Ostrowski's type inequalities for complex functions defined on unit circle with applications for unitary operators in Hilbert spaces

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OSTROWSKI'S TYPE INEQUALITIES FOR COMPLEX FUNCTIONS DEFINED ON UNIT CIRCLE WITH APPLICATIONS FOR UNITARY OPERATORS IN HILBERT SPACES

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Abstract. Some Ostrowski's type inequalities for the Riemann-Stieltjes integral $\int_a^b f\left(e^{it}\right) du\left(t\right)$ of continuous complex valued integrands $f\colon \mathcal{C}\left(0,1\right) \to \mathbb{C}$ defined on the complex unit circle $\mathcal{C}\left(0,1\right)$ and various subclasses of integrators $u\colon \left[a,b\right] \subseteq \left[0,2\pi\right] \to \mathbb{C}$ of bounded variation are given. Natural applications for functions of unitary operators in Hilbert spaces are provided as well.

1. Introduction

The problem of approximating the Stieltjes integral $\int_a^b f\left(t\right) du\left(t\right)$ by the quantity $f\left(x\right)\left[u\left(b\right)-u\left(a\right)\right]$, which is a natural generalization of the Ostrowski problem analyzed in 1937 (see [6]), was apparently first considered in the literature by the author in 2000 (see [1]) where we obtained the following result:

$$\left|\left[u\left(b\right)-u\left(a\right)\right]f\left(x\right)-\int_a^b f\left(t\right)\,du\left(t\right)\right| \leq H\Big[\left(x-a\right)^r\bigvee_a^x\left(f\right)+\left(b-x\right)^r\bigvee_x^b\left(f\right)\Big]$$

$$\leq H\times\begin{cases}\left[\left(x-a\right)^r+\left(b-x\right)^r\right]\left[\frac{1}{2}\bigvee_a^b\left(f\right)+\frac{1}{2}\left|\bigvee_a^x\left(f\right)-\bigvee_x^b\left(f\right)\right|\right];\\ \left[\left(x-a\right)^{qr}+\left(b-x\right)^{qr}\right]^{\frac{1}{q}}\left[\left(\bigvee_a^x\left(f\right)\right)^p+\left(\bigvee_x^b\left(f\right)\right)^p\right]^{\frac{1}{p}}\\ \qquad\text{if}\quad p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ \left[\frac{1}{2}\left(b-a\right)+\left|x-\frac{a+b}{2}\right|\right]^r\bigvee_a^b\left(f\right).\end{cases} \tag{1.1}$$

for each $x\in\left[a,b\right]$, provided f is of bounded variation on $\left[a,b\right]$, $\bigvee_a^b\left(f\right)$ is its total variation on $\left[a,b\right]$, while $u\colon \left[a,b\right]\to\mathbb{R}$ is $r-H$-Hölder continuous, i.e., we recall that:

$$\left|u\left(x\right)-u\left(y\right)\right|\leq H\left|x-y\right|^r\quad\text{for each}\quad x,y\in\left[a,b\right]\,, \tag{1.2}$$

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where $H > 0$ and $r \in (0, 1]$.

The dual case, i.e., when the *integrand* f is $q - K$-Hölder continuous and the *integrator* u is of bounded variation was obtained by the author in 2001 and can be stated as [2]

$$\left| [u(b) - u(a)] f(x) - \int_a^b f(t)\, du(t) \right| \le K \left[\frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right]^q \bigvee_a^b (u) \tag{1.3}$$

for each $x \in [a, b]$.

The above inequalities provide, as important consequences, the following midpoint inequalities:

$$\left| [u(b) - u(a)] f\left(\frac{a+b}{2}\right) - \int_a^b f(t)\, du(t) \right| \le \begin{cases} \frac{1}{2^r}(b-a)^r H \bigvee_a^b (f) \\ \frac{1}{2^q}(b-a)^q K \bigvee_a^b (u) \end{cases} \tag{1.4}$$

which can be numerically implemented and provide a quadrature rule for approximating the Stieltjes integral $\int_a^b f(t)\, du(t)$.

Let U be a selfadjoint operator on the complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$ with the spectrum $\mathrm{Sp}(U)$ included in the interval $[m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its *spectral family.* Then for any continuous function $f\colon [m, M] \to \mathbb{R}$, it is well known that we have the following spectral representation in terms of the Riemann-Stieltjes integral:

$$\langle f(U)x, y \rangle = \int_{m-0}^{M} f(\lambda)\, d(\langle E_\lambda x, y \rangle), \tag{1.5}$$

for any x, $y \in H$. The function $g_{x,y}(\lambda) := \langle E_\lambda x, y \rangle$ is of *bounded variation* on the interval $[m, M]$ and

$$g_{x,y}(m-0) = 0 \quad \text{and} \quad g_{x,y}(M) = \langle x, y \rangle$$

for any $x, y \in H$. It is also well known that $g_x(\lambda) := \langle E_\lambda x, x \rangle$ is *monotonic nondecreasing* and *right continuous* on $[m, M]$.

On utilizing the spectral representation (1.5) and the Ostrowski's type inequality (1.3) we obtained the following result for continuous functions of selfadjoint operators:

Theorem 1 ([4])**.** *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $\mathrm{Sp}(A) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. If $f\colon [m, M] \to \mathbb{R}$ is $r - H$-Hölder continuous on $[m, M]$, then we have the inequality*

$$\begin{aligned} |f(s)\langle x, y\rangle - \langle f(A)x, y\rangle| &\le H \bigvee_m^M (\langle E_{(\cdot)}x, y\rangle) \left[\frac{1}{2}(M-m) + \left| s - \frac{m+M}{2} \right| \right]^r \\ &\le H \|x\| \|y\| \left[\frac{1}{2}(M-m) + \left| s - \frac{m+M}{2} \right| \right]^r \end{aligned} \tag{1.6}$$

for any x, $y \in H$ and $s \in [m, M]$.

The following dual result also holds:

Theorem 2 ([3]). *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. If $f\colon [m, M] \to \mathbb{R}$ is a continuous function of bounded variation on $[m, M]$, then we have the inequality*

$$\begin{aligned} |f(s)\langle x, y\rangle - \langle f(A)x, y\rangle| &\le \langle E_s x, x\rangle^{1/2} \langle E_s y, y\rangle^{1/2} \bigvee_m^s (f) \\ &\quad + \langle (1_H - E_s)x, x\rangle^{1/2} \langle (1_H - E_s)y, y\rangle^{1/2} \bigvee_s^M (f) \\ &\le \|x\|\,\|y\| \left(\frac{1}{2} \bigvee_m^M (f) + \frac{1}{2} \left| \bigvee_m^s (f) - \bigvee_s^M (f) \right| \right) \le \|x\|\,\|y\| \bigvee_m^M (f) \end{aligned} \tag{1.7}$$

for any x, $y \in H$ and for any $s \in [m, M]$, where 1_H is the identity operator on H.

Motivated by the above results, we investigate in the current paper the magnitude of the difference

$$f\left(e^{is}\right)[u(b) - u(a)] - \int_a^b f\left(e^{it}\right) du(t) \quad \text{with} \quad s \in [a, b] \subseteq [0, 2\pi]$$

for continuous complex valued function $f\colon \mathcal{C}(0,1) \to \mathbb{C}$ defined on the complex unit circle $\mathcal{C}(0,1)$ and various subclasses of functions $u\colon [a, b] \subseteq [0, 2\pi] \to \mathbb{C}$ of bounded variation. Natural applications for functions of unitary operators in Hilbert spaces are provided as well.

2. Scalar Ostrowski's type inequalities

Theorem 3. *Assume that $f\colon \mathcal{C}(0,1) \to \mathbb{C}$ satisfies the following Hölder's type condition*

$$|f(z) - f(w)| \le H\,|z - w|^r \tag{2.1}$$

for any $w, z \in \mathcal{C}(0,1)$, where $H > 0$ and $r \in (0, 1]$ are given.

If $[a, b] \subseteq [0, 2\pi]$ and the function $u\colon [a, b] \to \mathbb{C}$ is of bounded variation on $[a, b]$, then

$$\begin{aligned} &\left| f(e^{is})[u(b) - u(a)] - \int_a^b f(e^{it})\, du(t) \right| \\ &\quad \le 2^r H \max_{t \in [a,b]} \left| \sin\left(\frac{s-t}{2} \right) \right|^r \bigvee_a^b (u) \end{aligned} \tag{2.2}$$

for any $s \in [a, b]$, where $\bigvee_a^b (u)$ denotes the total variation of u on the interval $[a, b]$.

Proof. Observe that

$$f\left(e^{is}\right)[u(b) - u(a)] - \int_a^b f\left(e^{it}\right) du(t) = \int_a^b \left[f\left(e^{is}\right) - f\left(e^{it}\right) \right] du(t) \tag{2.3}$$

for any $s \in [a, b]$.

It is known that if $p\colon [c,d] \to \mathbb{C}$ is a continuous function and $v\colon [c,d] \to \mathbb{C}$ is of bounded variation, then the Riemann-Stieltjes integral $\int_c^d p(t)\,dv(t)$ exists and the following inequality holds

$$\left| \int_c^d p(t)\,dv(t) \right| \le \max_{t\in[c,d]} |p(t)| \bigvee_c^d (v) \tag{2.4}$$

where $\bigvee_c^d (v)$ denotes the total variation of v on $[c,d]$.

