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DE GRUYTER OPEN Open Math. 2015; 13: 631–638

Open Mathematics **Open Access**

Research Article

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DOI 10.1515/math-2015-0061
Received July 20, 2015; accepted September 10, 2015.

Abstract: We establish in this paper some Jensen's type inequalities for functions defined by power series with nonnegative coefficients. Applications for functions of selfadjoint operators on complex Hilbert spaces are provided as well.

Keywords: Jensen's inequality, Measurable functions, Lebesgue integral, Selfadjoint operators, Functions of selfadjoint operators

MSC: 26D15, 26D20, 47A63

1 Introduction

In 1906, J. L. W. V. Jensen [13] has proved the following remarkable inequality

$$f\left(\frac{\sum_{i=1}^{n} p_i x_i}{\sum_{i=1}^{n} p_i}\right) \leq \frac{\sum_{i=1}^{n} p_i f(x_i)}{\sum_{i=1}^{n} p_i} \tag{1}$$

for a convex function $f : I \subset R \to R$ in the interval I, elements $x_i \in I$ and nonnegative numbers p_i, $i \in \{1, ..., n\}$ with $\sum_{i=1}^{n} p_i > 0$. This inequality is important in various fields of mathematics due to the fact that one can obtain from it other important inequalities such as the triangle inequality, Hölder's inequality, Ky Fan's inequality, to name only a few.

Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space consisting of a set Ω, a σ – algebra $\mathcal{A}$ of subsets of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R} \cup \{\infty\}$. Assume, for simplicity, that $\int_\Omega d\mu = 1$. Consider the *Lebesgue space*

$$L(\Omega, \mu) := \{f : \Omega \to \mathbb{R},\ f \text{ is } \mu\text{-measurable and } \int_\Omega |f(t)|\, d\mu(t) < \infty\}.$$

For simplicity of notation we write everywhere in the sequel $\int_\Omega w d\mu$ instead of $\int_\Omega w(t)\, d\mu(t)$.

In order to provide a reverse of the celebrated Jensen's integral inequality for convex functions, the author obtained in [4] and [7] the following result:

Theorem 1.1. *Let $\Phi : [m, M] \subset \mathbb{R} \to \mathbb{R}$ be a differentiable convex function on (m, M) and $f : \Omega \to [m, M]$ so that $\Phi \circ f,\ f,\ \Phi' \circ f,\ (\Phi' \circ f) f \in L(\Omega, \mu)$. Then we have the inequality:*

$$0 \leq \int_\Omega \Phi \circ f d\mu - \Phi\left(\int_\Omega f d\mu\right) \leq \int_\Omega (\Phi' \circ f) f d\mu - \int_\Omega \Phi' \circ f d\mu \int_\Omega f d\mu \tag{2}$$

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$$\le \frac{1}{2}\left[\Phi'(M)-\Phi'(m)\right]\int_\Omega \left|f-\int_\Omega f d\mu\right| d\mu$$
$$\le \frac{1}{2}\left[\Phi'(M)-\Phi'(m)\right]\left[\int_\Omega f^2 d\mu-\left(\int_\Omega f d\mu\right)^2\right]^{\frac{1}{2}}$$
$$\le \frac{1}{4}\left[\Phi'(M)-\Phi'(m)\right](M-m).$$

Remark 1.2. *We notice that the inequality between the first and the second term in (2) in the discrete case was proved in 1994 by Dragomir & Ionescu, see [9].*

For other recent reverses of Jensen inequality and applications to divergence measures see [6], [7] and [8].

Motivated by the above results we establish in this paper some Jensen's type inequalities for functions defined by power series with nonnegative coefficients. Applications for functions of selfadjoint operators on complex Hilbert spaces are provided as well.

