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Some Grüss-type results via Pompeiu’s-like inequalities

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Abstract In this paper, some Grüss-type results via Pompeiu’s-like inequalities are proved.

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المخلص

في هذه الورقة، تم إثبات بعض النتائج من نوع جُرسُ باستخدام متباينات من عائلة متباينة بومبيوي.

1 Introduction

In 1946, Pompeiu [18] derived a variant of Lagrange’s mean value theorem, now known as *Pompeiu’s mean value theorem* (see also [18, p.83]).

Theorem 1.1 (Pompeiu [18]) *For every real valued function f differentiable on an interval $[a, b]$ not containing 0 and for all pairs $x_1 \neq x_2$ in $[a, b]$, there exists a point ξ between x_1 and x_2 such that*

$$\frac{x_1 f(x_2) - x_2 f(x_1)}{x_1 - x_2} = f(\xi) - \xi f'(\xi). \quad (1.1)$$

The following inequality is useful to derive some Ostrowski-type inequalities; see [9].

Corollary 1.2 (Pompeiu’s inequality) *With the assumptions of Theorem 1.1 and if $\|f - \ell f'\|_\infty = \sup_{t \in (a,b)} |f(t) - t f'(t)| < \infty$ where $\ell(t) = t$, $t \in [a, b]$, then*

$$|t f(x) - x f(t)| \leq \|f - \ell f'\|_\infty |x - t| \quad (1.2)$$

for any $t, x \in [a, b]$.

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The inequality (1.2) was obtained by the author in [9].

For other Ostrowski-type inequalities concerning the p -norms $\|f - \ell f'\|_p$, see [1, 2, 17, 19].

For two Lebesgue integrable functions $f, g : [a, b] \rightarrow \mathbb{R}$, consider the Čebyšev functional:

$$C(f, g) := \frac{1}{b-a} \int_a^b f(t)g(t)dt - \frac{1}{(b-a)^2} \int_a^b f(t)dt \int_a^b g(t)dt. \tag{1.3}$$

Grüss [10] showed that

$$|C(f, g)| \leq \frac{1}{4} (M - m)(N - n), \tag{1.4}$$

provided that there exists the real numbers m, M, n, N such that

$$m \leq f(t) \leq M \quad \text{and} \quad n \leq g(t) \leq N \quad \text{for a.e. } t \in [a, b]. \tag{1.5}$$

The constant $\frac{1}{4}$ is best possible in (1.3) in the sense that it cannot be replaced by a smaller quantity.

Another, however less known, result, though it was obtained by Čebyšev [7], states that

$$|C(f, g)| \leq \frac{1}{12} \|f'\|_\infty \|g'\|_\infty (b-a)^2, \tag{1.6}$$

provided that f', g' exist and are continuous on $[a, b]$ and $\|f'\|_\infty = \sup_{t \in [a, b]} |f'(t)|$. The constant $\frac{1}{12}$ cannot be improved in the general case.

The Čebyšev inequality (1.6) also holds if $f, g : [a, b] \rightarrow \mathbb{R}$ are assumed to be absolutely continuous and $f', g' \in L_\infty[a, b]$, while $\|f'\|_\infty = \text{ess sup}_{t \in [a, b]} |f'(t)|$.

A mixture between Grüss' result (1.4) and Čebyšev's one (1.6) is the following inequality obtained by Ostrowski [15]:

$$|C(f, g)| \leq \frac{1}{8} (b-a)(M-m) \|g'\|_\infty, \tag{1.7}$$

provided that f is Lebesgue integrable and satisfies (1.5), while g is absolutely continuous and $g' \in L_\infty[a, b]$. The constant $\frac{1}{8}$ is best possible in (1.7).

The case of Euclidean norms of the derivative was considered by Lupaş [12], in which he proved that

$$|C(f, g)| \leq \frac{1}{\pi^2} \|f'\|_2 \|g'\|_2 (b-a), \tag{1.8}$$

provided that f, g are absolutely continuous and $f', g' \in L_2[a, b]$. The constant $\frac{1}{\pi^2}$ is the best possible.