Applying the property (2.4) to the identity (2.3) and utilizing the Hölder's type condition (2.1) we have successively

$$\begin{aligned} \left| f(e^{is})[u(b)-u(a)] - \int_a^b f(e^{it})\,du(t) \right| &= \max_{t\in[a,b]} \left| f\left(e^{is}\right) - f\left(e^{it}\right) \right| \bigvee_a^b (u) \\ &\le H \max_{t\in[a,b]} \left| e^{is} - e^{it} \right|^r \bigvee_a^b (u) \,. \end{aligned} \tag{2.5}$$

Since

$$\left| e^{is} - e^{it} \right|^2 = \left| e^{is} \right|^2 - 2\operatorname{Re}\left(e^{i(s-t)} \right) + \left| e^{it} \right|^2 = 2 - 2\cos(s-t) = 4\sin^2\left(\frac{s-t}{2} \right)$$

for any $t,\ s \in \mathbb{R}$, then

$$\left| e^{is} - e^{it} \right|^r = 2^r \left| \sin\left(\frac{s-t}{2} \right) \right|^r \tag{2.6}$$

for any $t,\ s \in \mathbb{R}$.

Now, by (2.5) and (2.6) we deduce the desired result (2.2). □

Remark 1. If $a = 0$ and $b = 2\pi$, then for any $s \in [0,2\pi]$ there exists a unique $t \in [0,2\pi]$ such that $\frac{1}{2}|t-s| = \frac{\pi}{2}$, therefore $\max_{t\in[0,2\pi]} \left| \sin\left(\frac{s-t}{2} \right) \right| = 1$ for all $s \in [0,2\pi]$ and we deduce from (2.2) the following inequality of interest

$$\left| f\left(e^{is}\right) \left[u(2\pi) - u(0)\right] - \int_0^{2\pi} f\left(e^{it}\right)\,du(t) \right| \le 2^r H \bigvee_0^{2\pi} (u) \tag{2.7}$$

that holds for each $s \in [0,2\pi]$.

Remark 2. If $[a,b] \subset [0,2\pi]$ and $0 < b-a \le \pi$ then for all $t,s \in [a,b]$ we have $\frac{1}{2}|t-s| \le \frac{1}{2}(b-a) \le \frac{\pi}{2}$. Since the function sin is increasing on $\left[0,\frac{\pi}{2}\right]$, then we have successively that

$$\begin{aligned} \max_{t\in[a,b]} \left| \sin\left(\frac{s-t}{2} \right) \right| &= \sin\left(\max_{t\in[a,b]} \frac{1}{2}|t-s| \right) = \sin\left(\frac{1}{2}\max\{b-s, s-a\} \right) \\ &= \sin\left(\frac{1}{4}(b-a) + \frac{1}{2}\left| s - \frac{a+b}{2} \right| \right) \end{aligned} \tag{2.8}$$

for any $s \in [a,b]$.

Therefore, under the assumptions of Theorem 3 and if $[a,b] \subset [0,2\pi]$ with $0 < b-a \le \pi$, then

$$\left|f(e^{is})[u(b)-u(a)] - \int_a^b f\left(e^{it}\right) du\left(t\right)\right|$$

$$\le 2^r H \sin^r \left[\frac{1}{4}\left(b-a\right)+\frac{1}{2}\left|s-\frac{a+b}{2}\right|\right] \bigvee_a^b (u) \le 2^r H \sin^r \left[\frac{1}{2}\left(b-a\right)\right] \bigvee_a^b (u) \tag{2.9}$$

for all $s \in [a,b]$.

In particular, the best inequality we can get from (2.9) is incorporated in (2.10)

$$\left|f\left(e^{\frac{a+b}{2}i}\right)\left[u\left(b\right)-u\left(a\right)\right] - \int_a^b f\left(e^{it}\right)\, du\left(t\right)\right| \le 2^r H \sin^r \left[\frac{1}{4}\left(b-a\right)\right] \bigvee_a^b (u)\ .$$

The case when $f\colon \mathcal{C}(0,1) \to \mathbb{C}$ satisfies the Lipschitz condition $|f(z) - f(w)| \le L|z-w|$ for any $w,\ z \in \mathcal{C}(0,1)$, where $L > 0$ is given, is of interest due to various examples one can consider. Also in this case we can show that the corresponding version of the inequality (2.11) is sharp.

Corollary 1. *Assume that $f\colon \mathcal{C}(0,1) \to \mathbb{C}$ is Lipschitzian with the constant $L > 0$ on the circle $\mathcal{C}(0,1)$. If $[a,b] \subset [0,2\pi]$ with $0 < b-a \le \pi$ and the function $u\colon [a,b] \to \mathbb{C}$ is of bounded variation on $[a,b]$, then we have*

$$\left|f\left(e^{\frac{a+b}{2}i}\right)\left[u\left(b\right)-u\left(a\right)\right] - \int_a^b f\left(e^{it}\right) du\left(t\right)\right| \le 2L \sin \left[\frac{1}{4}\left(b-a\right)\right] \bigvee_a^b (u)\ . \tag{2.11}$$

The constant 2 cannot be replaced by a smaller quantity.

Proof. We need to prove only the sharpness of the constant.

If we consider the function $f\colon \mathbb{C} \to \mathbb{C}$, $f(z) = z$, then obviously f is Lipschitzian with the constant $L = 1$. Also, consider in (2.11) $a = 0$ and $b = \pi$ to get

$$\left|i\left[u\left(\pi\right)-u\left(0\right)\right] - \int_0^\pi e^{it} du\left(t\right)\right| \le \sqrt{2} \bigvee_0^\pi (u)\ . \tag{2.12}$$

Utilising the integration by parts formula for the Riemann-Stieltjes integral we have

$$\int_0^\pi e^{it} du\left(t\right) = e^{it} u\left(t\right)\Big|_0^\pi - i\int_0^\pi e^{it} u\left(t\right)\, dt = -u\left(\pi\right) - u\left(0\right) - i\int_0^\pi e^{it} u\left(t\right)\, dt$$

and replacing into the inequality (2.12) we deduce

$$\left|i\left[u\left(\pi\right)-u\left(0\right)\right] + u\left(\pi\right) + u\left(0\right) + i\int_0^\pi e^{it} u\left(t\right)\, dt\right| \le \sqrt{2} \bigvee_0^\pi (u)$$

which is equivalent with

$$\left|\left(i-1\right)u\left(\pi\right) + \left(i+1\right)u\left(0\right) - \int_0^\pi e^{it} u\left(t\right) dt\right| \le \sqrt{2} \bigvee_0^\pi (u) \tag{2.13}$$

that holds for any functions of bounded variation $u\colon [0,\pi] \to \mathbb{C}$ and is of interest in itself.

Now, assume that there exists a constant $C > 0$ such that

$$\left| (i-1)\,u(\pi) + (i+1)\,u(0) - \int_0^{\pi} e^{it} u(t)\,dt \right| \le C \bigvee_0^{\pi}(u) \tag{2.14}$$

for any functions of bounded variation $u\colon [0,\pi] \to \mathbb{C}$.

Consider the function $u\colon [0,\pi] \to \mathbb{R}$ with

$$u(t) = \begin{cases} 0 & \text{if} \quad 0 \le t < \pi \\ 1 & \text{if} \quad t = \pi\,. \end{cases}$$

Then u is of bounded variation, $\int_0^{\pi} e^{it} u(t)\,dt = 0$, $\bigvee_0^{\pi}(u) = 1$ and from (2.14) we get $C \ge \sqrt{2}$ showing that (2.14) is sharp and therefore (2.11) is sharp. □

Remark 3. The case of Riemann integral, namely when $u(t) = t$, $t \in [a,b] \subseteq [0,2\pi]$, is as follows

$$\left| f\left(e^{is}\right) - \frac{1}{b-a}\int_a^b f\left(e^{it}\right)\,du(t) \right| \le 2^r H \max_{t\in[a,b]} \left| \sin\left(\frac{s-t}{2}\right) \right|^r \tag{2.15}$$

for any $s \in [a,b]$ provided that $f\colon \mathcal{C}(0,1) \to \mathbb{C}$ satisfies the Hölder's type condition (2.1).

When u is an integral, then the following weighted integral inequality also holds.

Remark 4. If $w\colon [a,b] \subseteq [0,2\pi] \to \mathbb{C}$ is Lebesgue integrable on $[a,b]$ and $f\colon \mathcal{C}(0,1) \to \mathbb{C}$ satisfies the Hölder's type condition (2.1), then

$$\begin{aligned} &\left| f\left(e^{is}\right) \int_a^b w(t)\,dt - \int_a^b f\left(e^{it}\right) w(t)\,dt \right| \\ &\qquad \le 2^r H \max_{t\in[a,b]} \left| \sin\left(\frac{s-t}{2}\right) \right|^r \int_a^b |w(t)|\,dt \end{aligned} \tag{2.16}$$

for any $s \in [a,b]$.