2 Results

The most important power series with nonnegative coefficients are:

$$\exp(z)=\sum_{n=0}^{\infty}\frac{1}{n!}z^n,\ z\in\mathbb{C},\ \frac{1}{1-z}=\sum_{n=0}^{\infty}z^n,\ z\in D(0,1),\tag{3}$$
$$\ln\frac{1}{1-z}=\sum_{n=1}^{\infty}\frac{1}{n}z^n,\ z\in D(0,1),\ \cosh z=\sum_{n=0}^{\infty}\frac{1}{(2n)!}z^{2n},\ z\in\mathbb{C},$$
$$\sinh z=\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}z^{2n+1},\ z\in\mathbb{C},$$
$$\frac{1}{2}\ln\left(\frac{1+z}{1-z}\right)=\sum_{n=1}^{\infty}\frac{1}{2n-1}z^{2n-1},\ z\in D(0,1),$$

where by $D(0,R)$ we denote the open disk centered in 0 with radius $R>0$.

The following results that improve Jensen inequality as well as provide some reverse inequalities can be stated:

Theorem 2.1. *Let $\Phi(z)=\sum_{n=0}^{\infty}a_n z^n$ be a power series with nonnegative coefficients and convergent on $D(0,R)$ with $R>0$ or $R=\infty$. Assume that $f:\Omega\to\mathbb{R}$ is μ-measurable and with $0<f(u)<R$ for μ-almost every u in Ω and such that $\Phi\circ f$, $(\Phi'\circ f)f$, $(\Phi'\circ f)f^{-1}\in L(\Omega,\mu)$. Then we have the inequalities*

$$0\le\frac{1}{2}\left[\int_\Omega f^2 d\mu-\left(\int_\Omega f d\mu\right)^2\right]\Phi''(0)\tag{4}$$
$$\le\frac{1}{2}\left[\int_\Omega f^2 d\mu-\left(\int_\Omega f d\mu\right)^2\right]\frac{\Phi'\left(\int_\Omega f d\mu\right)-\Phi'(0)}{\int_\Omega f d\mu}\le\int_\Omega\Phi\circ f d\mu-\Phi\left(\int_\Omega f d\mu\right)$$
$$\le\frac{1}{2}\left[\int_\Omega\left[\Phi'\circ f-\Phi'(0)\right]f d\mu-\left(\int_\Omega f d\mu\right)^2\int_\Omega\left[\Phi'\circ f-\Phi'(0)\right]f^{-1}d\mu\right]$$
$$\le\frac{1}{2}\left[\int_\Omega\left(\Phi''\circ f\right)f^2 d\mu-\Phi''(0)\left(\int_\Omega f d\mu\right)^2\right].$$

Proof. If $g : I \to \mathbb{R}$ is a differentiable convex function on the interior $\mathring{I}$ of the interval I then we have the *gradient inequality*

$$g'(t)(t-s) \geq g(t) - g(s) \geq g'(s)(t-s) \tag{5}$$

for any $t, s \in \mathring{I}$.

If we write the inequality (5) for the power function $g(t) = t^r$, $r \geq 1$ on the interval $(0, \infty)$, then we have

$$rt^{r-1}(t-s) \geq g(t) - g(s) \geq rs^{r-1}(t-s) \tag{6}$$

for any $s, t > 0$.

Let $n \geq 2$ be a natural number, then $g(t) = t^{n/2}$ is convex on $(0, \infty)$ and by taking $t = x^2$ and $s = y^2$ then we get from (6) that

$$\frac{n}{2} x^{n-2}\left(x^2 - y^2\right) \geq x^n - y^n \geq \frac{n}{2} y^{n-2}\left(x^2 - y^2\right) \tag{7}$$

for any $n \geq 2$ and any $x, y \geq 0$.

From (7) we have

$$\begin{aligned}\frac{n}{2}[f(u)]^{n-2}\left[[f(u)]^2 - \left(\int_\Omega f d\mu\right)^2\right] &\geq [f(u)]^n - \left(\int_\Omega f d\mu\right)^n \\ &\geq \frac{n}{2}\left(\int_\Omega f d\mu\right)^{n-2}\left[[f(u)]^2 - \left(\int_\Omega f d\mu\right)^2\right]\end{aligned}$$

for any $n \geq 2$, or, equivalently

$$\begin{aligned}\frac{n}{2}[f(u)]^n - \frac{n}{2}[f(u)]^{n-2}\left(\int_\Omega f d\mu\right)^2 &\geq [f(u)]^n - \left(\int_\Omega f d\mu\right)^n \\ &\geq \frac{n}{2}\left(\int_\Omega f d\mu\right)^{n-2}\left[[f(u)]^2 - \left(\int_\Omega f d\mu\right)^2\right]\end{aligned} \tag{8}$$

for any $n \geq 2$.

