Ostrowski type inequalities for functions whose derivatives are h-convex in absolute value

ACTA ET COMMENTATIONES UNIVERSITATIS TARTUENSIS DE MATHEMATICA
Volume 19, Number 2, December 2015
Available online at http://acutm.math.ut.ee

Ostrowski type inequalities for functions whose derivatives are h-convex in absolute value

S. S. Dragomir

Abstract. Some new inequalities of Ostrowski type for functions whose derivatives are h-convex in modulus are given. Applications for midpoint inequalities are provided.

1. Introduction

1.1. Ostrowski type inequalities. Comparison between functions and integral means are incorporated in Ostrowski type inequalities as follows.

The first result in this direction is due to Ostrowski [38].

Theorem 1. *Let $f\colon [a,b] \to \mathbb{R}$ be a differentiable function on (a,b) with the property that $|f'(t)| \le M$ for all $t \in (a,b)$. Then, for all $x \in [a,b]$,*

$$\begin{aligned} C(f;a,b) :&= \left| f(x) - \frac{1}{b-a} \int_a^b f(t)\,dt \right| \\ &\le \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] (b-a) M. \end{aligned}$$

The constant $\frac{1}{4}$ is the best possible in the sense that it cannot be replaced by a smaller quantity.

The following results for absolutely continuous functions hold (see [29], [30]).

Received February 19, 2015.

2010 *Mathematics Subject Classification.* 26D15, 25D10.

Key words and phrases. Absolutely continuous functions, convex functions, integral inequalities, Ostrowski inequality, Lebesgue norms, h-convex functions.

http://dx.doi.org/10.12697/ACUTM.2015.19.13

18

Theorem 2. *Let $f\colon [a,b] \to \mathbb{R}$ be absolutely continuous on $[a,b]$. Then, for all $x \in [a,b]$, we have*

$$C(f;a,b) \leq \begin{cases} \left[\frac{1}{4} + \left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)\|f'\|_\infty & \text{if} \quad f' \in L_\infty[a,b], \\ \\ \frac{1}{(\alpha+1)^{\frac{1}{\alpha}}}\left[\left(\frac{x-a}{b-a}\right)^{\alpha+1} + \left(\frac{b-x}{b-a}\right)^{\alpha+1}\right]^{\frac{1}{\alpha}} & \\ \times (b-a)^{\frac{1}{\alpha}}\|f'\|_\beta & \text{if} \quad f' \in L_\beta[a,b], \\ \\ \left[\frac{1}{2} + \left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\|f'\|_1 & \text{if} \quad f' \in L_1[a,b], \end{cases}$$

where $1/\alpha + 1/\beta = 1$, $\alpha > 1$, and $\|\cdot\|_{[a,b],r}$ $(r \in [1,\infty])$ are the usual Lebesgue norms on $L_r[a,b]$, i.e.,

$$\|g\|_{[a,b],\infty} := \operatorname*{ess\,sup}_{t\in[a,b]} |g(t)|$$

and

$$\|g\|_{[a,b],r} := \left(\int_a^b |g(t)|^r\,dt\right)^{\frac{1}{r}}, \; r \in [1,\infty).$$

The constants $\frac{1}{4}$, $\frac{1}{(\alpha+1)^{\frac{1}{\alpha}}}$ and $\frac{1}{2}$, respectively, are sharp in the sense presented in Theorem 1.

The above inequalities can also be obtained from the Fink result in [33] by choosing $n = 1$ and performing some appropriate computations.

If one drops the condition of absolute continuity and assumes that f is Hölder continuous, then one can obtain the following result (see, e.g., [21] and the references therein for earlier contributions).

Theorem 3. *Let $f\colon [a,b] \to \mathbb{R}$ be of r-H- Hölder type, i.e.,*

$$|f(x) - f(y)| \leq H|x-y|^r \textit{ for all } x, y \in [a,b],$$

where $r \in (0,1]$ and $H > 0$ are fixed. Then, for all $x \in [a,b]$, we have

$$C(f;a,b) \leq \frac{H}{r+1}\left[\left(\frac{b-x}{b-a}\right)^{r+1} + \left(\frac{x-a}{b-a}\right)^{r+1}\right](b-a)^r.$$

The constant $\frac{1}{r+1}$ is sharp in the above sense.

Note that if $r = 1$, i.e., f is Lipschitz continuous, then we get the following version of Ostrowski's inequality for Lipschitz functions (with L instead of H)

(see, e.g., [13]):

$$C(f;a,b)| \leq \left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)\,L,$$

where $x \in [a,b]$. Here the constant $\frac{1}{4}$ is also the best possible.

Moreover, if one drops the condition of the continuity of the function, and assumes that it is of bounded variation, then the following result can be obtained (see [15]).

Theorem 4. *Assume that* $f\colon [a,b] \to \mathbb{R}$ *is of bounded variation and denote by* $\bigvee_a^b (f)$ *its total variation. Then*

$$C(f;a,b) \leq \left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\bigvee_a^b (f) \tag{1.1}$$

for all $x \in [a,b]$. *The constant* $\frac{1}{2}$ *is the best possible.*

If we assume more about f, e.g., that f is monotonically increasing, then the inequality (1.1) may be improved in the following manner [12] (see also the monograph [28]).