Recently, Cerone and Dragomir [3] have proved the following results:

$$|C(f, g)| \leq \inf_{\gamma \in \mathbb{R}} \|g - \gamma\|_q \cdot \frac{1}{b-a} \left(\int_a^b \left| f(t) - \frac{1}{b-a} \int_a^b f(s) ds \right|^p dt \right)^{\frac{1}{p}}, \tag{1.9}$$

where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$ or $p = 1$ and $q = \infty$, and

$$|C(f, g)| \leq \inf_{\gamma \in \mathbb{R}} \|g - \gamma\|_1 \cdot \frac{1}{b-a} \text{ess sup}_{t \in [a, b]} \left| f(t) - \frac{1}{b-a} \int_a^b f(s) ds \right|, \tag{1.10}$$

provided that $f \in L_p[a, b]$ and $g \in L_q[a, b]$ ($p > 1, \frac{1}{p} + \frac{1}{q} = 1; p = 1, q = \infty$ or $p = \infty, q = 1$).

Notice that for $q = \infty, p = 1$ in (1.9), we obtain

$$\begin{aligned} |C(f, g)| &\leq \inf_{\gamma \in \mathbb{R}} \|g - \gamma\|_\infty \cdot \frac{1}{b-a} \int_a^b \left| f(t) - \frac{1}{b-a} \int_a^b f(s) ds \right| dt \\ &\leq \|g\|_\infty \cdot \frac{1}{b-a} \int_a^b \left| f(t) - \frac{1}{b-a} \int_a^b f(s) ds \right| dt \end{aligned} \tag{1.11}$$

and, if g satisfies (1.5), then

$$\begin{aligned}
 |C(f, g)| &\leq \inf_{\gamma \in \mathbb{R}} \|g - \gamma\|_\infty \cdot \frac{1}{b-a} \int_a^b \left| f(t) - \frac{1}{b-a} \int_a^b f(s) ds \right| dt \\
 &\leq \left\| g - \frac{n+N}{2} \right\|_\infty \cdot \frac{1}{b-a} \int_a^b \left| f(t) - \frac{1}{b-a} \int_a^b f(s) ds \right| dt \\
 &\leq \frac{1}{2} (N-n) \cdot \frac{1}{b-a} \int_a^b \left| f(t) - \frac{1}{b-a} \int_a^b f(s) ds \right| dt.
 \end{aligned}
 \tag{1.12}$$

The inequality between the first and the last term in (1.12) has been obtained by Cheng and Sun [8]. However, the sharpness of the constant $\frac{1}{2}$, a generalization for the abstract Lebesgue integral and the discrete version of it have been obtained in [4].

For other recent results on the Grüss inequality, see [5,6,11,13,14,16,20] and the references therein. In this paper, some Grüss-type results via Pompeiu’s-like inequalities are proved.

2 Some Pompeiu’s-type inequalities

We can generalize the above inequality for the larger class of functions that are absolutely continuous and complex valued as well as for other norms of the difference $f - \ell f'$.

Theorem 2.1 *Let $f : [a, b] \rightarrow \mathbb{C}$ be an absolutely continuous function on the interval $[a, b]$ with $b > a > 0$. Then for any $t, x \in [a, b]$, we have*

$$|tf(x) - xf(t)| \leq \begin{cases} \|f - \ell f'\|_\infty |x-t| & \text{if } f - \ell f' \in L_\infty[a, b], \\ \left(\frac{1}{2q-1}\right)^{1/q} \|f - \ell f'\|_p \left| \frac{x^q}{t^{q-1}} - \frac{t^q}{x^{q-1}} \right|^{1/q} & \text{if } f - \ell f' \in L_p[a, b] \\ & p > 1, \\ & \frac{1}{p} + \frac{1}{q} = 1, \\ \|f - \ell f'\|_1 \frac{\max\{t,x\}}{\min\{t,x\}}, & \end{cases}
 \tag{2.1}$$

or equivalently

$$\left| \frac{f(x)}{x} - \frac{f(t)}{t} \right| \leq \begin{cases} \|f - \ell f'\|_\infty \left| \frac{1}{t} - \frac{1}{x} \right| & \text{if } f - \ell f' \in L_\infty[a, b], \\ \left(\frac{1}{2q-1}\right)^{1/q} \|f - \ell f'\|_p \left| \frac{1}{t^{2q-1}} - \frac{1}{x^{2q-1}} \right|^{1/q} & \text{if } f - \ell f' \in L_p[a, b] \\ & p > 1, \\ & \frac{1}{p} + \frac{1}{q} = 1, \\ \|f - \ell f'\|_1 \frac{1}{\min\{t^2, x^2\}}. & \end{cases}
 \tag{2.2}$$

Proof If f is absolutely continuous, then f/ℓ is absolutely continuous on the interval $[a, b]$ that does not contain 0 and

$$\int_t^x \left(\frac{f(s)}{s} \right)' ds = \frac{f(x)}{x} - \frac{f(t)}{t}$$

for any $t, x \in [a, b]$ with $x \neq t$.