Integrating the inequality over u on Ω we get

$$\begin{aligned}\frac{n}{2}\int_\Omega f^n d\mu - \frac{n}{2}\int_\Omega f^{n-2} d\mu\left(\int_\Omega f d\mu\right)^2 &\geq \int_\Omega f^n d\mu - \left(\int_\Omega f d\mu\right)^n \\ &\geq \frac{n}{2}\left(\int_\Omega f d\mu\right)^{n-2}\left[\int_\Omega f^2 d\mu - \left(\int_\Omega f d\mu\right)^2\right]\end{aligned} \tag{9}$$

for any $n \geq 2$, which is an inequality of interest in itself.

Let $m \geq 2$. If we multiply (9) by $a_n \geq 0$ and sum over n from 2 to m we get

$$\begin{aligned}&\frac{1}{2}\int_\Omega\left(\sum_{n=2}^m n a_n f^n\right) d\mu - \frac{1}{2}\int_\Omega\left(\sum_{n=2}^m n a_n f^{n-2}\right) d\mu\left(\int_\Omega f d\mu\right)^2 \\ &\geq \int_\Omega\left(\sum_{n=2}^m a_n f^n\right) d\mu - \sum_{n=2}^m a_n\left(\int_\Omega f d\mu\right)^n \\ &\geq \frac{1}{2}\sum_{n=2}^m n a_n\left(\int_\Omega f d\mu\right)^{n-2}\left[\int_\Omega f^2 d\mu - \left(\int_\Omega f d\mu\right)^2\right].\end{aligned} \tag{10}$$

Observe that

$$\int_\Omega \left(\sum_{n=0}^m a_n f^n\right) d\mu - \sum_{n=0}^m a_n \left(\int_\Omega f d\mu\right)^n = \int_\Omega a_0 d\mu - a_0 \left(\int_\Omega f d\mu\right)^0 + \int_\Omega (a_1 f)\, d\mu - a_1 \left(\int_\Omega f d\mu\right)^1$$
$$+ \int_\Omega \left(\sum_{n=2}^m a_n f^n\right) d\mu - \sum_{n=2}^m a_n \left(\int_\Omega f d\mu\right)^n = \int_\Omega \left(\sum_{n=2}^m a_n f^n\right) d\mu - \sum_{n=2}^m a_n \left(\int_\Omega f d\mu\right)^n$$

for any $m \geq 2$.

From (10) we get

$$\frac{1}{2}\int_\Omega \left(\sum_{n=2}^m n a_n f^n\right) d\mu - \frac{1}{2}\int_\Omega \left(\sum_{n=2}^m n a_n f^{n-2}\right) d\mu \left(\int_\Omega f d\mu\right)^2 \tag{11}$$
$$\geq \int_\Omega \left(\sum_{n=0}^m a_n f^n\right) d\mu - \sum_{n=0}^m a_n \left(\int_\Omega f d\mu\right)^n \geq \frac{1}{2}\sum_{n=2}^m n a_n \left(\int_\Omega f d\mu\right)^{n-2} \left[\int_\Omega f^2 d\mu - \left(\int_\Omega f d\mu\right)^2\right],$$

for any $m \geq 2$.

Observe that the power series $\sum_{n=2}^\infty n a_n z^n$ and $\sum_{n=2}^m n a_n z^{n-2}$ are convergent on $D(0, R)$ and

$$\sum_{n=2}^\infty n a_n z^n = z \sum_{n=2}^\infty n a_n z^{n-1} = z \left(\sum_{n=1}^\infty n a_n z^{n-1} - a_1\right) = z \left(\Phi'(z) - \Phi'(0)\right),\ z \in D(0, R)$$

while

$$\sum_{n=2}^\infty n a_n z^{n-2} = \frac{1}{z} \sum_{n=2}^\infty n a_n z^{n-1} = \frac{\Phi'(z) - \Phi'(0)}{z},\ z \in D(0, R) \setminus \{0\}.$$