Theorem 5. *Let* $f\colon [a,b] \to \mathbb{R}$ *be monotonic nondecreasing. Then, for all* $x \in [a,b]$, *we have*

$$\begin{aligned} C(f;a,b) &\leq \frac{1}{b-a}\left\{[2x-(a+b)]\,f(x)+\int_a^b sgn\,(t-x)\,f(t)\,dt\right\} \\ &\leq \frac{1}{b-a}\left\{(x-a)\,[f(x)-f(a)]+(b-x)\,[f(b)-f(x)]\right\} \qquad (1.2) \\ &\leq \left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right][f(b)-f(a)]\,. \end{aligned}$$

All the inequalities in (1.2) *are sharp and the constant* $\frac{1}{2}$ *is the best possible.*

The case for convex functions is as follows (see [17]).

Theorem 6. *Let* $f\colon [a,b] \subset \mathbb{R} \to \mathbb{R}$ *be a convex function on* $[a,b]$. *Then, for any* $x \in (a,b)$, *one has the inequality*

$$\begin{aligned} \frac{1}{2}\left[(b-x)^2 f'_+(x)-(x-a)^2 f'_-(x)\right] &\leq \int_a^b f(t)\,dt-(b-a)\,f(x) \\ &\leq \frac{1}{2}\left[(b-x)^2 f'_-(b)-(x-a)^2 f'_+(a)\right]. \end{aligned}$$

The constant $\frac{1}{2}$ *is sharp in both inequalities. The second inequality also holds for* $x=a$ *and* $x=b$.

For other Ostrowski's type inequalities for the Lebesgue integral, see [3] – [13] and [19].

Inequalities for the Riemann-Stieltjes integral may be found in [14] and [16] while the generalization for isotonic functionals was provided in [18].

For the case of functions of self-adjoint operators on complex Hilbert spaces, see the recent monograph [20].

1.2. The case of derivatives that are convex in modulus. In [18], the author pointed out the following identity in representing an absolutely continuous function. Due to the fact that we use it throughout the paper, we give here a short proof.

Lemma 1. *Let $f\colon [a,b] \to \mathbb{R}$ be an absolutely continuous function on $[a,b]$. Then, for any $x \in [a,b]$, one has the equality*

$$f(x) = \frac{1}{b-a}\int_a^b f(t)\,dt + \frac{1}{b-a}\int_a^b (x-t)\left(\int_0^1 f'[(1-\lambda)x + \lambda t]\,d\lambda\right)dt. \tag{1.3}$$

Proof. For any $t, x \in [a,b]$, $x \neq t$, one has

$$\frac{f(x)-f(t)}{x-t} = \frac{1}{x-t}\int_t^x f'(u)\,du = \int_0^1 f'[(1-\lambda)x + \lambda t]\,d\lambda,$$

thus

$$f(x) = f(t) + (x-t)\int_0^1 f'[(1-\lambda)x + \lambda t]\,d\lambda \tag{1.4}$$

for any $t, x \in [a,b]$.

Integrating (1.4) with respect to t on $[a,b]$ and dividing by $(b-a)$, we obtain the desired identity (1.3). □

Using the above lemma, the following result can be obtained improving Ostrowski's inequality [4].

Theorem 7. *Let $f\colon [a,b] \to \mathbb{C}$ be an absolutely continuous function on $[a,b]$ such that $|f'|$ is convex on (a,b).*

(i) *If $f' \in L_\infty[a,b]$, then, for any $x \in [a,b]$,*

$$C(f;a,b) \le \frac{1}{2}\left[\frac{1}{4} + \left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)\left[|f'(x)| + \|f'\|_\infty\right].$$

The constant $\frac{1}{2}$ is sharp in the sense that it cannot be replaced by a smaller quantity.

(ii) *If* $f' \in L_p[a,b]$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, *then, for any* $x \in [a,b]$,

$$C(f;a,b) \leq \frac{1}{2\,(q+1)^{\frac{1}{q}}} \left[\left(\frac{b-x}{b-a} \right)^{q+1} + \left(\frac{x-a}{b-a} \right)^{q+1} \right]^{\frac{1}{q}} \times (b-a)^{\frac{1}{q}} \left\| \left| f'(x) \right| + \left| f' \right| \right\|_p .$$

The constant $\frac{1}{2}$ *is sharp in the sense that it cannot be replaced by a smaller quantity.*

(iii) *If* $f' \in L_1[a,b]$, *then, for any* $x \in [a,b]$,

$$C(f;a,b) \leq \frac{1}{2} \left[\frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b-a} \right| \right] \left[(b-a) \left| f'(x) \right| + \left\| f' \right\|_1 \right] .$$

In order to extend this result to other classes of functions, we need the following preparatory section.

2. h-convex functions

2.1. Some definitions. We recall here some concepts of convexities that are well known in the literature. Let I be an interval in $\mathbb{R}$.

Definition 1 (see [32]). We say that $f\colon I \to \mathbb{R}$ is a Godunova–Levin function or that f belongs to the class $Q\,(I)$ if f is non-negative and, for all $x, y \in I$ and $t \in (0,1)$, we have

$$f\,(tx + (1-t)\,y) \leq \frac{1}{t} f\,(x) + \frac{1}{1-t} f\,(y)\,.$$

Some further properties of the class $Q(I)$ can be found in [24], [25], [27], [37], [40], and [41]. In particular, non-negative monotone and non-negative convex functions belong to this class of functions.