In particular, if $w(t) \ge 0$ for $t \in [a,b]$ and $\int_a^b w(t)\,dt > 0$ then

$$\left| f\left(e^{is}\right) - \frac{1}{\int_a^b w(t)\,dt}\int_a^b f\left(e^{it}\right) w(t)\,dt \right| \le 2^r H \max_{t\in[a,b]} \left| \sin\left(\frac{s-t}{2}\right) \right|^r \tag{2.17}$$

for any $s \in [a,b]$.

Theorem 4. *Assume that $f\colon \mathcal{C}(0,1) \to \mathbb{C}$ is Lipschitzian with the constant $L > 0$ on the circle $\mathcal{C}(0,1)$. If $[a,b] \subseteq [0,2\pi]$ and the function $u\colon [a,b] \to \mathbb{C}$ is Lipschitzian with the constant $K > 0$ on $[a,b]$, then*

$$\begin{aligned} \left| f\left(e^{is}\right) [u(b) - u(a)] - \int_a^b f\left(e^{it}\right)\,du(t) \right| &\le 4LK \left[\sin^2\left(\frac{s-a}{4}\right) + \sin^2\left(\frac{b-s}{4}\right) \right] \\ &\le 8LK \sin^2\left(\frac{b-a}{4}\right) \end{aligned} \tag{2.18}$$

for any $s \in [a, b]$.

Proof. It is well known that if $p\colon [a,b] \to \mathbb{C}$ is a Riemann integrable function and $v\colon [a,b] \to \mathbb{C}$ is Lipschitzian with the constant $M > 0$, i.e.,

$$|f(s) - f(t)| \le M |s - t| \text{ for any } t, s \in [a, b] ,$$

then the Riemann-Stieltjes integral $\int_a^b p(t)\, dv(t)$ exists and the following inequality holds

$$\left| \int_a^b p(t)\; dv(t) \right| \le M \int_a^b |p(t)|\; dt . \tag{2.19}$$

Utilising the property (2.19), we have from (2.3) that

$$\begin{aligned} \left| f\left(e^{is}\right)[u(b) - u(a)] - \int_a^b f\left(e^{it}\right)\, du(t) \right| &= \left| \int_a^b \left[f\left(e^{is}\right) - f\left(e^{it}\right) \right]\, du(t) \right| \\ &\le K \int_a^b \left| f\left(e^{is}\right) - f\left(e^{it}\right) \right|\, dt \le KL \int_a^b \left| e^{is} - e^{it} \right|\, dt \end{aligned} \tag{2.20}$$

for any $s \in [a, b]$.

Since, by (2.6), $\left| e^{is} - e^{it} \right| = 2 \left| \sin\left(\frac{s-t}{2} \right) \right|$ for any t, $s \in \mathbb{R}$, then

$$\begin{aligned} \int_a^b \left| e^{is} - e^{it} \right|\, dt &= 2 \int_a^b \left| \sin\left(\frac{s-t}{2} \right) \right| dt \\ &= 2 \left[\int_a^s \sin\left(\frac{s-t}{2} \right) dt + \int_s^b \sin\left(\frac{t-s}{2} \right) dt \right] \\ &= 2 \left[1 - \cos\left(\frac{s-a}{2} \right) \right] + 2 \left[1 - \cos\left(\frac{b-s}{2} \right) \right] \\ &= 4 \left[\sin^2\left(\frac{s-a}{4} \right) + \sin^2\left(\frac{b-s}{4} \right) \right] \le 8 \sin^2\left(\frac{b-a}{4} \right) \end{aligned} \tag{2.21}$$

for any $s \in [a, b] \subseteq [0, 2\pi]$, and the inequality (2.18) is proved. □

The best inequality we can get from (2.18) is incorporated in

Corollary 2. *With the assumptions in Theorem 4 we have the inequality*

$$\left| f\left(e^{\frac{a+b}{2} i} \right) [u(b) - u(a)] - \int_a^b f\left(e^{it}\right)\, du(t) \right| \le 8LK \sin^2\left(\frac{b-a}{8} \right) . \tag{2.22}$$

The multiplicative constant 8 *cannot be replaced by a smaller quantity.*

Proof. We need to prove only the sharpness of the constant.

If we consider the function $f\colon \mathbb{C} \to \mathbb{C}$, $f(z) = z$, then obviously f is Lipschitzian with the constant $L = 1$. Also, consider in (2.22) $a = 0$ and $b = 2\pi$ to get

$$\left| -[u(2\pi) - u(0)] - \int_0^{2\pi} e^{it} du(t) \right| \le 4K . \tag{2.23}$$

Utilising the integration by parts formula for the Riemann-Stieltjes integral, we have

$$\int_0^{2\pi} e^{it} du(t) = e^{it} u(t) \Big|_0^{2\pi} - i \int_0^{2\pi} e^{it} u(t)\; dt = u(2\pi) - u(0) - i \int_0^{2\pi} e^{it} u(t)\; dt ,$$

which inserted in (2.23) produces the inequality

$$\Big| -2\left[u(2\pi)-u(0)\right]+i\int_0^{2\pi} e^{it}u(t)\,dt\Big| \le 4K$$

which is equivalent with

$$\Big| \int_0^{2\pi} e^{it}u(t)\,dt-\frac{2}{i}\left[u(2\pi)-u(0)\right]\Big| \le 4K \tag{2.24}$$

that holds for any K-Lipschitzian function $u\colon [0,2\pi]\to\mathbb{C}$ and is of interest in itself.

Now, assume that the inequality (2.24) holds with a constant $D>0$, namely

$$\Big| \int_0^{2\pi} e^{it}u(t)\,dt-\frac{2}{i}\left[u(2\pi)-u(0)\right]\Big| \le DK \tag{2.25}$$

for any K-Lipschitzian function $u\colon [0,2\pi]\to\mathbb{C}$.

Consider $u\colon [0,2\pi]\to\mathbb{R}$, $u(t)=|t-\pi|$. Then, by the continuity property of the modulus we have that u is Lipschitzian with the constant $K=1$.

We also have that

$$\begin{aligned}\int_0^{2\pi} e^{it}u(t)\,dt &= \int_0^{2\pi} e^{it}|t-\pi|\,dt=\int_0^{2\pi}|t-\pi|(\cos t+i\sin t)\,dt\\ &=\int_0^{2\pi}|t-\pi|\cos t\,dt+i\int_0^{2\pi}|t-\pi|\sin t\,dt\,.\end{aligned}$$

Observe that by symmetry reasons $\int_0^{2\pi}|t-\pi|\sin t\,dt=0$ and

$$\int_0^{2\pi}|t-\pi|\cos t\,dt=2\int_0^{\pi}(\pi-t)\cos t\,dt=2\Big[(\pi-t)\sin t\big|_0^{\pi}+\int_0^{\pi}\sin t\,dt\Big]=4$$

and by (2.25) we get $D\ge 4$ which proves the desired sharpness of the constant 8 in (2.22). □

Remark 5. If $u(t)=t$, $t\in[a,b]$, then we get from (2.18) and (2.22) the following inequalities for the Riemann integral

$$\begin{aligned}\Big|f\left(e^{is}\right)(b-a)-\int_a^b f\left(e^{it}\right)dt\Big| &\le 4L\Big[\sin^2\Big(\frac{s-a}{4}\Big)+\sin^2\Big(\frac{b-s}{4}\Big)\Big]\\ &\le 8L\sin^2\Big(\frac{b-a}{4}\Big)\end{aligned} \tag{2.26}$$

for any $s\in[a,b]$ and

$$\Big|f\big(e^{\frac{a+b}{2}i}\big)(b-a)-\int_a^b f\left(e^{it}\right)dt\Big| \le 8L\sin^2\Big(\frac{b-a}{8}\Big) \tag{2.27}$$

provided that $f\colon \mathcal{C}(0,1)\to\mathbb{C}$ is Lipschitzian with the constant $L>0$ on the circle $\mathcal{C}(0,1)$.

Remark 6. If $w\colon [a,b] \subseteq [0,2\pi] \to \mathbb{C}$ is essentially bounded on $[a,b]$ and $f\colon \mathcal{C}(0,1) \to \mathbb{C}$ is Lipschitzian with the constant $L>0$ on the circle $\mathcal{C}(0,1)$, then we have the following weighted integral inequality

$$\left|f\left(e^{is}\right)\int_a^b w(t)\,dt - \int_a^b f\left(e^{it}\right)w(t)\,dt\right| \le 4L\,\|w\|_\infty\left[\sin^2\left(\frac{s-a}{4}\right)+\sin^2\left(\frac{b-s}{4}\right)\right]$$

$$\le 8L\,\|w\|_\infty \sin^2\left(\frac{b-a}{4}\right) \tag{2.28}$$

for any $s\in[a,b]$ where $\|w\|_\infty := \operatorname{ess\,sup}_{t\in[a,b]}|w(t)|$.