Since $0 < f(u) < R$ for μ-almost every u in Ω, then $0 < \int_\Omega f d\mu < R$, the series $\sum_{n=2}^m n a_n [f(u)]^n$, $\sum_{n=2}^m n a_n [f(u)]^{n-2}$, $\sum_{n=0}^m a_n [f(u)]^n$ are convergent for μ-almost every u in Ω, $\sum_{n=0}^\infty a_n \left(\int_\Omega f d\mu\right)^n$ and $\sum_{n=2}^\infty n a_n \left(\int_\Omega f d\mu\right)^{n-2}$ are convergent and

$$\sum_{n=2}^m n a_n [f(u)]^n = f(u) \left(\Phi'(f(u)) - \Phi'(0)\right),$$

$$\sum_{n=2}^m n a_n [f(u)]^{n-2} = \frac{\Phi'(f(u)) - \Phi'(0)}{f(u)},\quad \sum_{n=0}^m a_n [f(u)]^n = \Phi(f(u))$$

for μ-almost every u in Ω.

We also have

$$\sum_{n=0}^\infty a_n \left(\int_\Omega f d\mu\right)^n = \Phi\left(\int_\Omega f d\mu\right)$$

and

$$\sum_{n=2}^\infty n a_n \left(\int_\Omega f d\mu\right)^{n-2} = \frac{\Phi'\left(\int_\Omega f d\mu\right) - \Phi'(0)}{\int_\Omega f d\mu}.$$

By taking the limit in (11) over $m \to \infty$, interchanging the limit with the integral, we get the third and fourth inequalities in (4).

Since Φ' is also a convex function on $(0, R)$ then we have by (5)

$$\Phi'\left(\int_\Omega f d\mu\right) - \Phi'(0) \geq \Phi''(0) \int_\Omega f d\mu$$

and since $\int_\Omega f d\mu > 0$, we obtain the second inequality in (4). The first inequality is obvious.

By the inequality (5) applied for Φ' we also have

$$\Phi''(0) \le \frac{\Phi'(f(u)) - \Phi'(0)}{f(u)} \le \Phi''(f(u))$$

for μ-almost every u in Ω.

This implies that

$$\int_\Omega \left(\Phi' \circ f - \Phi'(0)\right) f^{-1} d\mu \ge \Phi''(0)$$

and

$$\int_\Omega \left(\Phi' \circ f - \Phi'(0)\right) f d\mu \le \int_\Omega \left(\Phi'' \circ f\right) f^2 d\mu,$$

which prove the fifth inequality in (4). □

Remark 2.2. *Let $\Phi(z) = \sum_{n=0}^\infty a_n z^n$ be a power series with nonnegative coefficients and convergent on $D(0,R)$ with $R > 0$ or $R = \infty$. If $x_i \in (0,R)$ and $w_i \ge 0$ $(i = 1,\dots,n)$ with $W_n := \sum_{i=1}^n w_i = 1$, then we have the inequalities*

$$\begin{aligned} 0 &\le \frac{1}{2}\left[\sum_{i=1}^n w_i x_i^2 - \left(\sum_{i=1}^n w_i x_i\right)^2\right]\Phi''(0) \qquad (12)\\ &\le \frac{1}{2}\left[\sum_{i=1}^n w_i x_i^2 - \left(\sum_{i=1}^n w_i x_i\right)^2\right]\frac{\Phi'\left(\sum_{i=1}^n w_i x_i\right) - \Phi'(0)}{\sum_{i=1}^n w_i x_i}\\ &\le \sum_{i=1}^n w_i \Phi(x_i) - \Phi\left(\sum_{i=1}^n w_i x_i\right)\\ &\le \frac{1}{2}\left[\sum_{i=1}^n w_i x_i \left[\Phi'(x_i) - \Phi'(0)\right] - \sum_{i=1}^n \frac{w_i}{x_i}\left[\Phi'(x_i) - \Phi'(0)\right]\left(\sum_{i=1}^n w_i x_i\right)^2\right]\\ &\le \frac{1}{2}\left[\sum_{i=1}^n w_i x_i^2 \Phi''(x_i) - \Phi''(0)\left(\sum_{i=1}^n w_i x_i\right)^2\right]. \end{aligned}$$

We have the following particular inequalities of interest.