Since

$$\int_t^x \left(\frac{f(s)}{s} \right)' ds = \int_t^x \frac{f'(s)s - f(s)}{s^2} ds,$$

we get the following identity:

$$tf(x) - xf(t) = xt \int_t^x \frac{f'(s)s - f(s)}{s^2} ds
 \tag{2.3}$$

for any $t, x \in [a, b]$.

We notice that the equality (2.3) was proved for the smaller class of differentiable function and in a different manner in [17].

Taking the modulus in (2.3), we have

$$\begin{aligned} |tf(x) - xf(t)| &= \left| xt \int_t^x \frac{f'(s)s - f(s)}{s^2} ds \right| \\ &\leq xt \left| \int_t^x \left| \frac{f'(s)s - f(s)}{s^2} \right| ds \right| := I, \end{aligned} \quad (2.4)$$

and utilizing Hölder's integral inequality we deduce

$$\begin{aligned} I &\leq xt \begin{cases} \sup_{s \in [t, x] \setminus \{x, t\}} |f'(s)s - f(s)| \left| \int_t^x \frac{1}{s^2} ds \right|, \\ \left| \int_t^x |f'(s)s - f(s)|^p ds \right|^{1/p} \left| \int_t^x \frac{1}{s^{2q}} ds \right|^{1/q} & \begin{matrix} p > 1, \\ \frac{1}{p} + \frac{1}{q} = 1, \end{matrix} \\ \left| \int_t^x |f'(s)s - f(s)| ds \right| \sup_{s \in [t, x] \setminus \{x, t\}} \left\{ \frac{1}{s^2} \right\}, \end{cases} \\ &= \begin{cases} \|f - \ell f'\|_\infty |x - t|, \\ \left(\frac{1}{2q-1} \right)^{1/q} \|f - \ell f'\|_p \left| \frac{x^q}{t^{q-1}} - \frac{t^q}{x^{q-1}} \right|^{1/q} & \begin{matrix} p > 1, \\ \frac{1}{p} + \frac{1}{q} = 1, \end{matrix} \\ \|f - \ell f'\|_1 \frac{\max\{t, x\}}{\min\{t, x\}}, \end{cases} \end{aligned} \quad (2.5)$$

and the inequality (2.2) is proved. \square

Remark 2.2 The first inequality in (2.1) also holds in the same form for $0 > b > a$.

3 Some Grüss-type inequalities

We have the following result of Grüss type.

Theorem 3.1 *Let $f, g : [a, b] \rightarrow \mathbb{C}$ be absolutely continuous functions on the interval $[a, b]$ with $b > a > 0$. If $f', g' \in L_\infty[a, b]$, then*

$$\begin{aligned} &\left| \frac{b^3 - a^3}{3} \int_a^b f(t)g(t) dt - \int_a^b tf(t) dt \int_a^b tg(t) dt \right| \\ &\leq \frac{1}{12} (b - a)^4 \|f - \ell f'\|_\infty \|g - \ell g'\|_\infty. \end{aligned} \quad (3.1)$$

The constant $\frac{1}{12}$ is best possible.

Proof From the first inequality in (2.1), we have

$$\begin{aligned} &\left| \int_a^b \int_a^b (tf(x) - xf(t))(tg(x) - xg(t)) dt dx \right| \\ &\leq \int_a^b \int_a^b |(tf(x) - xf(t))(tg(x) - xg(t))| dt dx \\ &\leq \|f - \ell f'\|_\infty \|g - \ell g'\|_\infty \int_a^b \int_a^b (x - t)^2 dt dx. \end{aligned} \quad (3.2)$$