In particular, we have

$$\left|f\left(e^{\frac{a+b}{2}i}\right)\int_a^b w(t)\,dt - \int_a^b f\left(e^{it}\right)w(t)\,dt\right| \le 8L\,\|w\|_\infty \sin^2\left(\frac{b-a}{8}\right). \tag{2.29}$$

The case of monotonic integrators is as follows:

Theorem 5. *Assume that $f\colon \mathcal{C}(0,1)\to\mathbb{C}$ is Lipschitzian with the constant $L>0$ on the circle $\mathcal{C}(0,1)$. If $[a,b]\subseteq[0,2\pi]$ and the function $u\colon [a,b]\to\mathbb{R}$ is monotonic nondecreasing on $[a,b]$, then*

$$\left|f\left(e^{is}\right)[u(b)-u(a)]-\int_a^b f\left(e^{it}\right)\,du(t)\right| \le 2L\left[\sin\left(\frac{b-s}{2}\right)u(b)-\sin\left(\frac{s-a}{2}\right)u(a)\right]$$

$$+L\int_a^b \operatorname{sgn}(s-t)\cos\left(\frac{s-t}{2}\right)u(t)\,dt \tag{2.30}$$

for any $s\in[a,b]$.

In particular, we have

$$\left|f\left(e^{\frac{a+b}{2}i}\right)[u(b)-u(a)]-\int_a^b f\left(e^{it}\right)\,du(t)\right| \le 2L\sin\left(\frac{b-a}{4}\right)[u(b)-u(a)]$$

$$+L\int_a^b \operatorname{sgn}\left(\frac{a+b}{2}-t\right)\cos\left(\frac{\frac{a+b}{2}-t}{2}\right)u(t)\,dt. \tag{2.31}$$

Proof. It is well known that if $p\colon [a,b]\to\mathbb{C}$ is a continuous function and $v\colon [a,b]\to\mathbb{R}$ is monotonic nondecreasing on $[a,b]$, then the Riemann-Stieltjes integral $\int_a^b p(t)\,dv(t)$ exists and the following inequality holds

$$\left|\int_a^b p(t)\,dv(t)\right| \le \int_a^b |p(t)|\,dv(t). \tag{2.32}$$

Utilising the property (2.32), we have from (2.3) that

$$\left|f\left(e^{is}\right)[u(b)-u(a)]-\int_a^b f\left(e^{it}\right)\,du(t)\right| = \left|\int_a^b \left[f\left(e^{is}\right)-f\left(e^{it}\right)\right]du(t)\right|$$

$$\le \int_a^b \left|f\left(e^{is}\right)-f\left(e^{it}\right)\right|du(t) \le L\int_a^b \left|e^{is}-e^{it}\right|du(t) \tag{2.33}$$

for any $s\in[a,b]$.

Since, by (2.6), $\left|e^{is}-e^{it}\right| = 2\left|\sin\left(\frac{s-t}{2}\right)\right|$ for any $t, s \in \mathbb{R}$, then

$$\int_a^b \left|e^{is}-e^{it}\right| du(t) = 2\int_a^b \left|\sin\left(\frac{s-t}{2}\right)\right| du(t)$$

$$= 2\Big[\int_a^s \sin\left(\frac{s-t}{2}\right) du(t) + \int_s^b \sin\left(\frac{t-s}{2}\right) du(t)\Big] \tag{2.34}$$

for any $s \in [a,b] \subseteq [0, 2\pi]$.

Utilising the integration by parts formula for the Riemann-Stieltjes integral, we have

$$\int_a^s \sin\left(\frac{s-t}{2}\right) du(t) = \sin\left(\frac{s-t}{2}\right) u(t)\Big|_a^s + \frac{1}{2}\int_a^s \cos\left(\frac{s-t}{2}\right) u(t)\, dt$$

$$= -\sin\left(\frac{s-a}{2}\right) u(a) + \frac{1}{2}\int_a^s \cos\left(\frac{s-t}{2}\right) u(t)\, dt$$

and

$$\int_s^b \sin\left(\frac{t-s}{2}\right) du(t) = \sin\left(\frac{t-s}{2}\right) u(t)\Big|_s^b - \frac{1}{2}\int_s^b \cos\left(\frac{t-s}{2}\right) u(t)\, dt$$

$$= \sin\left(\frac{b-s}{2}\right) u(b) - \frac{1}{2}\int_s^b \cos\left(\frac{t-s}{2}\right) u(t)\, dt\,,$$

which, by (2.34), produce the equality

$$\int_a^b \left|e^{is}-e^{it}\right| du(t) = 2\Big[\sin\left(\frac{b-s}{2}\right) u(b) - \sin\left(\frac{s-a}{2}\right) u(a)\Big]$$

$$+ \int_a^s \cos\left(\frac{s-t}{2}\right) u(t)\, dt - \int_s^b \cos\left(\frac{t-s}{2}\right) u(t)\, dt$$

$$= 2\Big[\sin\left(\frac{b-s}{2}\right) u(b) - \sin\left(\frac{s-a}{2}\right) u(a)\Big]$$

$$+ \int_a^b \operatorname{sgn}(s-t) \cos\left(\frac{s-t}{2}\right) u(t)\, dt\,. \tag{2.35}$$

Utilising (2.33) we deduce the desired result (2.30). □

Remark 7. We remark that if $a = 0$ and $b = 2\pi$, then we get from (2.30) and (2.31) that

$$\left|f\left(e^{is}\right)\left[u(2\pi) - u(0)\right] - \int_0^{2\pi} f\left(e^{it}\right) du(t)\right| \le 2L\sin\left(\frac{s}{2}\right)\left[u(2\pi) - u(0)\right]$$

$$+ L\int_0^{2\pi} \operatorname{sgn}(s-t) \cos\left(\frac{s-t}{2}\right) u(t)\, dt \tag{2.36}$$

for any $s \in [a,b]$.

In particular, we have

$$\left| f(-1)\left[u(2\pi)-u(0)\right] - \int_0^{2\pi} f\left(e^{it}\right) du(t) \right| \leq \sqrt{2}L\left[u(2\pi)-u(0)\right] \tag{2.37}$$
$$+ L\int_0^{2\pi} \operatorname{sgn}(\pi - t)\sin\left(\frac{t}{2}\right)u(t)\,dt\,.$$

Corollary 3. *Assume that f and u are as in Theorem 5 then for any $[a,b]\subset[0,2\pi]$ with $0<b-a\leq\pi$ we have the sequence of inequalities*

$$\left| f(e^{is})[u(b)-u(a)] - \int_a^b f(e^{it})\,du(t)\right| \leq 2L\left[\sin\left(\frac{b-s}{2}\right)u(b) - \sin\left(\frac{s-a}{2}\right)u(a)\right]$$
$$+ L\int_a^b \operatorname{sgn}(s-t)\cos\left(\frac{s-t}{2}\right)u(t)\,dt$$
$$\leq 2L\left[\sin\left(\frac{b-s}{2}\right)[u(b)-u(s)] + \sin\left(\frac{s-a}{2}\right)[u(s)-u(a)]\right] =: B(s) \tag{2.38}$$

where

$$B(s) \leq 2L \times \begin{cases} \sin\left[\frac{1}{4}(b-a) + \frac{1}{2}\left|s-\frac{a+b}{2}\right|\right][u(b)-u(a)] \\ 2\sin\left(\frac{b-a}{4}\right)\cos\left(\frac{s-\frac{a+b}{2}}{2}\right)\left[\frac{u(b)-u(a)}{2} + \left|u(s)-\frac{u(b)+u(a)}{2}\right|\right] \end{cases}$$

for any $s\in[a,b]$.

In particular, we have

$$\left| f\left(e^{\frac{a+b}{2}i}\right)[u(b)-u(a)] - \int_a^b f\left(e^{it}\right) du(t) \right| \leq 2L\sin\left(\frac{b-a}{4}\right)[u(b)-u(a)] \tag{2.39}$$
$$+ L\int_a^b \operatorname{sgn}\left(\frac{a+b}{2}-t\right)\cos\left(\frac{\frac{a+b}{2}-t}{2}\right)u(t)\,dt =: M$$

where

$$M \leq 2L\sin\left(\frac{b-a}{4}\right)[u(b)-u(a)]\,.$$

Proof. Since $0<b-a\leq\pi$, then $\frac{|s-t|}{2}\leq\frac{\pi}{2}$ for $s,t\in[a,b]$. Utilising the fact that u is monotonic nondecreasing on $[a,b]$ and $\cos\left(\frac{|s-t|}{2}\right)\geq 0$ for $s,t\in[a,b]$, then

$$\int_a^s \cos\left(\frac{s-t}{2}\right)u(t)\,dt \leq u(s)\int_a^s \cos\left(\frac{s-t}{2}\right)dt = 2u(s)\sin\left(\frac{s-a}{2}\right) \tag{2.40}$$

and

$$\int_s^b \cos\left(\frac{s-t}{2}\right)u(t)\,dt \geq u(s)\int_s^b \cos\left(\frac{s-t}{2}\right)dt = 2u(s)\sin\left(\frac{b-s}{2}\right)$$

i.e.,

$$-\int_s^b \cos\left(\frac{s-t}{2}\right)u(t)\,dt \leq -2u(s)\sin\left(\frac{b-s}{2}\right). \tag{2.41}$$

Summing (2.40) with (2.41) we deduce that

$$\int_a^b \operatorname{sgn}(s-t)\cos\left(\frac{s-t}{2}\right)u(t)\,dt \le 2u(s)\sin\left(\frac{s-a}{2}\right) - 2u(s)\sin\left(\frac{b-s}{2}\right)$$

giving that

$$2L\Big[\sin\left(\frac{b-s}{2}\right)u(b) - \sin\left(\frac{s-a}{2}\right)u(a)\Big] + L\int_a^b \operatorname{sgn}(s-t)\cos\left(\frac{s-t}{2}\right)u(t)\,dt$$
$$\le 2L\Big[\sin\left(\frac{b-s}{2}\right)[u(b)-u(s)] + \sin\left(\frac{s-a}{2}\right)[u(s)-u(a)]\Big],$$

which proves the second inequality in (2.38).