Corollary 2.3. *Assume that $f : \Omega \to \mathbb{R}$ is μ-measurable and with $0 < f(u)$ for μ-almost every u in Ω and such that $\exp \circ f$, $(\exp \circ f) f$, $(\exp \circ f) f^{-1} \in L(\Omega,\mu)$. Then we have the inequalities*

$$\begin{aligned} 0 &\le \frac{1}{2}\left[\int_\Omega f^2 d\mu - \left(\int_\Omega f d\mu\right)^2\right] \le \frac{1}{2}\left[\int_\Omega f^2 d\mu - \left(\int_\Omega f d\mu\right)^2\right]\frac{\exp\left(\int_\Omega f d\mu\right) - 1}{\int_\Omega f d\mu} \qquad (13)\\ &\le \int_\Omega (\exp \circ f)\, d\mu - \exp\left(\int_\Omega f d\mu\right)\\ &\le \frac{1}{2}\left[\int_\Omega (\exp \circ f - 1) f d\mu - \left(\int_\Omega f d\mu\right)^2 \int_\Omega (\exp \circ f - 1) f^{-1} d\mu\right]\\ &\le \frac{1}{2}\left[\int_\Omega f^2 (\exp \circ f)\, d\mu - \left(\int_\Omega f d\mu\right)^2\right]. \end{aligned}$$

The inequality (13) follows by (4) for $\Phi(z) = \exp(z) = \sum_{n=0}^{\infty} \frac{1}{n!} z^n$, $z \in \mathbb{C}$.

If we use the inequality (4) for $\Phi(z) = \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$, $z \in D(0,1)$, then we can state:

Corollary 2.4. *Assume that $f : \Omega \to \mathbb{R}$ is μ-measurable and with $0 < f(u) < 1$ for μ-almost every u in Ω and such that $(1-f)^{-1}, (1-f)^{-2} f, (1-f)^{-2} f^{-1} \in L(\Omega, \mu)$. Then we have the inequalities*

$$\begin{aligned} 0 &\leq \int_{\Omega} f^2 d\mu - \left(\int_{\Omega} f d\mu \right)^2 \qquad (14)\\ &\leq \frac{1}{2} \left[\int_{\Omega} f^2 d\mu - \left(\int_{\Omega} f d\mu \right)^2 \right] \frac{2 - \int_{\Omega} f d\mu}{\left(1 - \int_{\Omega} f d\mu\right)^2} \\ &\leq \int_{\Omega} (1-f)^{-1} d\mu - \left(1 - \int_{\Omega} f d\mu \right)^{-1} \\ &\leq \frac{1}{2} \left[\int_{\Omega} \frac{(2-f) f^2}{(1-f)^2} d\mu - \left(\int_{\Omega} f d\mu \right)^2 \int_{\Omega} \frac{2-f}{(1-f)^2} d\mu \right] \\ &\leq \int_{\Omega} \frac{f^2}{(1-f)^3} d\mu - \left(\int_{\Omega} f d\mu \right)^2 . \end{aligned}$$

3 Applications for functions of selfadjoint operators

Let A be a selfadjoint operator on the complex Hilbert space $(H, \langle .,. \rangle)$ with the spectrum $Sp(A)$ included in the interval $[m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its *spectral family*. Then for any continuous function $f : [m, M] \to \mathbb{R}$, it is well known that we have the following *spectral representation in terms of the Riemann-Stieltjes integral* (see for instance [12, p. 257]):

$$\langle f(A) x, y \rangle = \int_{m-0}^{M} f(\lambda) \, d(\langle E_\lambda x, y \rangle), \qquad (15)$$

for any $x, y \in H$.

The function $g_{x,y}(\lambda) := \langle E_\lambda x, y \rangle$ is of *bounded variation* on the interval $[m, M]$ and $g_{x,y}(m-0) = 0$ while $g_{x,y}(M) = \langle x, y \rangle$ for any $x, y \in H$. It is also well known that $g_x(\lambda) := \langle E_\lambda x, x \rangle$ is *monotonic nondecreasing* and *right continuous* on $[m, M]$ for any $x \in H$.

