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SUMS OF QUADRATIC HALF INTEGER HARMONIC NUMBERS OF ALTERNATING TYPE

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Abstract. Half integer values of quadratic harmonic numbers and reciprocal binomial coefficients sums are investigated in this paper. Closed form representations of double integral expressions are developed in terms of special functions.

1. Introduction and preliminaries

We express, amongst other results, the double integral representation

$$\int_0^1 \int_0^1 \frac{1}{(1-x)(1-y)} \left(\begin{array}{c} \Phi(-1,1,1+r) - \sqrt{x}\,\Phi(-x,1,1+r) \\ +\sqrt{y}\,\Phi(-y,1,1+r) + \frac{\sqrt{xy}}{1+r}\,{}_2F_1\left[\begin{array}{c} 1,1+r \\ 2+r \end{array}\middle| -xy\right] \end{array} \right) dxdy$$

in terms of special functions and a connection with alternating quadratic half integer harmonic numbers of the form

$$M(k,p) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\, H^2_{n-\frac{1}{2}}}{n^p \binom{n+k}{k}} \tag{1}$$

where k is a positive integer and $p = \{0,1\}$. Here $\Phi(z,t,a) = \sum_{m=0}^{\infty} \frac{z^m}{(m+a)^t}$ is the Lerch transcendent defined for $|z| < 1$ and $\mathbb{R}(a) > 0$ and satisfies the recurrence

$$\Phi(z,t,a) = z\,\Phi(z,t,a+1) + a^{-t}.$$

The Lerch transcendent generalizes the Hurwitz zeta function at $z = 1$,

$$\Phi(1,t,a) = \sum_{m=0}^{\infty} \frac{1}{(m+a)^t}$$

and the Polylogarithm, or de Jonquière's function, when $a = 1$,

$$Li_t(z) := \sum_{m=1}^{\infty} \frac{z^m}{m^t},\ t \in \mathbb{C} \text{ when } |z| < 1;\ \mathbb{R}(t) > 1 \text{ when } |z| = 1.$$

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Also, $H^2_{n-\frac{1}{2}}$ is a quadratic half integer harmonic number and ${}_2F_1\left[\begin{matrix}\cdot,\cdot\\ \cdot\end{matrix}\middle| z\right]$ is the classical Gauss hypergeometric function. It is known that the harmonic number H_n has the usual definition

$$H_n = \sum_{r=1}^{n} \frac{1}{r} = \sum_{j=1}^{\infty} \frac{n}{j(j+n)} = \int_0^1 \frac{1-x^n}{1-x}\,dx \qquad (H_0 := 0) \tag{2}$$

for $n \in \mathbb{N}$ where $\mathbb{N} := \{1,2,3,\cdots\}$ is the set of positive integers, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. An unusual, but intriguing representation has recently been given by Ciaurri et. al. [6], as

$$H_n = \pi \int_0^1 \left(x - \frac{1}{2}\right) \left(\frac{\cos\left(\frac{(4n+1)\pi x}{2}\right) - \cos\left(\frac{\pi x}{2}\right)}{\sin\left(\frac{\pi x}{2}\right)} \right) dx.$$

Choi [2] has also given the definition, in terms of log-sine functions

$$\begin{aligned} H_n &= -4n \int_0^{\frac{\pi}{2}} \ln(\sin x) \sin x (\cos x)^{2n-1}\,dx \\ &= -4n \int_0^{\frac{\pi}{2}} \ln(\cos x) \cos x (\sin x)^{2n-1}\,dx. \end{aligned}$$

Let $\mathbb{R}$ and $\mathbb{C}$ denote, respectively the sets of real and complex numbers. We define harmonic numbers at half integer values as $H_{n-\frac{1}{2}}$, which may be expressed in terms of the digamma (or Psi) function $\psi(z), z \in \mathbb{R}$ and the Euler-Mascheroni constant, γ as $H_{n-\frac{1}{2}} = \gamma + \psi\left(n + \frac{1}{2}\right)$. The digamma function is defined by

$$\psi(z) := \frac{d}{dz}\{\log \Gamma(z)\} = \frac{\Gamma'(z)}{\Gamma(z)} \qquad \text{or} \qquad \log \Gamma(z) = \int_1^z \psi(t)\,dt,$$

