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## Ostrowski Via a Two Functions Pompeiu's Inequality

S. S. Dragomir

### Abstract

In this paper, some generalizations of Pompeiu's inequality for two complex-valued absolutely continuous functions are provided. They are applied to obtain some new Ostrowski type results. Reverses for the integral Cauchy-Bunyakovsky-Schwarz inequality are provided as well.

### 1 Introduction

In 1946, Pompeiu [6] derived a variant of *Lagrange's mean value theorem*, now known as *Pompeiu's mean value theorem* (see also [8, p. 83]).

**Theorem 1** (Pompeiu, 1946 [6]). *For every real valued function  $f$  differentiable on an interval  $[a, b]$  not containing 0 and for all pairs  $x_1 \neq x_2$  in  $[a, b]$ , there exists a point  $\xi$  between  $x_1$  and  $x_2$  such that*

$$\frac{x_1 f(x_2) - x_2 f(x_1)}{x_1 - x_2} = f(\xi) - \xi f'(\xi). \quad (1)$$

The following inequality is useful to derive some Ostrowski type inequalities.

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**Corollary 1** (Pompeiu's Inequality). *With the assumptions of Theorem 1 and if  $\|f - \ell f'\|_\infty = \sup_{t \in (a,b)} |f(t) - t f'(t)| < \infty$  where  $\ell(t) = t$ ,  $t \in [a, b]$ , then*

$$|t f(x) - x f(t)| \leq \|f - \ell f'\|_\infty |x - t| \quad (2)$$

for any  $t, x \in [a, b]$ .

The inequality (2) was stated by the author in [3].

In 1938, A. Ostrowski [4] proved the following result in the estimating the integral mean:

**Theorem 2** (Ostrowski, 1938 [4]). *Let  $f : [a, b] \rightarrow \mathbb{R}$  be continuous on  $[a, b]$  and differentiable on  $(a, b)$  with  $|f'(t)| \leq M < \infty$  for all  $t \in (a, b)$ . Then for any  $x \in [a, b]$ , we have the inequality*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left[ \frac{1}{4} + \left( \frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] M (b-a). \quad (3)$$

The constant  $\frac{1}{4}$  is best possible in the sense that it cannot be replaced by a smaller quantity.

In order to provide another approximation of the integral mean, by making use of the Pompeiu's mean value theorem, the author proved the following result:

**Theorem 3** (Dragomir, 2005 [3]). *Let  $f : [a, b] \rightarrow \mathbb{R}$  be continuous on  $[a, b]$  and differentiable on  $(a, b)$  with  $[a, b]$  not containing 0. Then for any  $x \in [a, b]$ , we have the inequality*

$$\begin{aligned} & \left| \frac{a+b}{2} \cdot \frac{f(x)}{x} - \frac{1}{b-a} \int_a^b f(t) dt \right| \\ & \leq \frac{b-a}{|x|} \left[ \frac{1}{4} + \left( \frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] \|f - \ell f'\|_\infty, \end{aligned} \quad (4)$$

where  $\ell(t) = t$ ,  $t \in [a, b]$ .

The constant  $\frac{1}{4}$  is sharp in the sense that it cannot be replaced by a smaller quantity.

In [7], E. C. Popa using a mean value theorem obtained a generalization of (4) as follows:

**Theorem 4** (Popa, 2007 [7]). *Let  $f : [a, b] \rightarrow \mathbb{R}$  be continuous on  $[a, b]$  and differentiable on  $(a, b)$ . Assume that  $\alpha \notin [a, b]$ . Then for any  $x \in [a, b]$ , we have the inequality*

$$\begin{aligned} & \left| \left( \frac{a+b}{2} - \alpha \right) f(x) + \frac{\alpha - x}{b-a} \int_a^b f(t) dt \right| \\ & \leq \left[ \frac{1}{4} + \left( \frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] (b-a) \|f - \ell_\alpha f'\|_\infty, \end{aligned} \quad (5)$$

where  $\ell_\alpha(t) = t - \alpha$ ,  $t \in [a, b]$ .

In [5], J. Pečarić and S. Ungar have proved a general estimate with the  $p$ -norm,  $1 \leq p \leq \infty$  which for  $p = \infty$  gives Dragomir's result.