Observe that

$$\begin{aligned} & \int_a^b \int_a^b (tf(x) - xf(t))(tg(x) - xg(t)) dt dx \\ &= \int_a^b \int_a^b [t^2 f(x)g(x) + x^2 f(t)g(t) - tg(t)xf(x) - f(t)txg(x)] dt dx \\ &= 2 \left[\int_a^b t^2 dt \int_a^b f(t)g(t) dt - \int_a^b tf(t) dt \int_a^b tg(t) dt \right] \end{aligned}$$

and

$$\int_a^b \int_a^b (x - t)^2 dt dx = \frac{1}{3} \int_a^b [(b - x)^3 + (x - a)^3] dx = \frac{1}{6} (b - a)^4.$$

Utilizing the inequality (3.2), we deduce the desired result (3.1).

Now, assume that the inequality (3.1) holds with a constant $B > 0$ instead of $\frac{1}{12}$, i.e.,

$$\begin{aligned} & \left| \frac{b^3 - a^3}{3} \int_a^b f(t)g(t) dt - \int_a^b tf(t) dt \int_a^b tg(t) dt \right| \\ & \leq B (b - a)^4 \|f - \ell f'\|_\infty \|g - \ell g'\|_\infty. \end{aligned} \tag{3.3}$$

If we take $f(t) = g(t) = 1, t \in [a, b]$, then

$$\begin{aligned} & \frac{b^3 - a^3}{3} \int_a^b f(t)g(t) dt - \int_a^b tf(t) dt \int_a^b tg(t) dt \\ &= \frac{b^3 - a^3}{3} (b - a) - \left(\frac{b^2 - a^2}{2}\right)^2 = \frac{1}{12} (b - a)^4 \end{aligned}$$

and

$$\|f - \ell f'\|_\infty = \|g - \ell g'\|_\infty = 1$$

and by (3.3) we get $B \geq \frac{1}{12}$, which proves the sharpness of the constant. □

The following result for the complementary (p, q) -norms, with $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, holds.

Theorem 3.2 *Let $f, g : [a, b] \rightarrow \mathbb{C}$ be absolutely continuous functions on the interval $[a, b]$ with $b > a > 0$. If $f' \in L_p[a, b], g' \in L_q[a, b]$ with $p, q > 1, p, q \neq 2$ and $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\begin{aligned} & \left| \frac{b^3 - a^3}{3} \int_a^b f(t)g(t) dt - \int_a^b tf(t) dt \int_a^b tg(t) dt \right| \\ & \leq \frac{1}{2(2q - 1)^{1/q} (2p - 1)^{1/p}} \|f - \ell f'\|_p \|g - \ell g'\|_q M_q^{1/q}(a, b) M_p^{1/p}(a, b), \end{aligned} \tag{3.4}$$

where

$$M_q(a, b) := \int_a^b \int_a^b \left| \frac{x^q}{t^{q-1}} - \frac{t^q}{x^{q-1}} \right| dt dx.$$

We have the bounds

$$M_q(a, b) \leq (b - a) N_q^{1/2}(a, b)$$

and

$$M_p(a, b) \leq (b - a) N_p^{1/2}(a, b)$$

where, for $r > 1$,

$$N_r(a, b) := \begin{cases} 2 \left(\frac{b^{2r+1} - a^{2r+1}}{2r+1} \cdot \frac{b^{-2r+3} - a^{-2r+3}}{-2r+3} - \left(\frac{b^2 - a^2}{2} \right)^2 \right), & r \neq \frac{3}{2} \\ (b^2 - a^2) \left(\frac{b^2 + a^2}{2} \cdot \ln \frac{b}{a} - \frac{b^2 - a^2}{2} \right), & r = \frac{3}{2}. \end{cases}$$

Proof From the second inequality in (2.1), we have

$$|tf(x) - xf(t)| \leq \frac{1}{(2q-1)^{1/q}} \|f - \ell f'\|_p \left| \frac{x^q}{t^{q-1}} - \frac{t^q}{x^{q-1}} \right|^{1/q}$$

and

$$|tg(x) - xg(t)| \leq \frac{1}{(2p-1)^{1/p}} \|g - \ell g'\|_q \left| \frac{x^p}{t^{p-1}} - \frac{t^p}{x^{p-1}} \right|^{1/p}$$

for any $t, x \in [a, b]$.