where $z \in \mathbb{C}$ such that $\mathbb{R}(z) > 0$. A generalized binomial coefficient $\binom{\lambda}{\mu}$ $(\lambda, \mu \in \mathbb{C})$ is defined, in terms of the familiar (Euler's) gamma function, by

$$\binom{\lambda}{\mu} := \frac{\Gamma(\lambda+1)}{\Gamma(\mu+1)\Gamma(\lambda-\mu+1)}, \qquad (\lambda, \mu \in \mathbb{C}),$$

which, in the special case when $\mu = n,\ n \in \mathbb{N}_0$, yields

$$\binom{\lambda}{0} := 1 \quad \text{and} \quad \binom{\lambda}{n} := \frac{\lambda(\lambda-1)\cdots(\lambda-n+1)}{n!} = \frac{(-1)^n(-\lambda)_n}{n!} \quad (n \in \mathbb{N}),$$

where $(\lambda)_\nu$ $(\lambda, \nu \in \mathbb{C})$ is the Pochhammer symbol defined, also in terms of the gamma function, by

$$(\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu = 0;\ \lambda \in \mathbb{C} \setminus \{0\}) \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (\nu = n \in \mathbb{N};\ \lambda \in \mathbb{C}), \end{cases}$$

it being understood *conventionally* that $(0)_0 := 1$ and assumed that the Γ-quotient exists. A *generalized harmonic number* $H_n^{(m)}$ of order m is defined, for positive integers n and m, as follows:

$$H_n^{(m)} := \sum_{r=1}^{n} \frac{1}{r^m}, \quad (m,n \in \mathbb{N}) \qquad \text{and} \qquad H_0^{(m)} := 0 \qquad (m \in \mathbb{N})$$

and the polygamma function is defined by

$$\psi^{(n)}(z) := \frac{d^n}{dz^n}\{\psi(z)\} = \frac{d^{n+1}}{dz^{n+1}}\{\log\Gamma(z)\} \qquad (n \in \mathbb{N}_0).$$

In the case of non integer values of the argument $z = \frac{r}{q}$, we may write the generalized harmonic numbers, $H_z^{(\alpha+1)}$, in terms of polygamma functions

$$H_{\frac{r}{q}}^{(\alpha+1)} = \zeta(\alpha+1) + \frac{(-1)^\alpha}{\alpha!}\psi^{(\alpha)}\left(\frac{r}{q}+1\right), \quad \frac{r}{q} \neq \{-1,-2,-3,...\},$$

where $\zeta(z)$ is the zeta function. We also define

$$H_{\frac{r}{q}} = H_{\frac{r}{q}}^{(1)} = \gamma + \psi\left(\frac{r}{q}+1\right).$$

The evaluation of the polygamma function $\psi^{(\alpha)}\left(\frac{r}{a}\right)$ at rational values of the argument can be explicitly done via a formula as given by Kölbig [9], or Choi and Cvijovic [3] in terms of the polylogarithm or other special functions. Some specific values are given as

$$H_{-\frac{3}{2}}^{(1)} = 2 - 2\ln 2, \quad H_{\frac{1}{2}}^{(2)} = 4 - 2\zeta(2), \quad H_{-\frac{1}{2}}^{(3)} = -6\zeta(3),$$

many others are listed in the books [16], [24] and [25]. In this paper we will develop identities, closed form representations of alternating quadratic half integer harmonic numbers and reciprocal binomial coefficients. While there are many results for sums of harmonic numbers with positive terms, see for example [1], [4], [5], [7], [8], [10], [11], [12], [13], [14], [15], [17], [19], [20], [21], [26], [27], [28], [29] and references therein. There are fewer results for sums of the type (1).

The following Lemma will be useful in the proofs of the main Theorems.