**Theorem 5** (Pečarić & Ungar, 2006 [5]). *Let  $f : [a, b] \rightarrow \mathbb{R}$  be continuous on  $[a, b]$  and differentiable on  $(a, b)$  with  $0 < a < b$ . Then for  $1 \leq p, q \leq \infty$  with  $\frac{1}{p} + \frac{1}{q} = 1$  we have the inequality*

$$\left| \frac{a+b}{2} \cdot \frac{f(x)}{x} - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq PU(x, p) \|f - \ell f'\|_p, \quad (6)$$

for  $x \in [a, b]$ , where

$$\begin{aligned} PU(x, p) : &= (b-a)^{\frac{1}{p}-1} \left[ \left( \frac{a^{2-q} - x^{2-q}}{(1-2q)(2-q)} + \frac{x^{2-q} - a^{1+q}x^{1-2q}}{(1-2q)(1+q)} \right)^{1/q} \right. \\ & \quad \left. + \left( \frac{b^{2-q} - x^{2-q}}{(1-2q)(2-q)} + \frac{x^{2-q} - b^{1+q}x^{1-2q}}{(1-2q)(1+q)} \right)^{1/q} \right]. \end{aligned}$$

In the cases  $(p, q) = (1, \infty), (\infty, 1)$  and  $(2, 2)$  the quantity  $PU(x, p)$  has to be taken as the limit as  $p \rightarrow 1, \infty$  and 2, respectively.

For other inequalities in terms of the  $p$ -norm of the quantity  $f - \ell_\alpha f'$ , where  $\ell_\alpha(t) = t - \alpha$ ,  $t \in [a, b]$  and  $\alpha \notin [a, b]$  see [1] and [2].

In this paper, some new Pompeiu's type inequalities for two complex-valued absolutely continuous functions are provided. They are applied to obtain some Ostrowski type inequalities. Reverses for the integral Cauchy-Bunyakovsky-Schwarz inequality are provided as well.

## 2 A General Pompeiu's Inequality

We start with the following generalization of (2).

**Theorem 6.** Let  $f, g : [a, b] \rightarrow \mathbb{C}$  be absolutely continuous functions on the interval  $[a, b]$  with  $g(t) \neq 0$  for all  $t \in [a, b]$ . Then for any  $t, x \in [a, b]$  we have

$$\left| \frac{f(x)}{g(x)} - \frac{f(t)}{g(t)} \right| \leq \begin{cases} \|f'g - fg'\|_{\infty} \left| \int_t^x \frac{1}{|g(s)|^2} ds \right| & \text{if } f'g - fg' \in L_{\infty}[a, b], \\ \|f'g - fg'\|_p \left| \int_t^x \frac{1}{|g(s)|^{2q}} ds \right|^{1/q} & \begin{aligned} &\text{if } f'g - fg' \in L_p[a, b] \\ &p > 1, \\ &\frac{1}{p} + \frac{1}{q} = 1, \end{aligned} \\ \|f'g - fg'\|_1 \sup_{s \in [t, x] \setminus \{x, t\}} \left\{ \frac{1}{|g(s)|^2} \right\} & \end{cases} \quad (7)$$

or, equivalently

$$|g(t)f(x) - f(t)g(x)| \leq \begin{cases} \|f'g - fg'\|_{\infty} |g(t)g(x)| \left| \int_t^x \frac{1}{|g(s)|^2} ds \right| & \text{if } f'g - fg' \in L_{\infty}[a, b], \\ \|f'g - fg'\|_p |g(t)g(x)| \left| \int_t^x \frac{1}{|g(s)|^{2q}} ds \right|^{1/q} & \begin{aligned} &\text{if } f'g - fg' \in L_p[a, b] \\ &p > 1, \\ &\frac{1}{p} + \frac{1}{q} = 1, \end{aligned} \\ \|f'g - fg'\|_1 |g(t)g(x)| \sup_{s \in [t, x] \setminus \{x, t\}} \left\{ \frac{1}{|g(s)|^2} \right\} & \end{cases} \quad (8)$$

*Proof.* If  $f$  and  $g$  are absolutely continuous and  $g(t) \neq 0$  for all  $t \in [a, b]$ , then  $f/g$  is absolutely continuous on the interval  $[a, b]$  and

$$\int_t^x \left( \frac{f(s)}{g(s)} \right)' ds = \frac{f(x)}{g(x)} - \frac{f(t)}{g(t)}$$

for any  $t, x \in [a, b]$  with  $x \neq t$ .