The bounds for $B(s)$ follows from the elementary property stating that

$$\alpha x + \beta y \le \max\{\alpha,\beta\}(x+y)$$

where α, β, x, $y \ge 0$. □

3. A quadrature rule

We consider the following partition of the interval $[a,b]$

$$\Delta_n : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$$

and the intermediate points $\xi_k \in [x_k, x_{k+1}]$ where $0 \le k \le n-1$. Define $h_k := x_{k+1} - x_k$, $0 \le k \le n-1$ and $\nu(\Delta_n) = \max\{h_k : 0 \le k \le n-1\}$ the norm of the partition Δ_n.

For the continuous function $f\colon \mathcal{C}(0,1) \to \mathbb{C}$ and the function $u\colon\ [a,b] \subseteq [0,2\pi] \to \mathbb{C}$ of bounded variation on $[a,b]$, define the quadrature rule

$$O_n(f,u,\Delta_n,\xi) := \sum_{k=0}^{n-1} f\left(e^{i\xi_k}\right)[u(x_{k+1}) - u(x_k)] \tag{3.1}$$

and the remainder $R_n(f,u,\Delta_n,\xi)$ in approximating the Riemann-Stieltjes integral $\int_a^b f\left(e^{it}\right)du(t)$ by $O_n(f,u,\Delta_n,\xi)$. Then we have

$$\int_a^b f\left(e^{it}\right)du(t) = O_n(f,u,\Delta_n,\xi) + R_n(f,u,\Delta_n,\xi)\ . \tag{3.2}$$

The following result provides a priory bounds for $R_n(f,u,\Delta_n,\xi)$ in several instances of f and u as above.

Proposition 1. *Assume that $f\colon \mathcal{C}(0,1) \to \mathbb{C}$ satisfies the following Hölder's type condition*

$$|f(z) - f(w)| \le H|z-w|^r$$

for any w, $z \in \mathcal{C}(0,1)$, where $H > 0$ and $r \in (0,1]$ are given.

If $[a,b] \subseteq [0,2\pi]$ and the function $u\colon\ [a,b] \to \mathbb{C}$ is of bounded variation on $[a,b]$, then for any partition $\Delta_n\colon a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ with the norm

$\nu(\Delta_n) \le \pi$ we have the error bound

$$\begin{aligned}|R_n(f,u,\Delta_n,\xi)| &\le 2^r H \sum_{k=0}^{n-1} \sin^r \left[\frac{1}{4}(x_{k+1}-x_k) + \frac{1}{2}\left|\xi_k - \frac{x_k+x_{k+1}}{2}\right|\right] \bigvee_{x_k}^{x_{k+1}}(u) \\ &\le 2^r H \sum_{k=0}^{n-1} \sin^r \left[\frac{1}{2}(x_{k+1}-x_k)\right] \bigvee_{x_k}^{x_{k+1}}(u) \\ &\le H \sum_{k=0}^{n-1} (x_{k+1}-x_k)^r \bigvee_{x_k}^{x_{k+1}}(u) \le H\nu^r(\Delta_n) \bigvee_a^b (u) \qquad (3.3)\end{aligned}$$

for any intermediate points $\xi_k \in [x_k, x_{k+1}]$ where $0 \le k \le n-1$.

Proof. Since $\nu(\Delta_n) \le \pi$, then on writing inequality (2.9) on each interval $[x_k, x_{k+1}]$ and for any intermediate points $\xi_k \in [x_k, x_{k+1}]$ where $0 \le k \le n-1$, we have

$$\begin{aligned}&\left| f(e^{i\xi_k})[u(x_{k+1})-u(x_k)] - \int_{x_k}^{x_{k+1}} f\left(e^{it}\right) du(t) \right| \\ &\le 2^r H \sin^r \left[\frac{1}{4}(x_{k+1}-x_k) + \frac{1}{2}\left|\xi_k - \frac{x_k+x_{k+1}}{2}\right|\right] \bigvee_{x_k}^{x_{k+1}}(u) \\ &\le 2^r H \sin^r \left[\frac{1}{2}(x_{k+1}-x_k)\right] \bigvee_{x_k}^{x_{k+1}}(u) \le H(x_{k+1}-x_k)^r \bigvee_{x_k}^{x_{k+1}}(u) \qquad (3.4)\end{aligned}$$

where for the last inequality we have used the fact that $\sin x \le x$ for $x \in \left[0, \frac{\pi}{2}\right]$.

Summing over k from 0 to $n-1$ in (3.4) and utilizing the generalized triangle inequality, we deduce the first part of (3.3). The second part is obvious. □

Corollary 4. *Assume that f, u and Δ_n are as in Proposition 1. Define the midpoint trapezoid type quadrature rule by*

$$T_n(f,u,\Delta_n) := \sum_{k=0}^{n-1} f\left(e^{\frac{x_{k+1}+x_k}{2}i}\right)[u(x_{k+1}) - u(x_k)] \qquad (3.5)$$

and the error $E_n(f,u,\Delta_n)$ by

$$\int_a^b f\left(e^{it}\right) du(t) = T_n(f,u,\Delta_n) + E_n(f,u,\Delta_n). \qquad (3.6)$$

Then we have the error bounds

$$\begin{aligned}|E_n(f,u,\Delta_n)| &\le 2^r H \sum_{k=0}^{n-1} \sin^r \left[\frac{1}{4}(x_{k+1}-x_k)\right] \bigvee_{x_k}^{x_{k+1}}(u) \\ &\le \frac{1}{2^r} H \sum_{k=0}^{n-1} (x_{k+1}-x_k)^r \bigvee_{x_k}^{x_{k+1}}(u) \le \frac{1}{2^r} H\nu^r(\Delta_n) \bigvee_a^b (u). \qquad (3.7)\end{aligned}$$

The case of both integrator and integrand being Lipschitzian is incorporated in the following result:

Proposition 2. *Assume that* $f\colon \mathcal{C}(0,1) \to \mathbb{C}$ *is Lipschitzian with the constant* $L > 0$ *on the circle* $\mathcal{C}(0,1)$*. If* $[a,b] \subseteq [0,2\pi]$ *and the function* $u\colon [a,b] \to \mathbb{C}$ *is Lipschitzian with the constant* $K > 0$ *on* $[a,b]$*, then for any partition* $\Delta_n : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ *we have the error bound*

$$\begin{aligned} |R_n(f,u,\Delta_n,\xi)| &\le 4LK\sum_{k=0}^{n-1}\left[\sin^2\left(\frac{\xi_k - x_k}{4}\right) + \sin^2\left(\frac{x_{k+1}-\xi_k}{4}\right)\right] \\ &\le 8LK\sum_{k=0}^{n-1}\sin^2\left(\frac{x_{k+1}-x_k}{4}\right) \le \frac{1}{2}LK\sum_{k=0}^{n-1}(x_{k+1}-x_k)^2 \\ &\le \frac{1}{2}LK(b-a)\,\nu(\Delta_n) \end{aligned} \tag{3.8}$$

for any intermediate points $\xi_k \in [x_k, x_{k+1}]$ *where* $0 \le k \le n-1$*.*

In particular, we have

$$\begin{aligned} |E_n(f,u,\Delta_n)| &\le 8LK\sum_{k=0}^{n-1}\sin^2\left(\frac{x_{k+1}-x_k}{8}\right) \\ &\le \frac{1}{8}LK\sum_{k=0}^{n-1}(x_{k+1}-x_k)^2 \le \frac{1}{8}LK(b-a)\,\nu(\Delta_n)\,. \end{aligned} \tag{3.9}$$

The proof follows by Theorem 4 and the details are omitted.