Definition 2 (see [27]). We say that a function $f\colon I \to \mathbb{R}$ belongs to the class $P\,(I)$ if it is non-negative and, for all $x, y \in I$ and $t \in [0,1]$, we have

$$f\,(tx + (1-t)\,y) \leq f\,(x) + f\,(y)\,.$$

Obviously $Q\,(I)$ contains $P\,(I)$ and, for applications, it is important to note that also $P\,(I)$ contains all non-negative monotone, convex and *quasi convex functions,* i.e., non-negative functions satisfying

$$f\,(tx + (1-t)\,y) \leq \max\left\{ f\,(x)\,, f\,(y) \right\}$$

for all $x, y \in I$ and $t \in [0,1]$.

For results on P-functions, see [27] and [39] while, for quasi convex functions, the reader can consult [26].

19

Definition 3 (see [6])**.** Let $s \in (0,1]$. A function $f\colon [0,\infty) \to [0,\infty)$ is said to be s-convex (in the second sense) or Breckner s-convex if

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y)$$

for all $x, y \in [0,\infty)$ and $t \in [0,1]$.

For properties of this class of functions see [2], [1], [6], [7], [22], [23], [34], [35], and [44].

In order to unify the above concepts, S. Varošanec introduced the concept of h-convex functions as follows.

Assume that I and J are intervals in $\mathbb{R}$, $(0,1) \subseteq J$, and functions h and f are real non-negative functions defined in J and I, respectively.

Definition 4 (see [46])**.** Let $h\colon J \to [0,\infty)$ with h not identical to 0. We say that $f\colon I \to [0,\infty)$ is an h-convex function if, for all $x, y \in I$, we have

$$f(tx + (1-t)y) \leq h(t) f(x) + h(1-t) f(y)$$

for all $t \in (0,1)$.

For some results concerning this class of functions see [46], [5], [36], [43], [42], and [45].

2.2. Inequalities of Hermite–Hadamard type. In [42] the authors proved the following Hermite–Hadamard type inequality for integrable h-convex functions.

Theorem 8. *Assume that $f\colon I \to [0,\infty)$ is an h-convex function, $h \in L[0,1]$ and $f \in L[a,b]$, where $a, b \in I$ with $a < b$. Then*

$$\frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)\,dt \leq [f(a) + f(b)] \int_0^1 h(t)\,dt. \tag{HH}$$

If we write (HH) for $h(t) = t$, then we get the classical Hermite–Hadamard inequality for convex functions.

If we write it for the case of P-type functions, i.e., $h(t) = 1$, then we get the inequality

$$\frac{1}{2} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)\,dt \leq f(a) + f(b)$$

provided $f \in L[a,b]$, which has been obtained in [27].

If f is integrable on $[a,b]$ and Breckner s-convex on $[a,b]$, for $s \in (0,1)$, then, by taking $h(t) = t^s$ in (HH), we get

$$2^{s-1} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)\,dt \leq \frac{f(a) + f(b)}{s+1},$$

which was obtained in [22].

Since for the case of Godunova–Levin class of functions we have $h(t) = \frac{1}{t}$, which is not Lebesgue integrable on $(0,1)$, we can not apply the left inequality in (HH).

We now introduce another class of functions.

Definition 5. We say that a function $f\colon I \to [0,\infty)$ is of s-Godunova–Levin type with $s \in [0,1]$, if

$$f(tx + (1-t)y) \leq \frac{1}{t^s} f(x) + \frac{1}{(1-t)^s} f(y)$$

for all $t \in (0,1)$ and $x, y \in I$.

We observe that for $s = 0$ we obtain the class of P-functions while for $s = 1$ we obtain the class of Godunova–Levin. If we denote by $Q_s(I)$ the class of s-Godunova–Levin functions defined on I, then we obviously have

$$P(I) = Q_0(I) \subseteq Q_{s_1}(I) \subseteq Q_{s_2}(I) \subseteq Q_1(I) = Q(I)$$

for $0 \leq s_1 \leq s_2 \leq 1$.

We have the following Hermite–Hadamard type inequality.

Theorem 9. *Assume that $f\colon I \to [0,\infty)$ is of s-Godunova–Levin type with $s \in [0,1)$. If $f \in L[a,b]$ where $a, b \in I$ and $a < b$, then*

$$\frac{1}{2^{s+1}} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(t)\,dt \leq \frac{f(a) + f(b)}{1-s}. \tag{2.1}$$

We notice that, for $s = 1$, the first inequality in (2.1) still holds. This was obtained for the first time in [27].

3. Inequalities for functions whose derivatives are h-convex in modulus

3.1. The case when $|f'|$ is h-convex. The following result holds.

Theorem 10. *Let $f\colon [a,b] \to \mathbb{C}$ be an absolutely continuous function on $[a,b]$ such that $|f'|$ is h-convex on (a,b) with $h \in L[0,1]$.*

(i) *If $f' \in L_\infty[a,b]$, then, for any $x \in [a,b]$,*

$$\begin{aligned} C(f;a,b) \leq &\left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a}\right)^2\right] \\ &\times (b-a)\left[\left|f'(x)\right| + \left\|f'\right\|_\infty\right] \int_0^1 h(t)\,dt. \end{aligned} \tag{3.1}$$