If we multiply these inequalities and integrate, then we get

$$\begin{aligned} & \left| \int_a^b \int_a^b (tf(x) - xf(t))(tg(x) - xg(t)) dt dx \right| \\ & \leq \int_a^b \int_a^b |(tf(x) - xf(t))(tg(x) - xg(t))| dt dx \\ & \leq \frac{1}{(2q-1)^{1/q} (2p-1)^{1/p}} \|f - \ell f'\|_p \|g - \ell g'\|_q \\ & \quad \times \int_a^b \int_a^b \left| \frac{x^q}{t^{q-1}} - \frac{t^q}{x^{q-1}} \right|^{1/q} \left| \frac{x^p}{t^{p-1}} - \frac{t^p}{x^{p-1}} \right|^{1/p} dt dx. \end{aligned} \quad (3.5)$$

Utilizing Hölder's integral inequality for double integrals, we have

$$\begin{aligned} & \int_a^b \int_a^b \left| \frac{x^q}{t^{q-1}} - \frac{t^q}{x^{q-1}} \right|^{1/q} \left| \frac{x^p}{t^{p-1}} - \frac{t^p}{x^{p-1}} \right|^{1/p} dt dx \\ & \leq \left(\int_a^b \int_a^b \left| \frac{x^q}{t^{q-1}} - \frac{t^q}{x^{q-1}} \right| dt dx \right)^{1/q} \left(\int_a^b \int_a^b \left| \frac{x^p}{t^{p-1}} - \frac{t^p}{x^{p-1}} \right| dt dx \right)^{1/p} \\ & = M_q^{1/q}(a, b) M_p^{1/p}(a, b) \end{aligned} \quad (3.6)$$

for $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Utilizing Cauchy–Bunyakowsky–Schwarz integral inequality for double integrals, we have

$$\begin{aligned} M_q(a, b) & = \int_a^b \int_a^b \left| \frac{x^q}{t^{q-1}} - \frac{t^q}{x^{q-1}} \right| dt dx \\ & \leq \left(\int_a^b \int_a^b dt dx \right)^{1/2} \left(\int_a^b \int_a^b \left(\frac{x^q}{t^{q-1}} - \frac{t^q}{x^{q-1}} \right)^2 dt dx \right)^{1/2} \\ & = (b-a) \left(\int_a^b \int_a^b \left(\frac{x^q}{t^{q-1}} - \frac{t^q}{x^{q-1}} \right)^2 dt dx \right)^{1/2}. \end{aligned}$$



Observe that

$$\begin{aligned}
 N_q(a, b) &:= \int_a^b \int_a^b \left(\frac{x^q}{t^{q-1}} - \frac{t^q}{x^{q-1}} \right)^2 dt dx \\
 &= \int_a^b \int_a^b \frac{x^{2q}}{t^{2(q-1)}} dt dx - 2 \int_a^b \int_a^b \frac{x^q}{t^{q-1}} \frac{t^q}{x^{q-1}} dt dx + \int_a^b \int_a^b \frac{t^{2q}}{x^{2(q-1)}} dt dx \\
 &= 2 \int_a^b x^{2q} dx \int_a^b t^{-2(q-1)} dt - 2 \left(\int_a^b x dx \right)^2 \\
 &= 2 \left(\frac{b^{2q+1} - a^{2q+1}}{2q+1} \cdot \frac{b^{-2q+3} - a^{-2q+3}}{-2q+3} - \left(\frac{b^2 - a^2}{2} \right)^2 \right),
 \end{aligned}$$

provided $q \neq \frac{3}{2}$.

If $q = \frac{3}{2}$, then

$$N_q(a, b) = (b^2 - a^2) \left[\frac{b^2 + a^2}{2} \cdot \ln \frac{b}{a} - \frac{b^2 - a^2}{2} \right].$$

Therefore,

$$M_q(a, b) \leq (b - a) N_q^{1/2}(a, b)$$

and, similarly,

$$M_p(a, b) \leq (b - a) N_p^{1/2}(a, b).$$

□

Remark 3.3 The double integral

$$M_q(a, b) := \int_a^b \int_a^b \left| \frac{x^q}{t^{q-1}} - \frac{t^q}{x^{q-1}} \right| dt dx$$

can be computed exactly by iterating the integrals. However, the final form is too complicated to be stated here.