Proposition 3. *Assume that* $f\colon \mathcal{C}(0,1) \to \mathbb{C}$ *is Lipschitzian with the constant* $L > 0$ *on the circle* $\mathcal{C}(0,1)$*. If* $[a,b] \subseteq [0,2\pi]$ *and the function* $u\colon [a,b] \to \mathbb{R}$ *is monotonic nondecreasing on* $[a,b]$*, then for any partition* $\Delta_n : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ *with the norm* $\nu(\Delta_n) \le \pi$ *we have the error bound*

$$\begin{aligned} |R_n(f,u,\Delta_n,\xi)| &\le 2L\sum_{k=0}^{n-1}\left[\sin\left(\frac{x_{k+1}-\xi_k}{2}\right)u(x_{k+1}) - \sin\left(\frac{\xi_k - x_k}{2}\right)u(x_k)\right] \\ &\quad + L\sum_{k=0}^{n-1}\int_{x_k}^{x_{k+1}}\operatorname{sgn}(\xi_k - t)\cos\left(\frac{\xi_k - t}{2}\right)u(t)\,dt \\ &\le 2L\sum_{k=0}^{n-1}\left[\sin\left(\frac{x_{k+1}-\xi_k}{2}\right)[u(x_{k+1}) - u(\xi_k)]\right. \\ &\quad \left. + \sin\left(\frac{\xi_k - x_k}{2}\right)[u(\xi_k) - u(x_k)]\right] \\ &\le 2L\sum_{k=0}^{n-1}\sin\left[\frac{1}{4}(x_{k+1}-x_k) + \frac{1}{2}\left|\xi_k - \frac{x_k + x_{k+1}}{2}\right|\right][u(x_{k+1}) - u(x_k)] \\ &\le 2L\sum_{k=0}^{n-1}\sin\left[\frac{1}{2}(x_{k+1}-x_k)\right][u(x_{k+1}) - u(x_k)] \end{aligned}$$

$$\leq L \sum_{k=0}^{n-1} (x_{k+1} - x_k) [u(x_{k+1}) - u(x_k)] \leq \nu(\Delta_n) L [u(b) - u(a)] \tag{3.10}$$

for any intermediate points $\xi_k \in [x_k, x_{k+1}]$ *where* $0 \leq k \leq n-1$.

In particular, we have

$$\begin{aligned} |E_n(f,u,\Delta_n)| &\leq 2L \sum_{k=0}^{n-1} \sin\left(\frac{x_{k+1}-x_k}{4}\right) [u(x_{k+1}) - u(x_k)] \\ &\quad + L \sum_{k=0}^{n-1} \int_{x_k}^{x_{k+1}} \operatorname{sgn}\left(\frac{x_k + x_{k+1}}{2} - t\right) \cos\left(\frac{\frac{x_k+x_{k+1}}{2} - t}{2}\right) u(t)\, dt \\ &\leq 2L \sum_{k=0}^{n-1} \sin\left(\frac{x_{k+1}-x_k}{4}\right) [u(x_{k+1}) - u(x_k)] \end{aligned}$$

$$\leq \frac{1}{2} L \sum_{k=0}^{n-1} (x_{k+1} - x_k) [u(x_{k+1}) - u(x_k)] \leq \frac{1}{2} L \nu(\Delta_n) [u(b) - u(a)] . \tag{3.11}$$

The proof follows by Corollary 3 and the details are omitted.

4. Applications for functions of unitary operators

We recall that the bounded linear operator U on the Hilbert space H is unitary iff $U^* = U^{-1}$.

It is well known that (see for instance [5, p. 275-p. 276]), if U is a unitary operator, then there exists a family of projections $\{E_\lambda\}_{\lambda \in [0,2\pi]}$, called the spectral family of U with the following properties

a) $E_\lambda \leq E_\mu$ for $0 \leq \lambda \leq \mu \leq 2\pi$;

b) $E_0 = 0$ and $E_{2\pi} = 1_H$ (the identity operator on H);

c) $E_{\lambda+0} = E_\lambda$ for $0 \leq \lambda < 2\pi$;

d) $U = \int_0^{2\pi} e^{i\lambda} dE_\lambda$ where the integral is of Riemann-Stieltjes type.

Moreover, if $\{F_\lambda\}_{\lambda \in [0,2\pi]}$ is a family of projections satisfying the requirements a)-d) above for the operator U, then $F_\lambda = E_\lambda$ for all $\lambda \in [0, 2\pi]$.

Also, for every continuous complex valued function $f\colon \mathcal{C}(0,1) \to \mathbb{C}$ on the complex unit circle, we have

$$f(U) = \int_0^{2\pi} f\left(e^{i\lambda}\right) dE_\lambda \tag{4.1}$$

where the integral is taken in the Riemann-Stieltjes sense.

In particular, we have the equalities

$$f(U)x = \int_0^{2\pi} f\left(e^{i\lambda}\right) dE_\lambda x\,, \tag{4.2}$$

$$\langle f(U)x, y\rangle = \int_0^{2\pi} f\left(e^{i\lambda}\right) d\langle E_\lambda x, y\rangle \tag{4.3}$$

and

$$\|f(U)x\|^2 = \int_0^{2\pi} \left|f\left(e^{i\lambda}\right)\right|^2 d\|E_\lambda x\|^2 , \tag{4.4}$$

for any x, $y \in H$.

We consider the following partition of the interval $[a, b]$

$$\Delta_n : 0 = \lambda_0 < \lambda_1 < \cdots < \lambda_{n-1} < \lambda_n = 2\pi$$

and the intermediate points $\xi_k \in [\lambda_k, \lambda_{k+1}]$ where $0 \le k \le n-1$. Define $h_k := \lambda_{k+1} - \lambda_k$, $0 \le k \le n-1$ and $\nu(\Delta_n) = \max\{h_k : 0 \le k \le n-1\}$ the norm of the partition Δ_n.

If U is a unitary operator on the Hilbert space H and $\{E_\lambda\}_{\lambda\in[0,2\pi]}$, the spectral family of U, then we can introduce the following sums

$$O_n(f, U, \Delta_n, \xi; x, y) := \sum_{k=0}^{n-1} f\left(e^{i\xi_k}\right)\left\langle\left(E_{\lambda_{k+1}} - E_{\lambda_k}\right)x, y\right\rangle \tag{4.5}$$

and

$$T_n(f, U, \Delta_n; x, y) := \sum_{k=0}^{n-1} f\left(e^{\frac{\lambda_{k+1}+\lambda_k}{2}i}\right)\left\langle\left(E_{\lambda_{k+1}} - E_{\lambda_k}\right)x, y\right\rangle \tag{4.6}$$

where x, $y \in H$.

Theorem 6. *With the above assumptions for $U, \{E_\lambda\}_{\lambda\in[0,2\pi]}, \Delta_n$ with $\nu(\Delta_n) \le \pi$ and if $f\colon \mathcal{C}(0,1) \to \mathbb{C}$ satisfies the Hölder's type condition $|f(z) - f(w)| \le H|z-w|^r$ for any w, $z \in \mathcal{C}(0,1)$, where $H > 0$ and $r \in (0,1]$ are given, then we have the representation*

$$\langle f(U)x, y\rangle = O_n(f, U, \Delta_n, \xi; x, y) + R_n(f, U, \Delta_n, \xi; x, y) \tag{4.7}$$

with the error $R_n(f, U, \Delta_n, \xi; x, y)$ satisfying the bounds

$$\begin{aligned}
&\left|R_n(f, U, \Delta_n, \xi; x, y)\right| \\
&\quad\le 2^r H \sum_{k=0}^{n-1} \sin^r\left[\frac{1}{4}(\lambda_{k+1} - \lambda_k) + \frac{1}{2}\left|\xi_k - \frac{\lambda_k + \lambda_{k+1}}{2}\right|\right] \bigvee_{\lambda_k}^{\lambda_{k+1}}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right) \\
&\quad\le 2^r H \sum_{k=0}^{n-1} \sin^r\left[\frac{1}{2}(\lambda_{k+1} - \lambda_k)\right] \bigvee_{\lambda_k}^{\lambda_{k+1}}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right) \\
&\quad\le H \sum_{k=0}^{n-1} (\lambda_{k+1} - \lambda_k)^r \bigvee_{\lambda_k}^{\lambda_{k+1}}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right) \le H\nu^r(\Delta_n) \bigvee_0^{2\pi}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right) \\
&\quad\le H\nu^r(\Delta_n)\|x\|\|y\|
\end{aligned} \tag{4.8}$$

for any x, $y \in H$ and the intermediate points $\xi_k \in [\lambda_k, \lambda_{k+1}]$ where $0 \le k \le n-1$.

In particular we have

$$\langle f(U)x, y\rangle = T_n(f, U, \Delta_n; x, y) + E_n(f, U, \Delta_n; x, y) \tag{4.9}$$

with the error

$$\begin{aligned}
&\left|E_n\left(f, U, \Delta_n; x, y\right)\right| \\
&\leq 2^r H \sum_{k=0}^{n-1} \sin^r\left[\frac{1}{4}\left(\lambda_{k+1}-\lambda_k\right)\right] \bigvee_{\lambda_k}^{\lambda_{k+1}}\left(\left\langle E_{(\cdot)} x, y\right\rangle\right) \\
&\leq \frac{1}{2^r} H \sum_{k=0}^{n-1}\left(\lambda_{k+1}-\lambda_k\right)^r \bigvee_{\lambda_k}^{\lambda_{k+1}}\left(\left\langle E_{(\cdot)} x, y\right\rangle\right) \\
&\leq \frac{1}{2^r} H \nu^r\left(\Delta_n\right) \bigvee_0^{2\pi}\left(\left\langle E_{(\cdot)} x, y\right\rangle\right) \leq \frac{1}{2^r} H \nu^r\left(\Delta_n\right)\|x\|\|y\| \qquad (4.10)
\end{aligned}$$

for any x, $y \in H$.