(ii) *If* $f' \in L_p[a,b]$, $p > 1$, $1/p + 1/q = 1$, *then, for any* $x \in [a,b]$,

$$C(f;a,b) \leq \frac{1}{(q+1)^{\frac{1}{q}}}\left[\left(\frac{b-x}{b-a}\right)^{q+1} + \left(\frac{x-a}{b-a}\right)^{q+1}\right]^{\frac{1}{q}} \times (b-a)^{\frac{1}{q}} \left\| |f'(x)| + |f'| \right\|_p \int_0^1 h(t)\,dt. \tag{3.2}$$

(iii) *If* $f' \in L_1[a,b]$, *then, for any* $x \in [a,b]$,

$$C(f;a,b) \leq \left[\frac{1}{2} + \left|\frac{x - \frac{a+b}{2}}{b-a}\right|\right]\left[(b-a)|f'(x)| + \|f'\|_1\right]\int_0^1 h(t)\,dt. \tag{3.3}$$

Proof. (i). Using (1.3) and taking the modulus, we have

$$\begin{aligned} C(f;a,b) &= \frac{1}{b-a}\left|\int_a^b \int_0^1 (x-t) f'[(1-\lambda)x + \lambda t]\,d\lambda dt\right| \\ &\leq \frac{1}{b-a}\int_a^b \int_0^1 |x-t| \left|f'[(1-\lambda)x + \lambda t]\right| d\lambda dt \\ &=: K. \end{aligned}$$

Utilizing the h-convexity of $|f'|$ we have

$$\begin{aligned} K &\leq \frac{1}{b-a}\int_a^b \int_0^1 |x-t| \left[h(1-\lambda)|f'(x)| + h(\lambda)|f'(t)|\right] d\lambda dt \\ &= \frac{1}{b-a}\int_a^b |x-t| \left[|f'(x)| \int_0^1 h(1-\lambda)\,d\lambda + |f'(t)| \int_0^1 h(\lambda)\,d\lambda\right] dt \\ &= \frac{1}{b-a}\int_0^1 h(\lambda)\,d\lambda \int_a^b |x-t| \left[|f'(x)| + |f'(t)|\right] dt := M(x) \int_0^1 h(\lambda)\,d\lambda \\ &\leq \frac{1}{b-a}\int_0^1 h(\lambda)\,d\lambda \; ess \sup_{t\in[a,b]} \left[|f'(x)| + |f'(t)|\right] \int_a^b |x-t|\,dt \\ &= \left[\frac{(x-a)^2 + (b-x)^2}{2(b-a)}\right]\left[|f'(x)| + \|f'\|_\infty\right]\int_0^1 h(\lambda)\,d\lambda \\ &= \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a}\right)^2\right](b-a)\left[|f'(x)| + \|f'\|_\infty\right]\int_0^1 h(\lambda)\,d\lambda \end{aligned}$$

for any $x \in [a,b]$. The inequality (3.1) is proved.

(ii). As above, we have

$$C(f;a,b) \leq \frac{1}{b-a}\int_a^b |x-t| \left[|f'(x)| + |f'(t)|\right] dt =: M(x) \int_0^1 h(\lambda)\,d\lambda.$$

Using Hölder's integral inequality for $p > 1,\ 1/p + 1/q = 1$, we get that

$$\begin{aligned} M(x) &\leq \frac{1}{b-a}\left(\int_a^b |x-t|^q\, dt\right)^{\frac{1}{q}}\left(\int_a^b \left(\left|f'(x)\right| + \left|f'(t)\right|\right)^p dt\right)^{\frac{1}{p}} \\ &= \frac{1}{b-a}\left[\frac{(b-x)^{q+1} + (x-a)^{q+1}}{q+1}\right]^{\frac{1}{q}} \left\|\left|f'(x)\right| + \left|f'\right|\right\|_p, \end{aligned}$$

and the inequality (3.2) is proved.

(iii). We have

$$\begin{aligned} M(x) &\leq \sup_{t\in[a,b]} |x-t| \frac{1}{b-a}\int_a^b \left[\left|f'(x)\right| + \left|f'(t)\right|\right] dt \\ &= \frac{1}{b-a)} \max(x-a, b-x)\left[(b-a)\left|f'(x)\right| + \int_a^b \left|f'(t)\right| dt\right] \\ &= \left[\frac{1}{2} + \left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\left[(b-a)\left|f'(x)\right| + \left\|f'\right\|_1\right], \end{aligned}$$

and the inequality (3.3) is proved. □

The following particular case is interesting.

Corollary 1. *Under the assumptions of Theorem 10, we have the mid-point inequality*

$$\begin{aligned} &\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)dt\right| \\ &\leq \frac{1}{4}(b-a)\left[\left|f'\left(\frac{a+b}{2}\right)\right| + \left\|f'\right\|_\infty\right]\int_0^1 h(t)\, dt \end{aligned}$$

provided $f' \in L_\infty[a,b]$.

If $f' \in L_p[a,b]$, $p > 1,\ 1/p + 1/q = 1$, then

$$\begin{aligned} &\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)dt\right| \\ &\leq \frac{1}{2}(b-a)^{\frac{1}{q}}\left(\int_a^b \left[\left|f'\left(\frac{a+b}{2}\right)\right| + \left|f'(t)\right|\right]^p dt\right)^{\frac{1}{p}}\int_0^1 h(t)\, dt. \end{aligned}$$

If $f' \in L_1[a,b]$, then

$$\begin{aligned} &\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)dt\right| \\ &\leq \frac{1}{2}\left[(b-a)\left|f'\left(\frac{a+b}{2}\right)\right| + \int_a^b \left|f'(t)\right| dt\right]\int_0^1 h(t)\, dt. \end{aligned}$$

20

Remark 1. We observe that if $|f'|$ is convex on (a,b), then Theorem 10 reduces to Theorem 7.