The following result holds:

Theorem 3.1. *Let $\Phi(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with nonnegative coefficients and convergent on $D(0, R)$ with $R > 0$ or $R = \infty$ and A a bonded selfadjoint operator on the Hilbert space H with $m = \min Sp(A)$ and $M = \max Sp(A)$. Assume that $f : I \to \mathbb{R}$ is continuous on I with $[m, M] \subset \mathring{I}$ and $0 < f(u) < R$ for any $u \in \mathring{I}$. Then we have the inequalities*

$$\begin{aligned} 0 &\leq \frac{1}{2} \left[\left\langle f^2(A) x, x \right\rangle - \langle f(A) x, x \rangle^2 \right] \Phi''(0) \qquad (16)\\ &\leq \frac{1}{2} \left[\left\langle f^2(A) x, x \right\rangle - \langle f(A) x, x \rangle^2 \right] \frac{\Phi'(\langle f(A) x, x \rangle) - \Phi'(0)}{\langle f(A) x, x \rangle} \\ &\leq \langle (\Phi \circ f)(A) x, x \rangle - \Phi(\langle f(A) x, x \rangle) \\ &\leq \frac{1}{2} \left[\left\langle \left((\Phi' \circ f)(A) - \Phi'(0) I \right) f(A) x, x \right\rangle \right. \end{aligned}$$

$$-\left\langle\left(\left(\Phi'\circ f\right)(A)-\Phi'(0)\,I\right)f^{-1}(A)\,x,x\right\rangle\langle f(A)\,x,x\rangle^{2}\Big]$$
$$\le\frac{1}{2}\left[\left\langle\left(\Phi''\circ f\right)(A)\,f^{2}(A)\,x,x\right\rangle-\Phi''(0)\left(\langle f(A)\,x,x\rangle\right)^{2}\right]$$

for any $x\in H$, $\|x\|=1$.

In particular, if $0<m\le M<R$, *then*

$$\begin{aligned}0&\le\frac{1}{2}\left[\left\langle A^{2}x,x\right\rangle-\langle Ax,x\rangle^{2}\right]\Phi''(0)\\&\le\frac{1}{2}\left[\left\langle A^{2}x,x\right\rangle-\langle Ax,x\rangle^{2}\right]\frac{\Phi'(\langle Ax,x\rangle)-\Phi'(0)}{\langle Ax,x\rangle}\\&\le\langle\Phi(A)\,x,x\rangle-\Phi(\langle Ax,x\rangle)\\&\le\frac{1}{2}\left[\left\langle\left(\Phi'(A)-\Phi'(0)\,I\right)Ax,x\right\rangle-\left\langle\left(\Phi'(A)-\Phi'(0)\,I\right)A^{-1}x,x\right\rangle\langle Ax,x\rangle^{2}\right]\\&\le\frac{1}{2}\left[\left\langle\Phi''(A)\,A^{2}x,x\right\rangle-\Phi''(0)\,\langle Ax,x\rangle^{2}\right]\end{aligned}\tag{17}$$

for any $x\in H$, $\|x\|=1$.

Proof. Let $x\in H$, $\|x\|=1$. For small $\varepsilon>0$, consider $f:[m-\varepsilon,M]\to\mathbb{R}$ continuous and $g(\lambda)=\langle E_{\lambda}x,x\rangle$ monotonic nondecreasing on $[m-\varepsilon,M]$. Utilising the inequality (4) for the positive measure $d\mu=dg$ we have

$$0\le\frac{1}{2}\left[\int_{m-\varepsilon}^{M}f^{2}(\lambda)\,d\,\langle E_{\lambda}x,x\rangle-\left(\int_{m-\varepsilon}^{M}f(\lambda)\,d\,\langle E_{\lambda}x,x\rangle\right)^{2}\right]\Phi''(0)$$
$$\le\frac{1}{2}\left[\int_{m-\varepsilon}^{M}f^{2}(\lambda)\,d\,\langle E_{\lambda}x,x\rangle-\left(\int_{m-\varepsilon}^{M}f(\lambda)\,d\,\langle E_{\lambda}x,x\rangle\right)^{2}\right]$$
$$\times\frac{\Phi'\left(\int_{m-\varepsilon}^{M}f(\lambda)\,d\,\langle E_{\lambda}x,x\rangle\right)-\Phi'(0)}{\int_{m-\varepsilon}^{M}f(\lambda)\,d\,\langle E_{\lambda}x,x\rangle}$$