LEMMA 1. *Let r be a positive integer.*
(a) Then for $p \in \mathbb{N}$

$$\sum_{j=1}^{r} \frac{(-1)^j}{j^p} = \frac{1}{2^p}\left(H_{\left[\frac{r}{2}\right]}^{(p)} + H_{\left[\frac{r-1}{2}\right]}^{(p)}\right) - H_{2\left[\frac{r+1}{2}\right]-1}^{(p)}. \tag{3}$$

For $p = 1$,

$$\sum_{j=1}^{r} \frac{(-1)^j}{j} = H_{\left[\frac{r}{2}\right]} - H_r,$$

where $[x]$ *is the integer part of* x*. We also have the known results, for* $0<t\leqslant 1$

$$\ln^2(1+t)=2\sum_{n=1}^{\infty}\frac{(-t)^{n+1}H_n}{n+1}$$

and when $t=1$,

$$\ln^2 2=2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}H_n}{n+1}=\zeta(2)-2Li_2\left(\frac{1}{2}\right). \tag{4}$$

Also

$$t\ln(1+t)=\sum_{n=1}^{\infty}\frac{(-t)^{n+1}}{n},\quad t\in(-1,1]$$

hence,

$$\ln 2=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}=\sum_{n=1}^{\infty}\frac{1}{n2^n}=\frac{1}{2}\sum_{n=1}^{\infty}\frac{H_n}{2^n}. \tag{5}$$

(b)

$$U(0)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}H_{n-\frac{1}{2}}}{n}=\frac{3}{4}\zeta(2)-2\ln^2 2,$$

$$V(0)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}H_{n-\frac{1}{2}}}{n^2}=2\pi G-\ln 2\zeta(2)-\frac{7}{2}\zeta(3),$$

and

$$X(0)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}H^2_{n-\frac{1}{2}}}{n}=\frac{7}{4}\zeta(3)+4\ln^3 2-3\ln 2\zeta(2).$$

Proof. From part (a) the proof of (3) is given in the paper [18].

Formulae (4) and (5) are standard known results. Next from, part (b), and the definition (2),

$$U(0)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}H_{n-\frac{1}{2}}}{n}=\int_0^1\frac{1}{1-x}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(1-x^{n-\frac{1}{2}}\right)}{n}dx$$
$$=\int_0^1\frac{1}{1-x}\left(\ln 2-\frac{\ln(1+x)}{\sqrt{x}}\right)dx=\frac{3}{4}\zeta(2)-2\ln^2 2.$$

Similarly

$$V(0)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}H_{n-\frac{1}{2}}}{n^2}=\int_0^1\frac{1}{1-x}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(1-x^{n-\frac{1}{2}}\right)}{n^2}dx$$
$$=\int_0^1\frac{1}{1-x}\left(\frac{\zeta(2)}{2}+\frac{Li_2(-x)}{\sqrt{x}}\right)dx=2\pi G-\ln 2\zeta(2)-\frac{7}{2}\zeta(3),$$

where $G:=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}=.91596...$ is Catalan's constant. □

LEMMA 2. *Let r be a positive integer.*
(a) *Then we have the recurrence relation*

$$U(r)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}H_{n-\frac{1}{2}}}{n+r}=-U(r-1)+\frac{\pi}{2r-1}+2\ln 2\left(\frac{(-1)^r}{2r-1}-\frac{1}{r}\right)+\frac{2(-1)^r}{2r-1}\left(H_{\left[\frac{r-1}{2}\right]}-H_{r-1}\right), \tag{6}$$

with solution

$$U(r)=(-1)^r U(0)+(-1)^r\left(2H_r-2H_{\left[\frac{r}{2}\right]}+H_{r-\frac{1}{2}}+2\ln 2\right)\ln 2+(-1)^r\pi\sum_{j=1}^{r}\frac{(-1)^j}{2j-1}+2(-1)^r\sum_{j=1}^{r}\frac{H_{\left[\frac{j-1}{2}\right]}-H_{j-1}}{2j-1} \tag{7}$$

and

$$U(0):=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}H_{n-\frac{1}{2}}}{n}=\frac{3}{4}\zeta(2)-2\ln^2 2.$$