Proof. For given $x, y \in H$, define the function $u(\lambda) := \langle E_\lambda x, y\rangle$, $\lambda \in [0, 2\pi]$. We will show that u is of bounded variation and

$$\bigvee_0^{2\pi}(u) =: \bigvee_0^{2\pi}\left(\left\langle E_{(\cdot)} x, y\right\rangle\right) \leq \|x\|\|y\| \, . \qquad (4.11)$$

It is well known that, if P is a nonnegative selfadjoint operator on H, i.e., $\langle Px, x\rangle \geq 0$ for any $x \in H$, then the following inequality is a *generalization of the Schwarz inequality* in H

$$|\langle Px, y\rangle|^2 \leq \langle Px, x\rangle \langle Py, y\rangle \, , \qquad (4.12)$$

for any x, $y \in H$.

Now, if $d\colon 0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 2\pi$ is an arbitrary partition of the interval $[0, 2\pi]$, then we have by Schwarz's inequality for nonnegative operators (4.12) that

$$\begin{aligned}
&\bigvee_0^{2\pi}\left(\left\langle E_{(\cdot)} x, y\right\rangle\right) = \sup_d\left\{\sum_{i=0}^{n-1}\left|\left\langle\left(E_{t_{i+1}}-E_{t_i}\right) x, y\right\rangle\right|\right\} \\
&\leq \sup_d\left\{\sum_{i=0}^{n-1}\left[\left\langle\left(E_{t_{i+1}}-E_{t_i}\right) x, x\right\rangle^{1/2}\left\langle\left(E_{t_{i+1}}-E_{t_i}\right) y, y\right\rangle^{1/2}\right]\right\} := I \, . \qquad (4.13)
\end{aligned}$$

By the Cauchy-Buniakovski-Schwarz inequality for sequences of real numbers we also have that

$$\begin{aligned}
I &\leq \sup_d\left[\sum_{i=0}^{n-1}\left\langle\left(E_{t_{i+1}}-E_{t_i}\right) x, x\right\rangle\right]^{1/2}\left[\sum_{i=0}^{n-1}\left\langle\left(E_{t_{i+1}}-E_{t_i}\right) y, y\right\rangle\right]^{1/2} \\
&\leq \sup_d\left[\sum_{i=0}^{n-1}\left\langle\left(E_{t_{i+1}}-E_{t_i}\right) x, x\right\rangle\right]^{1/2} \sup_d\left[\sum_{i=0}^{n-1}\left\langle\left(E_{t_{i+1}}-E_{t_i}\right) y, y\right\rangle\right]^{1/2} \\
&= \left[\bigvee_0^{2\pi}\left(\left\langle E_{(\cdot)} x, x\right\rangle\right)\right]^{1/2}\left[\bigvee_0^{2\pi}\left(\left\langle E_{(\cdot)} y, y\right\rangle\right)\right]^{1/2} = \|x\|\|y\| \qquad (4.14)
\end{aligned}$$

for any x, $y \in H$.

On making use of (4.13) and (4.14) we deduce the desired result (4.11).

Now, applying Proposition 1 to the spectral representation (4.3) we deduce the desired result (4.7) with the error bound (4.8). The details are omitted. □

Remark 8. In the case when the partition reduces to the whole interval $[0,2\pi]$, then utilizing the inequality (2.7) we deduce the bound

$$\left|f\left(e^{is}\right)\langle x,y\rangle-\langle f(U)x,y\rangle\right|\le 2^{r}H\bigvee_{0}^{2\pi}\left(\langle E_{(\cdot)}x,y\rangle\right)\le 2^{r}H\|x\|\|y\| \tag{4.15}$$

for any $s\in[0,2\pi]$ and any vectors $x,\ y\in H$.

In the case when the division is

$$\Delta_2:0=\lambda_0<\lambda_1=\pi<\lambda_2=2\pi$$

and we take the intermediate points $u\in[0,\pi]$ and $v\in[\pi,2\pi]$, then we get from Theorem 6 that

$$\begin{aligned}&\left|f(e^{iu})\langle E_\pi x,y\rangle+f(e^{iv})\langle (I-E_\pi)x,y\rangle-\langle f(U)x,y\rangle\right|\\&\le 2^{r}H\Big[\sin^{r}\left[\frac{1}{4}\pi+\frac{1}{2}\Big|u-\frac{\pi}{2}\Big|\right]\bigvee_{0}^{\pi}(\langle E_{(\cdot)}x,y\rangle)\\&\quad+\sin^{r}\left[\frac{1}{4}\pi+\frac{1}{2}\Big|v-\frac{3\pi}{2}\Big|\right]\bigvee_{\pi}^{2\pi}\left(\langle E_{(\cdot)}x,y\rangle\right)\Big]\end{aligned} \tag{4.16}$$

for any vectors $x,\ y\in H$.

The best inequality we can get from (4.17) is obtained for $u=\frac{\pi}{2}$ and $v=\frac{3\pi}{2}$, namely

$$\begin{aligned}&\left|f(i)\langle E_\pi x,y\rangle+f(-i)\langle (1_H-E_\pi)x,y\rangle-\langle f(U)x,y\rangle\right|\\&\le 2^{\frac{r}{2}}H\bigvee_{0}^{2\pi}\left(\langle E_{(\cdot)}x,y\rangle\right)\le 2^{\frac{r}{2}}H\|x\|\|y\|\end{aligned} \tag{4.17}$$

for any vectors $x,\ y\in H$.

If U is a unitary operator on the Hilbert space H and $\{E_\lambda\}_{\lambda\in[0,2\pi]}$, the spectral family of U, then we can introduce the following sums depending only of one vector $x\in H$

$$\tilde{O}_n(f,U,\Delta_n,\xi;x):=\sum_{k=0}^{n-1}f\left(e^{i\xi_k}\right)\left\langle\left(E_{\lambda_{k+1}}-E_{\lambda_k}\right)x,x\right\rangle \tag{4.18}$$

and

$$\tilde{T}_n(f,U,\Delta_n;x,y):=\sum_{k=0}^{n-1}f\left(e^{\frac{\lambda_{k+1}+\lambda_k}{2}i}\right)\left\langle\left(E_{\lambda_{k+1}}-E_{\lambda_k}\right)x,x\right\rangle. \tag{4.19}$$

Theorem 7. *With the above assumptions for $U,\{E_\lambda\}_{\lambda\in[0,2\pi]},\Delta_n$ with $\nu(\Delta_n)\le\pi$ and, if $f:\mathcal{C}(0,1)\to\mathbb{C}$ is Lipschitzian with the constant $L>0$ on the circle $\mathcal{C}(0,1)$, then we have the representation*

$$\langle f(U)x,x\rangle = \tilde{O}_n(f,U,\Delta_n,\xi;x) + \tilde{R}_n(f,U,\Delta_n,\xi;x) \tag{4.20}$$

with the error $\tilde{R}_n(f,U,\Delta_n,\xi;x)$ *satisfying the bounds*

$$\begin{aligned}
&\left|\tilde{R}_n(f,U,\Delta_n,\xi;x)\right| \\
&\le 2L\sum_{k=0}^{n-1}\left[\sin\left(\frac{\lambda_{k+1}-\xi_k}{2}\right)\langle E_{\lambda_{k+1}}x,x\rangle - \sin\left(\frac{\xi_k-\lambda_k}{2}\right)\langle E_{\lambda_k}x,x\rangle\right] \\
&\quad + L\sum_{k=0}^{n-1}\int_{\lambda_k}^{\lambda_{k+1}}\operatorname{sgn}(\xi_k-t)\cos\left(\frac{\xi_k-t}{2}\right)\langle E_t x,x\rangle\,dt \\
&\le 2L\sum_{k=0}^{n-1}\left[\sin\left(\frac{\lambda_{k+1}-\xi_k}{2}\right)\left[\left\langle\left(E_{\lambda_{k+1}}-E_{\xi_k}\right)x,x\right\rangle\right]\right. \\
&\quad \left.+\sin\left(\frac{\xi_k-\lambda_k}{2}\right)\langle(E_{\xi_k}-E_{\lambda_k})x,x\rangle\right] \\
&\le 2L\sum_{k=0}^{n-1}\sin\left[\frac{1}{4}(\lambda_{k+1}-\lambda_k)+\frac{1}{2}\left|\xi_k-\frac{\lambda_k+\lambda_{k+1}}{2}\right|\right]\left\langle\left(E_{\lambda_{k+1}}-E_{\lambda_k}\right)x,x\right\rangle \\
&\le 2L\sum_{k=0}^{n-1}\sin\left[\frac{1}{2}(\lambda_{k+1}-\lambda_k)\right]\left\langle\left(E_{\lambda_{k+1}}-E_{\lambda_k}\right)x,x\right\rangle
\end{aligned}$$

$$\le L\sum_{k=0}^{n-1}(\lambda_{k+1}-\lambda_k)\left\langle\left(E_{\lambda_{k+1}}-E_{\lambda_k}\right)x,x\right\rangle \le \nu(\Delta_n)L\|x\|^2 \tag{4.21}$$

for any $x\in H$ *and the intermediate points* $\xi_k\in[\lambda_k,\lambda_{k+1}]$ *where* $0\le k\le n-1$.