The Euclidian norms case is as follows:

Theorem 3.4 *Let $f, g : [a, b] \rightarrow \mathbb{C}$ be absolutely continuous functions on the interval $[a, b]$ with $b > a > 0$. If $f', g' \in L_2[a, b]$, then*

$$\begin{aligned}
 &\left| \frac{b^3 - a^3}{3} \int_a^b f(t) g(t) dt - \int_a^b t f(t) dt \int_a^b t g(t) dt \right| \\
 &\leq \frac{1}{9} \|f - \ell f'\|_2 \|g - \ell g'\|_2 \left[(b^3 + a^3) \ln \frac{b}{a} - \frac{2}{3} (b^3 - a^3) \right].
 \end{aligned} \tag{3.7}$$

Proof From the second inequality in (2.1), we have

$$|tf(x) - xf(t)| \leq \frac{1}{\sqrt{3}} \|f - \ell f'\|_2 \left| \frac{x^2}{t} - \frac{t^2}{x} \right|^{1/2}$$

and

$$|tg(x) - xg(t)| \leq \frac{1}{\sqrt{3}} \|g - \ell g'\|_2 \left| \frac{x^2}{t} - \frac{t^2}{x} \right|^{1/2}$$

for any $t, x \in [a, b]$.

If we multiply these inequalities and integrate, then we get

$$\begin{aligned} & \left| \int_a^b \int_a^b (tf(x) - xf(t))(tg(x) - xg(t)) dt dx \right| \\ & \leq \int_a^b \int_a^b |(tf(x) - xf(t))(tg(x) - xg(t))| dt dx \\ & \leq \frac{1}{3} \|f - \ell f'\|_2 \|g - \ell g'\|_2 \int_a^b \int_a^b \left| \frac{x^2}{t} - \frac{t^2}{x} \right| dt dx. \end{aligned} \quad (3.8)$$

Since

$$\begin{aligned} & \int_a^b \int_a^b \left| \frac{x^2}{t} - \frac{t^2}{x} \right| dt dx \\ & = \int_a^b \left(\int_a^x \left(\frac{x^2}{t} - \frac{t^2}{x} \right) dt + \int_x^b \left(\frac{t^2}{x} - \frac{x^2}{t} \right) dt \right) dx \\ & = \int_a^b \left(x^2 (2 \ln x - \ln a - \ln b) + \frac{b^3 + a^3 - 2x^3}{3x} \right) dx \end{aligned}$$

and

$$\begin{aligned} & \int_a^b x^2 (2 \ln x - \ln a - \ln b) dx \\ & = \int_a^b 2x^2 \ln x dx - \ln(ab) \int_a^b x^2 dx \\ & = \frac{(b^3 + a^3) \ln \frac{b}{a}}{3} - \frac{2}{9} (b^3 - a^3), \end{aligned}$$

while

$$\int_a^b \frac{b^3 + a^3 - 2x^3}{3x} dx = \frac{(b^3 + a^3) \ln \frac{b}{a}}{3} - \frac{2}{9} (b^3 - a^3),$$

then we conclude that

$$\int_a^b \int_a^b \left| \frac{x^2}{t} - \frac{t^2}{x} \right| dt dx = \frac{2}{3} \left[(b^3 + a^3) \ln \frac{b}{a} - \frac{2}{3} (b^3 - a^3) \right].$$

Making use of the inequality (3.8), we deduce the desired result (3.7). \square

Remark 3.5 It is an open question to the author if $\frac{1}{9}$ is best possible in (3.7).

Theorem 3.6 Let $f, g : [a, b] \rightarrow \mathbb{C}$ be absolutely continuous functions on the interval $[a, b]$ with $b > a > 0$. Then,

$$\begin{aligned} & \left| \frac{b^3 - a^3}{3} \int_a^b f(t) g(t) dt - \int_a^b tf(t) dt \int_a^b tg(t) dt \right| \\ & \leq \|f - \ell f'\|_1 \|g - \ell g'\|_1 \frac{2b^3 + a^3 - 3ab^2}{6a}. \end{aligned} \quad (3.9)$$