(b) *Similarly,*

$$V(r)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}H_{n-\frac{1}{2}}}{(n+r)^2}=-V(r-1)+\frac{2\pi}{(2r-1)^2}+2\ln 2\left(\frac{2(-1)^r}{(2r-1)^2}-\frac{1}{r^2}\right)+\frac{4(-1)^r}{(2r-1)^2}\left(H_{\left[\frac{r-1}{2}\right]}-H_{r-1}\right)+\frac{(-1)^r}{2(2r-1)}\left(H^{(2)}_{\left[\frac{r-1}{2}\right]}-H^{(2)}_{\left[\frac{r}{2}\right]-\frac{1}{2}}\right), \tag{8}$$

with solution

$$V(r)=(-1)^r V(0)+2(-1)^r\pi\sum_{j=1}^{r}\frac{(-1)^j}{(2j-1)^2}+4(-1)^r\sum_{j=1}^{r}\frac{H_{\left[\frac{j-1}{2}\right]}-H_{j-1}}{(2j-1)^2}+(-1)^r\left(3\zeta(2)+H^{(2)}_{r-\frac{1}{2}}-\frac{1}{2}\left(H^{(2)}_{\left[\frac{r}{2}\right]}-H^{(2)}_{\left[\frac{r+1}{2}\right]-\frac{1}{2}}\right)\right)\ln 2+\frac{(-1)^r}{2}\sum_{j=1}^{r}\frac{1}{2j-1}\left(H^{(2)}_{\left[\frac{j-1}{2}\right]}-H^{(2)}_{\left[\frac{j}{2}\right]-\frac{1}{2}}\right), \tag{9}$$

and

$$V(0)=2\pi G-\ln 2\,\zeta(2)-\frac{7}{2}\zeta(3).$$

(c) *Finally,*

$$X(r)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}H^2_{n-\frac{1}{2}}}{n+r}=-X(r-1)-\frac{2\pi\ln 2}{2r-1}-\frac{4U(r-1)}{2r-1}+\frac{4(-1)^r}{(2r-1)^2}\left(\ln 2+H_{\left[\frac{r-1}{2}\right]}-H_{r-1}\right)+\frac{2\pi}{(2r-1)^2}+\frac{4\ln^2 2}{r}, \tag{10}$$

with solution

$$
\begin{aligned}
X(r) = {} & (-1)^r X(0) + 2(-1)^r \pi \sum_{j=1}^{r} \frac{(-1)^j}{(2j-1)^2} + 4(-1)^r \sum_{j=1}^{r} \frac{\ln 2 + H_{\left[\frac{j-1}{2}\right]} - H_{j-1}}{(2j-1)^2} \\
& + 2(-1)^r \pi \ln 2 \sum_{j=1}^{r} \frac{(-1)^j}{2j-1} - 4(-1)^r \sum_{j=1}^{r} \frac{(-1)^j U(j-1)}{2j-1} \\
& + 4(-1)^r \ln^2 2 \left(H_{\left[\frac{r-1}{2}\right]} - H_{r-1}\right)
\end{aligned} \tag{11}
$$

and

$$X(0) = \frac{7}{4}\zeta(3) + 4\ln^3 2 - 3\ln 2\,\zeta(2).$$

Proof. The proof of (7) is presented in [22] and the proof of (9) is given in [23]. From part (c) we consider $X(r)$ and by changing the index of summation

$$
\begin{aligned}
X(r) &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H^2_{n-\frac{1}{2}}}{n+r} = \sum_{n=2}^{\infty} \frac{(-1)^n H^2_{n-\frac{3}{2}}}{n+r-1} \\
&= \sum_{n=2}^{\infty} \frac{(-1)^n}{n+r-1} \left(H_{n-\frac{1}{2}} - \frac{2}{2n-1}\right)^2 \\
&= -\sum_{n=1}^{\infty} \frac{(-1)^{n+1} H^2_{n-\frac{1}{2}}}{n+r-1} + 4\sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_{n-\frac{1}{2}}}{(2n-1)(n+r-1)} \\
&\quad -4\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^2(n+r-1)} + \frac{\left(H_{\frac{1}{2}} - 2\right)^2}{r} \\
&= -X(r-1) + \frac{4}{2r-1} \sum_{n=1}^{\infty} (-1)^{n+1} H_{n-\frac{1}{2}} \left(\frac{2}{2n-1} - \frac{1}{n+r-1}\right) \\
&\quad -4\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2r-1)^2} \left(\frac{1}{n+r-1} - \frac{2}{2n-1} + \frac{2(2r-1)}{(2n-1)^2}\right) + \frac{4\ln^2 2}{r}.
\end{aligned}
$$

