &\quad + \frac{1}{(b-a)^2} \int_a^b \left( \int_a^s |g(\tau)| d\tau \right) \bigvee_a^s(f) ds \\ &\leq \frac{1}{(b-a)^2} \int_a^b (2s-a-b) |g(s)| \bigvee_a^s(f) ds + \frac{1}{b-a} \int_a^b |g(\tau)| d\tau \cdot \bigvee_a^b(f). \end{aligned}$$

Now, if  $f$  is monotonic nondecreasing, then by (3.7) we have

$$\begin{aligned} |C(f; g)| &\leq \frac{1}{(b-a)^2} \left[ \int_a^b (2s-a-b) f(s) |g(s)| ds \right. \\ &\quad \left. + 2 \int_a^b \left( \int_a^s |g(\tau)| d\tau \right) f(s) ds - \int_a^b f(s) ds \cdot \int_a^b |g(\tau)| d\tau \right]. \end{aligned}$$

The case where  $f$  is  $L$ -Lipschitzian provides via (3.6) a simpler inequality

$$\begin{aligned} |C(f; g)| &\leq L \left[ \frac{1}{2} \int_a^b |g(\tau)| d\tau \right. \\ &\quad \left. - \frac{2}{(b-a)^2} \int_a^b \left( \int_a^s |g(\tau)| d\tau \right) \left( s - \frac{a+b}{2} \right) ds \right] \\ &\leq \frac{1}{2} L \int_a^b |g(s)| ds. \end{aligned}$$

Now, if  $f$  is of bounded variation and  $|g|$  is bounded above by  $M$ , i.e.,  $|g(t)| \leq M$  for a.e.  $t \in [a, b]$ , then by (4.4) we have

$$|C(f; g)| \leq \frac{4}{(b-a)^2} M \int_a^b \left(t - \frac{a+b}{2}\right) \bigvee_a^t(f) dt$$

$$\leq \begin{cases} M \bigvee_a^b(f); \\ \frac{2(b-a)^{\frac{1}{q}-1}}{(q+1)^{\frac{1}{q}}} M \left( \int_a^b \left[ \bigvee_a^t(f) \right]^p dt \right)^{\frac{1}{p}}, \quad p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{2M}{b-a} \int_a^b \left( \bigvee_a^t(f) \right) dt. \end{cases} \quad (6.1)$$

The constant 4 in (6.1) is the best possible.

Finally, if  $-\infty < \phi \leq g(t) \leq \Phi$  for a.e.  $t \in [a, b]$ , then  $\left|g(t) - \frac{\phi + \Phi}{2}\right| \leq \frac{1}{2}(\Phi - \phi)$ , and since

$$C\left(f; g - \frac{\phi + \Phi}{2}\right) = C(f; g),$$

by (6.1) we deduce the inequalities

$$|C(f; g)| \leq \frac{2}{(b-a)^2} (\Phi - \phi) \int_a^b \left(t - \frac{a+b}{2}\right) \bigvee_a^t(f) dt$$

$$\leq \begin{cases} \frac{1}{2} (\Phi - \phi) \bigvee_a^b(f); \\ \frac{(b-a)^{\frac{1}{q}-1}}{(q+1)^{\frac{1}{q}}} (\Phi - \phi) \left( \int_a^b \left[ \bigvee_a^t(f) \right]^p dt \right)^{\frac{1}{p}}, \quad p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{\Phi - \phi}{b-a} \int_a^b \left( \bigvee_a^t(f) \right) dt. \end{cases}$$

The constant 2 in the first inequality is the best possible.

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