Proof From the third inequality in (2.1), we have

$$\begin{aligned} & \left| \int_a^b \int_a^b (tf(x) - xf(t))(tg(x) - xg(t)) dt dx \right| \\ & \leq \int_a^b \int_a^b |(tf(x) - xf(t))(tg(x) - xg(t))| dt dx \\ & \leq \|f - \ell f'\|_1 \|g - \ell g'\|_1 \int_a^b \int_a^b \left(\frac{\max\{t, x\}}{\min\{t, x\}} \right)^2 dt dx. \end{aligned} \quad (3.10)$$



Observe that

$$\begin{aligned} & \int_a^b \int_a^b \left(\frac{\max\{t, x\}}{\min\{t, x\}} \right)^2 dt dx \\ &= \int_a^b \left[\int_a^x \left(\frac{\max\{t, x\}}{\min\{t, x\}} \right)^2 dt + \int_x^b \left(\frac{\max\{t, x\}}{\min\{t, x\}} \right)^2 dt \right] dx \\ &= \int_a^b \left[\int_a^x \left(\frac{x}{t} \right)^2 dt + \int_x^b \left(\frac{t}{x} \right)^2 dt \right] dx \\ &= \frac{2b^3 + a^3 - 3ab^2}{6a}, \end{aligned}$$

which together with (3.10) produces the desired inequality (3.9). □

4 Some related results

The following result holds.

Theorem 4.1 *Let $f, g : [a, b] \rightarrow \mathbb{C}$ be absolutely continuous functions on the interval $[a, b]$ with $b > a > 0$. If $f', g' \in L_\infty[a, b]$, then*

$$\begin{aligned} & \left| (b-a) \int_a^b \frac{f(t)g(t)}{t^2} dt - \int_a^b \frac{f(t)}{t} dt \int_a^b \frac{g(t)}{t} dt \right| \\ & \leq (b-a)^2 \frac{L^2(a, b) - G^2(a, b)}{L^2(a, b)G^2(a, b)} \|f - \ell f'\|_\infty \|g - \ell g'\|_\infty, \end{aligned} \tag{4.1}$$

where $G(a, b) := \sqrt{ab}$ is the geometric mean and

$$L(a, b) := \frac{b-a}{\ln b - \ln a}$$

is the Logarithmic mean.

The inequality (4.1) is sharp.

Proof From the first inequality in (2.2), we have

$$\begin{aligned} & \left| \left(\frac{f(x)}{x} - \frac{f(t)}{t} \right) \left(\frac{g(x)}{x} - \frac{g(t)}{t} \right) \right| \\ & \leq \|f - \ell f'\|_\infty \|g - \ell g'\|_\infty \left(\frac{1}{t} - \frac{1}{x} \right)^2 \end{aligned} \tag{4.2}$$

for any $t, x \in [a, b]$.

Integrating this inequality on $[a, b]^2$, we get

$$\begin{aligned} & \left| \int_a^b \int_a^b \left(\frac{f(x)}{x} - \frac{f(t)}{t} \right) \left(\frac{g(x)}{x} - \frac{g(t)}{t} \right) dt dx \right| \\ & \leq \int_a^b \int_a^b \left| \left(\frac{f(x)}{x} - \frac{f(t)}{t} \right) \left(\frac{g(x)}{x} - \frac{g(t)}{t} \right) \right| dt dx \\ & \leq \|f - \ell f'\|_\infty \|g - \ell g'\|_\infty \int_a^b \int_a^b \left(\frac{1}{t} - \frac{1}{x} \right)^2 dt dx. \end{aligned} \tag{4.3}$$

We have

$$\begin{aligned} & \int_a^b \int_a^b \left(\frac{f(x)}{x} - \frac{f(t)}{t} \right) \left(\frac{g(x)}{x} - \frac{g(t)}{t} \right) dt dx \\ &= 2 \left[(b-a) \int_a^b \frac{f(t)g(t)}{t^2} dt - \int_a^b \frac{f(t)}{t} dt \int_a^b \frac{g(t)}{t} dt \right] \end{aligned}$$

and

$$\int_a^b \int_a^b \left(\frac{1}{t} - \frac{1}{x} \right)^2 dt dx = 2(b-a)^2 \frac{L^2(a,b) - G^2(a,b)}{L^2(a,b)G^2(a,b)}.$$

Making use of (4.3), we get the desired result (4.1).