From Lemma 1 and using the known results

$$
\begin{aligned}
X(r) = {} & -X(r-1) + \frac{8}{2r-1}\left(G - \frac{\pi}{4}\ln 2\right) - \frac{4}{2r-1} U(r-1) \\
& - \frac{4(-1)^r}{(2r-1)^2}\left(-\ln 2 - H_{\left[\frac{r-1}{2}\right]} + H_{r-1}\right) + \frac{2\pi}{(2r-1)^2} - \frac{8G}{2r-1} + \frac{4\ln^2 2}{r}
\end{aligned}
$$

and upon simplification results in the recurrence relation (10). By the subsequent reduction of the $X(r)$, $X(r-1)$, $X(r-2)$,...,$X(1)$ terms in (10), we arrive at the identity (11). □

EXAMPLE 1. For $r=5$, we have from (11)

$$X(5)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}H_{n-\frac{1}{2}}^{2}}{n+5}=\frac{637279}{297675}+\frac{449018\pi}{99225}-\frac{563}{105}\zeta(2)-\frac{7}{4}\zeta(3)$$
$$-\frac{526\pi\ln 2}{315}+3\ln 2\zeta(2)+\frac{5491\ln^{2}2}{315}-4\ln^{3}2-\frac{1757806\ln 2}{99225}.$$

It is of some interest to note that $X(r)$ may be expanded in a slightly different way so that it gives rise to another unexpected harmonic series identity. This is pursued in the next Lemma.

LEMMA 3. *For $r\in\mathbb{N}$, we have the identity*

$$\Upsilon(r)=\sum_{n=1}^{\infty}\frac{H_{2n-\frac{1}{2}}}{2n+r-1}\left(\frac{H_{2n-\frac{1}{2}}}{2n+r}-\frac{4}{4n-1}\right)=\frac{4G-3\zeta(2)}{2r-1}$$
$$+X(r)+\frac{2}{(2r-1)^{2}}\left(H_{\frac{r-1}{2}}+3\ln 2-\frac{\pi}{2}\right).$$

For $r=0$

$$\Upsilon(0)=\sum_{n=1}^{\infty}\frac{H_{2n-\frac{1}{2}}^{2}}{2n(2n-1)}=\frac{7}{4}\zeta(3)-3\ln 2\zeta(2)+\pi$$
$$+\frac{1}{2}\zeta(2)-2\ln 2-\pi\ln 2+4\ln^{2}2+4\ln^{3}2. \tag{12}$$

Proof. We have, by expansion

$$X(r)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}H_{n-\frac{1}{2}}^{2}}{n+r}=\sum_{n=1}^{\infty}\left(\frac{H_{2n-\frac{3}{2}}^{2}}{2n+r-1}-\frac{H_{2n-\frac{1}{2}}^{2}}{2n+r}\right)$$
$$=\sum_{n=1}^{\infty}\left(\frac{H_{2n-\frac{1}{2}}}{2n+r-1}\left(\frac{H_{2n-\frac{1}{2}}}{2n+r}-\frac{4}{4n-1}\right)-\frac{4}{(4n-1)^{2}(2n+r-1)}\right),$$

and it follows that

$$\sum_{n=1}^{\infty}\frac{H_{2n-\frac{1}{2}}}{2n+r-1}\left(\frac{H_{2n-\frac{1}{2}}}{2n+r}-\frac{4}{4n-1}\right)$$
$$=X(r)+\sum_{n=1}^{\infty}\frac{4}{(4n-1)^{2}(2n+r-1)}$$
$$=X(r)+\frac{2}{(2r-1)^{2}}\left(H_{\frac{r-1}{2}}+3\ln 2-\frac{\pi}{2}\right)+\frac{4G-3\zeta(2)}{2r-1},$$

where $X(r)$ is given by (11). The $r=0$ case follows from

$$\sum_{n=1}^{\infty} \frac{H_{2n-\frac{1}{2}}}{2n-1} \left(\frac{H_{2n-\frac{1}{2}}}{2n} - \frac{4}{4n-1} \right) = X(0) + 2 \left(H_{-\frac{1}{2}} + 3\ln 2 - \frac{\pi}{2} \right) - 4G + 3\zeta(2).$$

We can evaluate

$$\sum_{n=1}^{\infty} \frac{4H_{2n-\frac{1}{2}}}{(2n-1)(4n-1)} = 4G + 2\pi - \frac{5}{2}\zeta(2) - 4\ln 2 - \pi \ln 2 + 4\ln^2 2,$$

therefore (12) is attained. □

The next few theorems relate the main results of this investigation, namely the closed form and integral representation of (1).