Assume that $|f'|$ is Breckner s-convex on $[a,b]$ for $s \in (0,1)$.

(a) If $f' \in L_\infty[a,b]$, then, for any $x \in [a,b]$,

$$C(f;a,b) \le \frac{1}{s+1}\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)\left[|f'(x)|+\|f'\|_\infty\right].$$

(aa) If $f' \in L_p[a,b]$, $p>1$, $1/p+1/q=1$, then, for any $x \in [a,b]$,

$$\begin{aligned}C(f;a,b) \le{}& \frac{1}{(s+1)(q+1)^{\frac{1}{q}}}\left[\left(\frac{b-x}{b-a}\right)^{q+1}+\left(\frac{x-a}{b-a}\right)^{q+1}\right]^{\frac{1}{q}}\\ &\times (b-a)^{\frac{1}{q}}\left\||f'(x)|+|f'|\right\|_p.\end{aligned}$$

(aaa) If $f' \in L_1[a,b]$, then, for any $x \in [a,b]$,

$$C(f;a,b) \le \frac{1}{s+1}\left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\left[(b-a)|f'(x)|+\|f'\|_1\right].$$

Assume that $|f'|$ is of s-Godunova-Levin type with $s \in [0,1)$.

(b) If $f' \in L_\infty[a,b]$, then, for any $x \in [a,b]$,

$$C(f;a,b) \le \frac{1}{1-s}\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)\left[|f'(x)|+\|f'\|_\infty\right].$$

(bb) If $f' \in L_p[a,b]$, $p>1$, $1/p+1/q=1$, then, for any $x \in [a,b]$,

$$\begin{aligned}C(f;a,b) \le{}& \frac{1}{(1-s)(q+1)^{\frac{1}{q}}}\left[\left(\frac{b-x}{b-a}\right)^{q+1}+\left(\frac{x-a}{b-a}\right)^{q+1}\right]^{\frac{1}{q}}\\ &\times (b-a)^{\frac{1}{q}}\left\||f'(x)|+|f'|\right\|_p.\end{aligned}$$

(bbb) If $f' \in L_1[a,b]$, then, for any $x \in [a,b]$,

$$C(f;a,b) \le \frac{1}{1-s}\left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\left[(b-a)|f'(x)|+\|f'\|_1\right].$$

3.2. The case when $|f'|^p$ is h-convex.

The following result holds.

Theorem 11. *Let $f\colon [a,b] \to \mathbb{C}$ be an absolutely continuous function on $[a,b]$ such that $|f'|^p$ with $p>1$ is h-convex on (a,b) where $h \in L[0,1]$.*

(i) *If* $f' \in L_\infty[a,b]$, *then, for any* $x \in [a,b]$,

$$C(f;a,b) \le \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a}\right)^2\right](b-a) \times \left[\left|f'(x)\right|^p + \left\|f'\right\|_\infty^p\right]^{1/p} \left(\int_0^1 h(t)\,dt\right)^{1/p}. \tag{3.4}$$

(ii) *If* $f' \in L_p[a,b]$, $p > 1$, $1/p + 1/q = 1$, *then, for any* $x \in [a,b]$,

$$C(f;a,b) \le \frac{(b-a)^{\frac{1}{q}}}{(q+1)^{1/q}} \left[\left(\frac{b-x}{b-a}\right)^{q+1} + \left(\frac{x-a}{b-a}\right)^{q+1}\right]^{1/q} \times \left[(b-a)\left|f'(x)\right|^p + \left\|f'\right\|_p^p\right]^{1/p} \left(\int_0^1 h(t)\,dt\right)^{1/p}. \tag{3.5}$$

(iii) *If* $f' \in L_p[a,b]$, *then, for any* $x \in [a,b]$,

$$\begin{aligned} C(f;a,b) &\le \left[\frac{1}{2} + \left|\frac{x - \frac{a+b}{2}}{b-a}\right|\right] \\ &\quad \times \left\|\left|f'(x)\right|^p + \left|f'\right|^p\right\|^p \left(\int_0^1 h(t)\,dt\right)^{1/p} \\ &\le \left[\frac{1}{2} + \left|\frac{x - \frac{a+b}{2}}{b-a}\right|\right] \\ &\quad \times \left((b-a)\left|f'(x)\right|^p + \left\|f'\right\|_p^p\right)^{1/p} \left(\int_0^1 h(t)\,dt\right)^{1/p}. \end{aligned}$$

Proof. As in the proof of Theorem 10, we have

$$\begin{aligned} C(f;a,b) &= \frac{1}{b-a}\left|\int_a^b \int_0^1 (x-t)\, f'\left[(1-\lambda)x + \lambda t\right] d\lambda dt\right| \\ &\le \frac{1}{b-a}\int_a^b |x-t| \left(\int_0^1 \left|f'\left[(1-\lambda)x + \lambda t\right]\right| d\lambda\right) dt =: K \end{aligned}$$

for any $x \in [a,b]$.