In particular we have

$$\langle f(U)x,x\rangle = \tilde{T}_n(f,U,\Delta_n;x) + \tilde{E}_n(f,U,\Delta_n;x) \tag{4.22}$$

with the error

$$\begin{aligned}
|\tilde{E}_n(f,U,\Delta_n;x)| &\le 2L\sum_{k=0}^{n-1}\sin\left(\frac{\lambda_{k+1}-\lambda_k}{4}\right)\left\langle\left(E_{\lambda_{k+1}}-E_{\lambda_k}\right)x,x\right\rangle \\
&\quad + L\sum_{k=0}^{n-1}\int_{x_k}^{x_{k+1}}\operatorname{sgn}\left(\frac{\lambda_k+\lambda_{k+1}}{2}-t\right)\cos\left(\frac{\frac{\lambda_k+\lambda_{k+1}}{2}-t}{2}\right)\langle E_t x,x\rangle\,dt \\
&\le 2L\sum_{k=0}^{n-1}\sin\left(\frac{\lambda_{k+1}-\lambda_k}{4}\right)\left\langle\left(E_{\lambda_{k+1}}-E_{\lambda_k}\right)x,x\right\rangle
\end{aligned}$$

$$\le \frac{1}{2}L\sum_{k=0}^{n-1}(\lambda_{k+1}-\lambda_k)\left\langle\left(E_{\lambda_{k+1}}-E_{\lambda_k}\right)x,x\right\rangle \le \frac{1}{2}L\nu(\Delta_n)\|x\|^2 \tag{4.23}$$

for any $x\in H$.

The proof follows by Proposition 3 applied for the monotonic nondecreasing function $u(t) := \langle E_t x, x\rangle$, $t \in [0, 2\pi]$.

Remark 9. We remark that if the partition reduces to the whole interval $[0, 2\pi]$ then we get from (2.36) that

$$\begin{aligned} \left| f(e^{is})\|x\|^2 - \langle f(U)x, x\rangle \right| &\le 2L \sin\left(\frac{s}{2}\right)\|x\|^2 \\ &+ L\int_0^{2\pi} \operatorname{sgn}(s-t)\cos\left(\frac{s-t}{2}\right)\langle E_t x, x\rangle\, dt \end{aligned} \tag{4.24}$$

for any $s \in [a, b]$ and $x \in H$.

In particular, we have

$$\begin{aligned} \left| f(-1)\|x\|^2 - \langle f(U)x, x\rangle \right| &\le \sqrt{2}L\|x\|^2 \\ &+ L\int_0^{2\pi} \operatorname{sgn}(\pi-t)\sin\left(\frac{t}{2}\right)\langle E_t x, x\rangle\, dt \end{aligned} \tag{4.25}$$

for any $x \in H$.

Example 1. In order to provide some simple examples for the inequalities above we choose two complex functions as follows.

a) Consider the power function $f\colon \mathbb{C}\setminus\{0\} \to \mathbb{C}$, $f(z) = z^m$ where m is a nonzero integer. Then, obviously, for any z, w belonging to the unit circle $\mathcal{C}(0,1)$ we have the inequality

$$|f(z) - f(w)| \le |m|\,|z - w|$$

which shows that f is Lipschitzian with the constant $L = |m|$ on the circle $\mathcal{C}(0,1)$.

Then from (4.15), we get for any unitary operator U that

$$\left|e^{ims}\langle x, y\rangle - \langle U^m x, y\rangle\right| \le 2|m| \bigvee_0^{2\pi}\left(\langle E_{(\cdot)}x, y\rangle\right) \le 2|m|\,\|x\|\,\|y\| \tag{4.26}$$

for any $s \in [0, 2\pi]$ and x, $y \in H$.

Also, from (4.16) and the intermediate points $u \in [0, \pi]$ and $v \in [\pi, 2\pi]$, we have for any unitary operator U

$$\begin{aligned} &\left|e^{imu}\langle E_\pi x, y\rangle + e^{imv}\langle (1_H - E_\pi)x, y\rangle - \langle U^m x, y\rangle\right| \\ &\le 2|m|\Big[\sin\left[\frac{1}{4}\pi + \frac{1}{2}\Big|u - \frac{\pi}{2}\Big|\right] \bigvee_0^{\pi}\left(\langle E_{(\cdot)}x, y\rangle\right) \\ &+ \sin\left[\frac{1}{4}\pi + \frac{1}{2}\Big|v - \frac{3\pi}{2}\Big|\right] \bigvee_\pi^{2\pi}\left(\langle E_{(\cdot)}x, y\rangle\right)\Big] \end{aligned} \tag{4.27}$$

for any vectors x, $y \in H$, where $\{E_\lambda\}_{\lambda\in[0,2\pi]}$ is the spectral family of U.

The best inequality we can get from (4.27) is obtained for $u = \frac{\pi}{2}$ and $v = \frac{3\pi}{2}$, namely

$$\left|i^m \langle E_\pi x, y\rangle + (-i)^m \langle (1_H - E_\pi) x, y\rangle - \langle U^m x, y\rangle \right| \\ \le \sqrt{2}\,|m| \bigvee_0^{2\pi} \left(\langle E_{(\cdot)} x, y\rangle\right) \le \sqrt{2}\,|m| \,\|x\|\,\|y\| \,, \tag{4.28}$$

for any vectors x, $y \in H$.

b) For $a \neq \pm 1, 0$ consider the function $f\colon \mathcal{C}(0,1) \to \mathbb{C}$, $f_a(z) = \frac{1}{1-az}$. Observe that

$$|f_a(z) - f_a(w)| = \frac{|a|\,|z-w|}{|1-az|\,|1-aw|} \tag{4.29}$$

for any z, $w \in \mathcal{C}(0,1)$.

If $z = e^{it}$ with $t \in [0, 2\pi]$, then we have

$$|1-az|^2 = 1 - 2a\operatorname{Re}(\bar{z}) + a^2 |z|^2 = 1 - 2a\cos t + a^2 \\ \ge 1 - 2|a| + a^2 = (1-|a|)^2$$

therefore

$$\frac{1}{|1-az|} \le \frac{1}{|1-|a||} \quad \text{and} \quad \frac{1}{|1-aw|} \le \frac{1}{|1-|a||} \tag{4.30}$$

for any z, $w \in \mathcal{C}(0,1)$.

Utilising (4.29) and (4.30) we deduce

$$|f_a(z) - f_a(w)| \le \frac{|a|}{(1-|a|)^2}\,|z-w| \tag{4.31}$$

for any $z, w \in \mathcal{C}(0,1)$, showing that the function f_a is Lipschitzian with the constant $L_a = \frac{|a|}{(1-|a|)^2}$ on the circle $\mathcal{C}(0,1)$.

Applying the inequality (4.15), we get for any unitary operator U that

$$\left|(1-ae^{is})^{-1}\langle x, y\rangle - \left\langle (1_H - aU)^{-1} x, y\right\rangle \right| \\ \le \frac{2|a|}{(1-|a|)^2} \bigvee_0^{2\pi} \left(\langle E_{(\cdot)} x, y\rangle\right) \le \frac{2|a|}{(1-|a|)^2}\,\|x\|\,\|y\| \tag{4.32}$$

for any $s \in [0, 2\pi]$ and x, $y \in H$.

Also, from (4.16) and the intermediate points $u \in [0,\pi]$ and $v \in [\pi, 2\pi]$, we have for any unitary operator U

$$\left|(1-ae^{iu})^{-1}\langle E_\pi x, y\rangle + (1-ae^{iv})^{-1}\langle (1_H - E_\pi) x, y\rangle - \left\langle (1_H - aU)^{-1} x, y\right\rangle \right| \\ \le \frac{2|a|}{(1-|a|)^2}\Big[\sin\left[\frac{1}{4}\pi + \frac{1}{2}\left|u - \frac{\pi}{2}\right|\right] \bigvee_0^{\pi} \left(\langle E_{(\cdot)} x, y\rangle\right) \\ + \sin\left[\frac{1}{4}\pi + \frac{1}{2}\left|v - \frac{3\pi}{2}\right|\right] \bigvee_\pi^{2\pi} (\langle E_{(\cdot)} x, y\rangle)\Big] \tag{4.33}$$

for any vectors $x, y \in H$, where $\{E_\lambda\}_{\lambda\in[0,2\pi]}$ is the spectral family of U.

The best inequality we can get from (4.27) is obtained for $u = \frac{\pi}{2}$ and $v = \frac{3\pi}{2}$, namely

$$\left|(1-ai)^{-1}\langle E_\pi x, y\rangle + (1+ai)^{-1}\langle (1_H - E_\pi)x, y\rangle - \left\langle (1_H - aU)^{-1}x, y\right\rangle\right|$$

$$\leq \frac{\sqrt{2}\,|a|}{(1-|a|)^2}\bigvee_0^{2\pi}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right) \leq \frac{\sqrt{2}\,|a|}{(1-|a|)^2}\|x\|\,\|y\| \tag{4.34}$$

for any vectors $x, y \in H$.

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