2. Closed form of harmonic sums and Integral identities

Now we prove the following theorems.

THEOREM 1. *Let k be a positive integer, then from (1) with $p=1$ we have*

$$M(k,1) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_{n-\frac{1}{2}}^2}{n \binom{n+k}{k}} = \frac{7}{4}\zeta(3) - 3\ln 2\zeta(2) + 4\ln^3 2 - \sum_{r=1}^{k} (-1)^{1+r} \binom{k}{r} X(r) \tag{13}$$

where $X(r)$ is given by (11).

Proof. Consider the expansion

$$\begin{aligned} M(k,1) &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_{n-\frac{1}{2}}^2}{n \binom{n+k}{k}} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} k!\, H_{n-\frac{1}{2}}^2}{n(n+1)_k} \\ &= \sum_{n=1}^{\infty} (-1)^{n+1} k!\, H_{n-\frac{1}{2}}^2 \left(\sum_{r=1}^{k} \frac{\Lambda_r}{n+r} + \frac{\Omega}{n} \right) \end{aligned}$$

where

$$\Lambda_r = \lim_{n \to -r} \left\{ \frac{n+r}{n \prod_{r=1}^{k} n+r} \right\} = -\frac{(-1)^{r+1}}{k!} \binom{k}{r}, \qquad \Omega = \frac{1}{k!}$$

hence

$$M(k,1) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_{n-\frac{1}{2}}^2}{n} - \sum_{r=1}^{k} (-1)^{r+1} \binom{k}{r} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_{n-\frac{1}{2}}^2}{n+r}$$
$$= X(0) - \sum_{r=1}^{k} (-1)^{r+1} \binom{k}{r} X(r). \quad \square$$

The other case of $M(k,0)$ can be evaluated by a similar technique. We list the result in the next corollary.

COROLLARY 1. *Under the assumptions of Theorem 1, we have,*

$$M(k,0) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_{n-\frac{1}{2}}^2}{\binom{n+k}{k}} = \sum_{r=1}^{k} (-1)^{r+1} r \binom{k}{r} X(r). \tag{14}$$

Proof. The proof follows directly from Theorem 1 and using the same technique. $\square$

It is possible to represent the alternating harmonic number sums (13), (14) and (10) in terms of an integral, which is given in the next Theorem.

THEOREM 2. *Let k be a positive integer, then we have:*

$$\int_0^1 \int_0^1 \frac{1}{(1-x)(1-y)} \left(\begin{array}{c} \sqrt{y}\, {}_2F_1\left[\left.\begin{array}{c}1,1\\2+k\end{array}\right| -y\right] - \sqrt{x}\, {}_2F_1\left[\left.\begin{array}{c}1,1\\2+k\end{array}\right| -x\right] \\ + {}_2F_1\left[\left.\begin{array}{c}1,1\\2+k\end{array}\right| -1\right] + \sqrt{xy}\, {}_2F_1\left[\left.\begin{array}{c}1,1\\2+k\end{array}\right| -xy\right] \end{array} \right) dxdy$$
$$= (1+k)\left(\frac{7}{4}\zeta(3) + 4\ln^3 2 - 3\ln 2\,\zeta(2)\right) - (1+k)\sum_{r=1}^{k} (-1)^{1+r} \binom{k}{r} X(r),$$

where $X(r)$ is given by (11).

Proof. From (2) we can write

$$M(k,1) = \int_0^1 \int_0^1 \frac{1}{(1-x)(1-y)} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \left(1-x^{n-\frac{1}{2}}\right)\left(1-y^{n-\frac{1}{2}}\right)}{n\binom{n+k}{k}} dxdy$$

therefore

$$\int_0^1\int_0^1 \frac{1}{(1-x)(1-y)} \left(\begin{array}{c} \sqrt{y}\,{}_2F_1\left[\left.\begin{array}{c}1,1\\2+k\end{array}\right| -y\right] - \sqrt{x}\,{}_2F_1\left[\left.\begin{array}{c}1,1\\2+k\end{array}\right| -x\right] \\ +{}_2F_1\left[\left.\begin{array}{c}1,1\\2+k\end{array}\right| -1\right] + \sqrt{xy}\,{}_2F_1\left[\left.\begin{array}{c}1,1\\2+k\end{array}\right| -xy\right] \end{array}\right) dxdy$$

$$= (1+k)\left(\frac{7}{4}\zeta(3) + 4\ln^3 2 - 3\ln 2\zeta(2) - \sum_{r=1}^{k}(-1)^{1+r}\binom{k}{r}X(r)\right). \quad \square$$

Similar integral representations can be evaluated for $M(k,0)$ and $X(r)$, the results are recorded in the next Theorem.