By Hölder's integral inequality, we have

$$\begin{aligned} \int_0^1 \left|f'\left[(1-\lambda)x + \lambda t\right]\right| d\lambda &\le \left(\int_0^1 1^q d\lambda\right)^{1/q} \left(\int_0^1 \left|f'\left[(1-\lambda)x + \lambda t\right]\right|^p d\lambda\right)^{1/p} \\ &= \left(\int_0^1 \left|f'\left[(1-\lambda)x + \lambda t\right]\right|^p d\lambda\right)^{1/p} \end{aligned}$$

for any $x \in [a,b]$, where $1/p + 1/q = 1$, $p > 1$.

Since $|f'|^p$ is h-convex on (a,b) with $h \in L[0,1]$, we have

$$\int_0^1 \left|f'[(1-\lambda)x+\lambda t]\right|^p d\lambda \le \left[\left|f'(x)\right|^p + \left|f'(t)\right|^p\right] \int_0^1 h(\lambda)\, d\lambda$$

for any $x \in [a,b]$.

Therefore,

$$K \le \frac{1}{b-a}\left(\int_0^1 h(\lambda)\, d\lambda\right)^{1/p} \int_a^b |x-t| \left[\left|f'(x)\right|^p + \left|f'(t)\right|^p\right]^{1/p} dt \tag{3.6}$$

for any $x \in [a,b]$.

(i). If $f' \in L_\infty[a,b]$, then

$$\begin{aligned}
&\int_a^b |x-t| \left[\left|f'(x)\right|^p + \left|f'(t)\right|^p\right]^{1/p} dt \\
&\le \operatorname*{ess\,sup}_{t\in[a,b]} \left[\left|f'(x)\right|^p + \left|f'(t)\right|^p\right]^{1/p} \int_a^b |x-t|\, dt \\
&= \left[\left|f'(x)\right|^p + \left\|f'\right\|_\infty^p\right]^{1/p} \frac{1}{2}\left[(x-a)^2 + (b-x)^2\right]
\end{aligned}$$

for any $x \in [a,b]$, and utilizing (3.6), the inequality (3.4) is proved.

(ii). If $f' \in L_p[a,b]$, $p>1$, $1/p+1/q=1$, then by Hölder's inequality we have

$$\begin{aligned}
&\int_a^b |x-t| \left[\left|f'(x)\right|^p + \left|f'(t)\right|^p\right]^{1/p} dt \\
&\le \left(\int_a^b |x-t|^q\, dt\right)^{1/q} \left(\int_a^b \left(\left[\left|f'(x)\right|^p + \left|f'(t)\right|^p\right]^{1/p}\right)^p dt\right)^{1/p} \\
&= \left[\frac{(b-x)^{q+1} + (x-a)^{q+1}}{q+1}\right]^{1/q} \left[(b-a)\left|f'(x)\right|^p + \left\|f'\right\|_p^p\right]^{1/p} \\
&= \frac{(b-a)^{1+\frac{1}{q}}}{(q+1)^{1/q}} \left[\left(\frac{b-x}{b-a}\right)^{q+1} + \left(\frac{x-a}{b-a}\right)^{q+1}\right]^{1/q} \\
&\quad \times \left[(b-a)\left|f'(x)\right|^p + \left\|f'\right\|_1^p\right]^{1/p}
\end{aligned}$$

for any $x \in [a,b]$, and by (3.6) we deduce the desired inequality (3.5).

(iii). If $f' \in L_p[a,b]$, then by Hölder's inequality we also have

$$
\begin{aligned}
&\int_a^b |x-t| \left[\left|f'(x)\right|^p + \left|f'(t)\right|^p \right]^{1/p} dt \\
&\leq \sup_{t\in[a,b]} |x-t| \int_a^b \left[\left|f'(x)\right|^p + \left|f'(t)\right|^p \right]^{1/p} dt \\
&= \max\{x-a, b-x\} \int_a^b \left[\left|f'(x)\right|^p + \left|f'(t)\right|^p \right]^{1/p} dt \\
&= (b-a)\left[\frac{1}{2} + \left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right] \left\| \left|f'(x)\right|^p + \left|f'\right|^p \right\|^p \\
&\leq (b-a)\left[\frac{1}{2} + \left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right] \left(\int_a^b \left[\left|f'(x)\right|^p + \left|f'(t)\right|^p \right] dt \right)^{1/p} \\
&= (b-a)\left[\frac{1}{2} + \left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right] \left((b-a)\left|f'(x)\right|^p + \left\|f'\right\|_p^p \right)^{1/p}
\end{aligned}
$$

for any $x \in [a,b]$. □

The following midpoint type inequalities are of interest.

Corollary 2. *Under the assumptions of Theorem 11, we have the inequality*

$$
\begin{aligned}
&\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)dt \right| \\
&\leq \frac{1}{4}(b-a)\left[\left| f'\left(\frac{a+b}{2}\right)\right|^p + \left\|f'\right\|_\infty^p \right]^{1/p} \left(\int_0^1 h(t)\,dt \right)^{1/p}
\end{aligned}
$$

provided $f' \in L_\infty[a,b]$.