If we take $f(t) = g(t) = 1$, then we have

$$\begin{aligned} & (b-a) \int_a^b \frac{f(t)g(t)}{t^2} dt - \int_a^b \frac{f(t)}{t} dt \int_a^b \frac{g(t)}{t} dt \\ &= (b-a)^2 \frac{L^2(a,b) - G^2(a,b)}{L^2(a,b)G^2(a,b)} \end{aligned}$$

and

$$\|f - \ell f'\|_\infty = \|g - \ell g'\|_\infty = 1,$$

and we obtain in both sides of (4.1) the same quantity

$$(b-a)^2 \frac{L^2(a,b) - G^2(a,b)}{L^2(a,b)G^2(a,b)}.$$

□

The case of Euclidian norms is as follows:

Theorem 4.2 Let $f, g : [a, b] \rightarrow \mathbb{C}$ be absolutely continuous functions on the interval $[a, b]$ with $b > a > 0$. If $f', g' \in L_2[a, b]$, then

$$\begin{aligned} & \left| (b-a) \int_a^b \frac{f(t)g(t)}{t^2} dt - \int_a^b \frac{f(t)}{t} dt \int_a^b \frac{g(t)}{t} dt \right| \\ & \leq \frac{1}{6} \|f - \ell f'\|_2 \|g - \ell g'\|_2 \frac{(b-a)^3}{a^2 b^2}. \end{aligned} \quad (4.4)$$

Proof From the second inequality in (2.2) for $p = q = 2$, we have

$$\left| \frac{f(x)}{x} - \frac{f(t)}{t} \right| \leq \frac{1}{\sqrt{3}} \|f - \ell f'\|_2 \left| \frac{1}{t^3} - \frac{1}{x^3} \right|^{1/2} \quad (4.5)$$

and

$$\left| \frac{g(x)}{x} - \frac{g(t)}{t} \right| \leq \frac{1}{\sqrt{3}} \|g - \ell g'\|_2 \left| \frac{1}{t^3} - \frac{1}{x^3} \right|^{1/2} \quad (4.6)$$

for any $t, x \in [a, b]$.

On multiplying (4.5) with (4.6), we derive

$$\left| \left(\frac{f(x)}{x} - \frac{f(t)}{t} \right) \left(\frac{g(x)}{x} - \frac{g(t)}{t} \right) \right| \leq \frac{1}{3} \|f - \ell f'\|_2 \|g - \ell g'\|_2 \left| \frac{1}{t^3} - \frac{1}{x^3} \right| \quad (4.7)$$

for any $t, x \in [a, b]$.



Integrating this inequality on $[a, b]^2$, we get

$$\begin{aligned} & \left| \int_a^b \int_a^b \left(\frac{f(x)}{x} - \frac{f(t)}{t} \right) \left(\frac{g(x)}{x} - \frac{g(t)}{t} \right) dt dx \right| \\ & \leq \int_a^b \int_a^b \left| \left(\frac{f(x)}{x} - \frac{f(t)}{t} \right) \left(\frac{g(x)}{x} - \frac{g(t)}{t} \right) \right| dt dx \\ & \leq \frac{1}{3} \|f - \ell f'\|_2 \|g - \ell g'\|_2 \int_a^b \int_a^b \left| \frac{1}{t^3} - \frac{1}{x^3} \right| dt dx. \end{aligned} \tag{4.8}$$

We have

$$\begin{aligned} \int_a^b \int_a^b \left| \frac{1}{t^3} - \frac{1}{x^3} \right| dt dx &= \int_a^b \left[\int_a^x \left(\frac{1}{t^3} - \frac{1}{x^3} \right) dt + \int_x^b \left(\frac{1}{x^3} - \frac{1}{t^3} \right) dt \right] dx \\ &= \int_a^b \left[\int_a^x \left(\frac{1}{t^3} - \frac{1}{x^3} \right) dt + \int_x^b \left(\frac{1}{x^3} - \frac{1}{t^3} \right) dt \right] dx = \frac{(b-a)^3}{a^2 b^2}. \end{aligned}$$

From (4.8), we then obtain the desired result (4.4). □

Remark 4.3 It is an open question to the author if $\frac{1}{6}$ is the best possible constant in (4.4).

The interested reader may obtain other similar results in terms of the norms $\|f - \ell f'\|_p \|g - \ell g'\|_q$ with $p, q > 1, p, q \neq 2$ and $\frac{1}{p} + \frac{1}{q} = 1$. However, the details are omitted.

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