THEOREM 3. *Let the conditions of Theorem 2 hold, then we have:*

$$\int_0^1\int_0^1 \frac{1}{(1-x)(1-y)} \left(\begin{array}{c} \sqrt{y}\,{}_2F_1\left[\left.\begin{array}{c}1,2\\2+k\end{array}\right| -y\right] - \sqrt{x}\,{}_2F_1\left[\left.\begin{array}{c}1,2\\2+k\end{array}\right| -x\right] \\ +{}_2F_1\left[\left.\begin{array}{c}1,2\\2+k\end{array}\right| -1\right] + \sqrt{xy}\,{}_2F_1\left[\left.\begin{array}{c}1,2\\2+k\end{array}\right| -xy\right] \end{array}\right) dxdy$$

$$= (1+k)\sum_{r=1}^{k}(-1)^{r+1} r\binom{k}{r}X(r)$$

Also for $X(r)$

$$\int_0^1\int_0^1 \frac{1}{(1-x)(1-y)} \left(\begin{array}{c} \Phi(-1,1,1+r) - \sqrt{x}\,\Phi(-x,1,1+r) \\ +\sqrt{y}\,\Phi(-y,1,1+r) + \frac{\sqrt{xy}}{1+r}\,{}_2F_1\left[\left.\begin{array}{c}1,1+r\\2+r\end{array}\right| -xy\right] \end{array}\right) dxdy$$

$$= X(r).$$

Proof. The proof follows the same pattern as that employed in Theorem 2. $\square$

EXAMPLE 2. Some illustrative examples follow.

$$\int_0^1\int_0^1 \frac{1}{(1-x)(1-y)} \left(\begin{array}{c} -1 + 2\ln 2 + \frac{\sqrt{x}+\sqrt{y}-1}{\sqrt{xy}} - \frac{(1+x)\ln(1+x)}{x^{3/2}} \\ -\frac{(1+y)\ln(1+y)}{y^{3/2}} + \frac{(1+xy)\ln(1+xy)}{(xy)^{3/2}} \end{array}\right) dxdy$$

$$= 7\zeta(3) + 6\zeta(2) - 12\ln 2\zeta(2) + 4\pi\ln 2 - 4\pi + 16\ln^3 2 - 24\ln^2 2 + 8\ln 2.$$

$$\int_0^1\int_0^1 \frac{1}{(1-x)(1-y)} \left(\begin{array}{c} 1 - \ln 2 - \frac{\sqrt{x}+\sqrt{y}-1}{\sqrt{xy}} + \frac{\ln(1+x)}{x^{3/2}} \\ +\frac{\ln(1+y)}{y^{3/2}} - \frac{\ln(1+xy)}{(xy)^{3/2}} \end{array}\right) dxdy$$

$$= -\frac{7}{2}\zeta(3) - 6\zeta(2) + 6\ln 2\zeta(2) - 4\pi\ln 2 + 4\pi - 8\ln^3 2 + 24\ln^2 2 - 8\ln 2.$$

$$\int_0^1\int_0^1 \frac{1}{(1-x)(1-y)}\left(\begin{array}{c} -\frac{1}{2}+\frac{2y^{3/2}-xy^{3/2}+2x^{3/2}-2-x^{3/2}y+xy}{2(xy)^{3/2}} \\ +\ln 2-\frac{\ln(1+x)}{x^{5/2}}-\frac{\ln(1+y)}{y^{5/2}}+\frac{\ln(1+xy)}{(xy)^{5/2}} \end{array}\right)dxdy$$
$$=\frac{7}{4}\zeta(3)+4\zeta(2)-3\ln 2\zeta(2)+\frac{4}{3}\pi\ln 2-\frac{28}{9}\pi+4\ln^3 2-\frac{38}{3}\ln^2 2+\frac{88}{9}\ln 2-\frac{4}{9}.$$

REMARK 1. It appears that, for $r\in\mathbb{N}_0$,

$$F(r)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\,H^2_{n-\frac{1}{2}}}{(n+r)^2}$$

may not have a closed form representation, in terms of some common special functions. Remarkably, however, the sum of two consecutive terms of $F(r)$ does have a closed form. This result is pursued in the next Lemma.