If $f' \in L_p[a,b]$, $p > 1$, $1/p + 1/q = 1$, *then*

$$
\begin{aligned}
&\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)dt \right| \\
&\leq \frac{1}{2(q+1)^{1/q}}(b-a)^{\frac{1}{q}} \\
&\quad \times \left[(b-a)\left| f'\left(\frac{a+b}{2}\right)\right|^p + \left\|f'\right\|_p^p \right]^{1/p} \left(\int_0^1 h(t)\,dt \right)^{1/p}.
\end{aligned}
$$

If $f' \in L_p[a,b]$, *then*

$$\begin{aligned}
&\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)dt \right| \\
&\leq \frac{1}{2}\left\| \left| f'\left(\frac{a+b}{2}\right)\right|^p + \left| f'\right|^p \right\|^p \left(\int_0^1 h\left(t\right)dt\right)^{1/p} \\
&\leq \frac{1}{2}\left((b-a)\left| f'\left(\frac{a+b}{2}\right)\right|^p + \left\| f'\right\|_p^p \right)^{1/p} \left(\int_0^1 h\left(t\right)dt\right)^{1/p}.
\end{aligned}$$

Remark 2. The interested reader can state the corresponding particular inequalities for different h-convex functions.

References

[1] M. Alomari and M. Darus, *Hadamard-type inequalities for s-convex functions*, Int. Math. Forum **3** (2008), no. 37-40, 1965–1975.

[2] M. Alomari and M. Darus, *The Hadamard's inequality for s-convex function*, Int. J. Math. Anal. (Ruse) **2** (2008), no. 13-16, 639–646.

[3] G. A. Anastassiou, *Univariate Ostrowski inequalities, revisited*, Monatsh. Math. **135** (2002), no. 3, 175–189.

[4] N. S. Barnett, P. Cerone, S. S. Dragomir, M. R. Pinheiro, and A. Sofo, *Ostrowski type inequalities for functions whose modulus of the derivatives are convex and applications*, in: Inequality Theory and Applications, Vol. 2 (Chinju/Masan, 2001), pp. 19–32, Nova Sci. Publ., Hauppauge, NY, 2003. Preprint: RGMIA Res. Rep. Coll. **5** (2002), No. 2, Art. 1 `[Online http://rgmia.org/papers/v5n2/Paperwapp2q.pdf]`.

[5] M. Bombardelli and S. Varošanec, *Properties of h-convex functions related to the Hermite-Hadamard-Fejér inequalities*, Comput. Math. Appl. **58** (2009), no. 9, 1869–1877.

[6] W. W. Breckner, *Stetigkeitsaussagen für eine Klasse verallgemeinerter konvexer Funktionen in topologischen linearen Räumen*, Publ. Inst. Math. (Beograd) (N.S.) **23(37)** (1978), 13–20.

[7] W. W. Breckner and G. Orbán, *Continuity Properties of Rationally s-Convex Mappings with Values in an Ordered Topological Linear Space*, Universitatea "Babeş-Bolyai", Facultatea de Matematica, Cluj-Napoca, 1978.

[8] P. Cerone and S. S. Dragomir, *Midpoint-type rules from an inequalities point of view*, in: Handbook of Analytic-Computational Methods in Applied Mathematics, Chapman & Hall/CRC, Boca Raton, FL, 2000, pp. 135–200.

[9] P. Cerone and S. S. Dragomir, *New bounds for the three-point rule involving the Riemann-Stieltjes integrals*, in: Advances in Statistics Combinatorics and Related Areas, World Science Publishing, River Edge, NJ, 2002, pp. 53–62.

[10] P. Cerone, S. S. Dragomir, and J. Roumeliotis, *Some Ostrowski type inequalities for n-time differentiable mappings and applications*, Demonstratio Math. **32**(2) (1999), 697–712.

[11] G. Cristescu, *Hadamard type inequalities for convolution of h-convex functions.* Ann. Tiberiu Popoviciu Semin, Funct. Equ. Approx. Convexity **8** (2010), 3–11.

[12] S. S. Dragomir, *Ostrowski's inequality for monotonous mappings and applications*, J. KSIAM **3**(1) (1999), 127–135.

[13] S. S. Dragomir, *The Ostrowski's integral inequality for Lipschitzian mappings and applications*, Comp. Math. Appl. **38** (1999), 33-37.

[14] S. S. Dragomir, *On the Ostrowski's inequality for Riemann-Stieltjes integral,* Korean J. Appl. Math. **7** (2000), 477–485.

[15] S. S. Dragomir, *On the Ostrowski's inequality for mappings of bounded variation and applications*, Math. Inequal. Appl. **4**(1) (2001), 33–40.

[16] S. S. Dragomir, *On the Ostrowski's inequality for Riemann-Stieltjes integral* $\int_a^b f(t)\,du(t)$ *where* f *is of Hölder type and* u *is of bounded variation and applications*, J. KSIAM **5**(1) (2001), 35–45.

[17] S. S. Dragomir, *An inequality improving the first Hermite-Hadamard inequality for convex functions defined on linear spaces and applications for semi-inner products*, J. Inequal. Pure Appl. Math. **3** (2002), no. 2, Article 31, 8 pp.

[18] S. S. Dragomir, *Ostrowski type inequalities for isotonic linear functionals*, J. Inequal. Pure Appl. Math. **3**(5) (2002), Art. 68.

[19] S. S. Dragomir, *An Ostrowski like inequality for convex functions and applications*, Rev. Mat. Complut. **16**(2) (2003), 373–382.

[20] S. S. Dragomir, *Operator Inequalities of Ostrowski and Trapezoidal Type*, Springer Briefs in Mathematics. Springer, New York, 2012.

[21] S. S. Dragomir, P. Cerone, J. Roumeliotis, and S. Wang, *A weighted version of Ostrowski inequality for mappings of Hölder type and applications in numerical analysis*, Bull. Math. Soc. Sci. Math. Roumanie **42(90)**(4) (1999), 301–314.