$$\le\int_{m-\varepsilon}^{M}\Phi(f(\lambda))\,d\,\langle E_{\lambda}x,x\rangle-\Phi\left(\int_{m-\varepsilon}^{M}f(\lambda)\,d\,\langle E_{\lambda}x,x\rangle\right)$$
$$\le\frac{1}{2}\left[\int_{m-\varepsilon}^{M}\left(\Phi'(f(\lambda))-\Phi'(0)\right)f(\lambda)\,d\,\langle E_{\lambda}x,x\rangle\right.$$
$$\left.-\int_{m-\varepsilon}^{M}\left(\Phi'(f(\lambda))-\Phi'(0)\right)f^{-1}(\lambda)\,d\,\langle E_{\lambda}x,x\rangle\left(\int_{m-\varepsilon}^{M}f(\lambda)\,d\,\langle E_{\lambda}x,x\rangle\right)^{2}\right]$$
$$\le\frac{1}{2}\left[\int_{m-\varepsilon}^{M}\Phi''(f(\lambda))\,f^{2}(\lambda)\,d\,\langle E_{\lambda}x,x\rangle-\Phi''(0)\left(\int_{m-\varepsilon}^{M}f(\lambda)\,d\,\langle E_{\lambda}x,x\rangle\right)^{2}\right].$$

Taking the limit over $\varepsilon\to0+$ we deduce the desired result (16). □

We can give some examples as follows:

Example 3.2. *If* $A>0$ *(is a positive definite operator) on* H, *then we have the exponential inequalities*

$$0\le\frac{1}{2}\left[\left\langle A^{2}x,x\right\rangle-\langle Ax,x\rangle^{2}\right]\tag{18}$$

$$\leq \frac{1}{2}\left[\left\langle A^2x,x\right\rangle - \langle Ax,x\rangle^2\right]\frac{\exp(\langle Ax,x\rangle)-1}{\langle Ax,x\rangle}$$
$$\leq \langle \exp(A)x,x\rangle - \exp(\langle Ax,x\rangle)$$
$$\leq \frac{1}{2}\left[\langle(\exp(A)-I)Ax,x\rangle - \left\langle(\exp(A)-I)A^{-1}x,x\right\rangle\langle Ax,x\rangle^2\right]$$
$$\leq \frac{1}{2}\left[\left\langle A^2\exp(A)x,x\right\rangle - \langle Ax,x\rangle^2\right]$$

for any $x \in H$, $\|x\| = 1$.

Example 3.3. *If* $0 < A < I$, *then we have*

$$\begin{aligned}0 &\leq \left\langle A^2x,x\right\rangle - \langle Ax,x\rangle^2 \\ &\leq \frac{1}{2}\left[\left\langle A^2x,x\right\rangle - \langle Ax,x\rangle^2\right]\frac{\langle(2I-A)x,x\rangle}{\langle(I-A)x,x\rangle^2} \\ &\leq \left\langle(I-A)^{-1}x,x\right\rangle - \langle(I-A)x,x\rangle^{-1} \\ &\leq \frac{1}{2}\left[\left\langle(2I-A)A^2(I-A)^{-2}x,x\right\rangle - \left\langle(2I-A)(I-A)^{-2}x,x\right\rangle\langle Ax,x\rangle^2\right] \\ &\leq \left\langle A^2(I-A)^{-3}x,x\right\rangle - \langle Ax,x\rangle^2\end{aligned} \tag{19}$$

for any $x \in H$, $\|x\| = 1$.

For recent inequalities for continuous functions of selfadjoint operators see the papers [1], [5], the monographs [10], [11] and the references therein.

Acknowledgement: The author would like to thank the anonymous referee for valuable suggestions that have been implemented in the final version of the paper.

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