LEMMA 4. *For $r\in\mathbb{N}$,*

$$\begin{aligned} F(r)+F(r+1) &= \sum_{n=1}^{\infty}(-1)^{n+1}\,H^2_{n-\frac{1}{2}}\left(\frac{1}{(n+r)^2}+\frac{1}{(n+r+1)^2}\right) \\ &= \left(\frac{2}{2r+1}\right)^3\pi-\left(\frac{2}{2r+1}\right)^2\pi\ln 2+\frac{4\ln^2 2}{(1+r)^2} \\ &\quad -2\left(\frac{2}{2r+1}\right)^3(-1)^r\left(\ln 2+H_{[\frac{r}{2}]}-H_r\right) \\ &\quad -2\left(\frac{2}{2r+1}\right)\left(V(r)+\left(\frac{2}{2r+1}\right)U(r)\right) \\ &\quad -\frac{(-1)^r}{(2r+1)^2}(-1)^r\left(H^{(2)}_{[\frac{r}{2}]}-H^{(2)}_{2[\frac{r+1}{2}]-1}\right), \end{aligned} \tag{15}$$

and for $r=0$,

$$\begin{aligned} F(0)+F(1) &= \sum_{n=1}^{\infty}(-1)^{n+1}\,H^2_{n-\frac{1}{2}}\left(\frac{1}{n^2}+\frac{1}{(n+1)^2}\right) \\ &= -8\pi G+8\pi-8\zeta(2)+14\zeta(3) \\ &\quad -4\pi\ln 2+4\ln 2\zeta(2)+20\ln^2 2-16\ln 2. \end{aligned}$$

Proof. Consider

$$F(r)+F(r+1)=\sum_{n=1}^{\infty}(-1)^{n+1}\,H^2_{n-\frac{1}{2}}\left(\frac{1}{(n+r)^2}+\frac{1}{(n+r+1)^2}\right)$$

and by changing the index of summation in the second sum we can write

$$F(r)+F(r+1)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\ H_{n-\frac{1}{2}}^{2}}{(n+r)^{2}}+\frac{4\ln^{2}2}{(r+1)^{2}}-\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(n+r)^{2}}\left(H_{n-\frac{1}{2}}-\frac{2}{2n-1}\right)^{2}$$

$$=\sum_{n=1}^{\infty}\frac{4(-1)^{n+1}\ H_{n-\frac{1}{2}}}{(2n-1)(n+r)^{2}}-\sum_{n=1}^{\infty}\frac{4(-1)^{n+1}}{(2n-1)^{2}(n+r)^{2}}+\frac{4\ln^{2}2}{(r+1)^{2}}$$

$$=4\sum_{n=1}^{\infty}(-1)^{n+1}H_{n-\frac{1}{2}}\left(\begin{array}{c}\left(\frac{2}{2r+1}\right)^{2}\frac{1}{2n-1}-\frac{1}{2}\left(\frac{2}{2r+1}\right)^{2}\frac{1}{n+r}\\-\left(\frac{1}{2r+1}\right)\frac{1}{(n+r)^{2}}\end{array}\right)$$

$$-4\sum_{n=1}^{\infty}(-1)^{n+1}\left(\begin{array}{c}-\left(\frac{2}{2r+1}\right)^{3}\frac{1}{2n-1}+\left(\frac{2}{2r+1}\right)^{2}\frac{1}{(2n-1)^{2}}\\+\frac{4}{(2r+1)^{3}(n+r)}+\frac{1}{(2r+1)^{2}(n+r)^{2}}\end{array}\right)+\frac{4\ln^{2}2}{(r+1)^{2}}$$

and a rearrangement leads to (15). The case of $r=0$ follows. □

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