[22] S. S. Dragomir and S. Fitzpatrick, *The Hadamard inequalities for s-convex functions in the second sense*, Demonstratio Math. **32** (1999), no. 4, 687–696.

[23] S. S. Dragomir and S. Fitzpatrick, *The Jensen inequality for s-Breckner convex functions in linear spaces*, Demonstratio Math. **33** (2000), no. 1, 43–49.

[24] S. S. Dragomir and B. Mond, *On Hadamard's inequality for a class of functions of Godunova and Levin*, Indian J. Math. **39** (1997), no. 1, 1–9.

[25] S. S. Dragomir and C. E. M. Pearce, *On Jensen's inequality for a class of functions of Godunova and Levin*, Period. Math. Hungar. **33** (1996), no. 2, 93–100.

[26] S. S. Dragomir and C. E. M. Pearce, *Quasi-convex functions and Hadamard's inequality*, Bull. Austral. Math. Soc. **57** (1998), 377–385.

[27] S. S. Dragomir, J. Pečarić, and L. Persson, *Some inequalities of Hadamard type*, Soochow J. Math. **21** (1995), no. 3, 335–341.

[28] S. S. Dragomir and Th. M. Rassias (Eds), *Ostrowski Type Inequalities and Applications in Numerical Integration*, Kluwer Academic Publisher, Dordrecht, 2002.

[29] S. S. Dragomir and S. Wang, *A new inequality of Ostrowski's type in L_1-norm and applications to some special means and to some numerical quadrature rules*, Tamkang J. Math. **28** (1997), 239–244.

[30] S. S. Dragomir and S. Wang, *A new inequality of Ostrowski's type in L_p-norm and applications to some special means and to some numerical quadrature rules*, Indian J. Math. **40**(3) (1998), 245–304.

[31] S. S. Dragomir and S. Wang, *Applications of Ostrowski's inequality to the estimation of error bounds for some special means and some numerical quadrature rules*, Appl. Math. Lett. **11** (1998), 105–109.

[32] E. K. Godunova and V. I. Levin, *Inequalities for functions of a broad class that contains convex, monotone and some other forms of functions*, in: Numerical Mathematics and Mathematical Physics, Moskov. Gos. Ped. Inst., Moscow, 1985, pp. 138–142. (Russian)

[33] A. M. Fink, *Bounds on the deviation of a function from its averages*, Czechoslovak Math. J. **42(117)** (1992), no. 2, 298–310.

[34] H. Hudzik and L. Maligranda, *Some remarks on s-convex functions*, Aequationes Math. **48** (1994), no. 1, 100–111.

[35] U. S. Kirmaci, M. Klaričić Bakula, M. E Özdemir, and J. Pečarić, *Hadamard-type inequalities for s-convex functions*, Appl. Math. Comput. **193** (2007), no. 1, 26–35.
[36] M. A. Latif, *On some inequalities for h-convex functions*, Int. J. Math. Anal. (Ruse) **4** (2010), no. 29-32, 1473–1482.
[37] D. S. Mitrinović and J. E. Pečarić, *Note on a class of functions of Godunova and Levin*, C. R. Math. Rep. Acad. Sci. Canada **12** (1990), no. 1, 33–36.
[38] A. Ostrowski, *Über die Absolutabweichung einer differentienbaren Funktionen von ihren Integralmittelwert*, Comment. Math. Hel. **10** (1938), 226–227.
[39] C. E. M. Pearce and A. M. Rubinov, *P-functions, quasi-convex functions, and Hadamard-type inequalities*, J. Math. Anal. Appl. **240** (1999), no. 1, 92–104.
[40] J. E. Pečarić and S. S. Dragomir, *On an inequality of Godunova-Levin and some refinements of Jensen integral inequality*, in: Itinerant Seminar on Functional Equations, Approximation and Convexity (Cluj-Napoca, 1989), pp. 263–268, Preprint, 89-6, Univ. "Babeş-Bolyai", Cluj-Napoca, 1989.
[41] M. Radulescu, S. Radulescu, and P. Alexandrescu, *On the Godunova-Levin-Schur class of functions*, Math. Inequal. Appl. **12** (2009), no. 4, 853–862.
[42] M. Z. Sarikaya, A. Saglam, and H. Yildirim, *On some Hadamard-type inequalities for h-convex functions*, J. Math. Inequal. **2** (2008), no. 3, 335–341.
[43] M. Z. Sarikaya, E. Set and M. E. Özdemir, *On some new inequalities of Hadamard type involving h-convex functions*, Acta Math. Univ. Comenian. (N.S.) **79** (2010), no. 2, 265–272.
[44] E. Set, M. E. Özdemir, and M. Z. Sarıkaya, *New inequalities of Ostrowski's type for s-convex functions in the second sense with applications*, Facta Univ. Ser. Math. Inform. **27** (2012), no. 1, 67–82.
[45] M. Tunç, *Ostrowski-type inequalities via h-convex functions with applications to special means*, J. Inequal. Appl. **2013**, 2013:326.
[46] S. Varošanec, *On h-convexity*, J. Math. Anal. Appl. **326** (2007), no. 1, 303–311.

Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia; School of Computational & Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa

E-mail address: sever.dragomir@vu.edu.au
URL: http://rgmia.org/dragomir