Since

$$\int_t^x \left( \frac{f(s)}{g(s)} \right)' ds = \int_t^x \frac{f'(s)g(s) - f(s)g'(s)}{g^2(s)} ds,$$

then we get the following identity

$$\frac{f(x)}{g(x)} - \frac{f(t)}{g(t)} = \int_t^x \frac{f'(s)g(s) - f(s)g'(s)}{g^2(s)} ds \quad (9)$$

for any  $t, x \in [a, b]$ .

Taking the modulus in (9) we have

$$\begin{aligned} \left| \frac{f(x)}{g(x)} - \frac{f(t)}{g(t)} \right| &= \left| \int_t^x \frac{f'(s)g(s) - f(s)g'(s)}{g^2(s)} ds \right| \\ &\leq \left| \int_t^x \frac{|f'(s)g(s) - f(s)g'(s)|}{|g(s)|^2} ds \right| := I \end{aligned} \quad (10)$$

and utilizing Hölder's integral inequality we deduce

$$\begin{aligned} I &\leq \begin{cases} \sup_{s \in [t, x]([x, t])} |f'(s)g(s) - f(s)g'(s)| \left| \int_t^x \frac{1}{|g(s)|^2} ds \right|, \\ \left| \int_t^x |f'(s)g(s) - f(s)g'(s)|^p ds \right|^{1/p} \left| \int_t^x \frac{1}{|g(s)|^{2q}} ds \right|^{1/q} \end{cases} \quad \begin{matrix} p > 1, \\ \frac{1}{p} + \frac{1}{q} = 1, \end{matrix} \\ &\leq \begin{cases} \|f'g - fg'\|_\infty \left| \int_t^x \frac{1}{|g(s)|^2} ds \right|, \\ \|f'g - fg'\|_p \left| \int_t^x \frac{1}{|g(s)|^{2q}} ds \right|^{1/q} \end{cases} \quad \begin{matrix} p > 1, \\ \frac{1}{p} + \frac{1}{q} = 1, \end{matrix} \\ &\quad \|f'g - fg'\|_1 \sup_{s \in [t, x]([x, t])} \left\{ \frac{1}{|g(s)|^2} \right\} \end{aligned}$$

and the inequality (7) is proved.  $\square$

The following particular case extends Pompeiu's inequality to other  $p$ -norms than  $p = \infty$  obtained in (2).

**Corollary 2.** *Let  $f : [a, b] \rightarrow \mathbb{C}$  be an absolutely continuous function on the interval  $[a, b]$  with  $b > a > 0$ . Then for any  $t, x \in [a, b]$  we have*

$$\begin{aligned}
& \left| \frac{f(x)}{x} - \frac{f(t)}{t} \right| \\
& \leq \begin{cases} \|f - \ell f'\|_{\infty} \left| \frac{1}{t} - \frac{1}{x} \right| & \text{if } f - \ell f' \in L_{\infty}[a, b], \\
\frac{1}{(2q-1)^{1/q}} \|f - \ell f'\|_p \left| \frac{1}{t^{2q-1}} - \frac{1}{x^{2q-1}} \right|^{1/q} & \text{if } f - \ell f' \in L_p[a, b] \\
& p > 1, \\
& \frac{1}{p} + \frac{1}{q} = 1, \\
\|f - \ell f'\|_1 \frac{1}{\min\{t^2, x^2\}} & \end{cases} \quad (11)
\end{aligned}$$

or, equivalently

$$\begin{aligned}
& |tf(x) - xf(t)| \\
& \leq \begin{cases} \|f - \ell f'\|_{\infty} |x - t| & \text{if } f - \ell f' \in L_{\infty}[a, b], \\
\frac{1}{(2q-1)^{1/q}} \|f - \ell f'\|_p \left| \frac{x^q}{t^{q-1}} - \frac{t^q}{x^{q-1}} \right|^{1/q} & \text{if } f - \ell f' \in L_p[a, b] \\
& p > 1, \frac{1}{p} + \frac{1}{q} = 1, \\
\|f - \ell f'\|_1 \frac{\max\{t, x\}}{\min\{t, x\}}, & \end{cases} \quad (12)
\end{aligned}$$

where  $\ell(t) = t, t \in [a, b]$ .

The proof follows by (7) for  $g(t) = \ell(t) = t, t \in [a, b]$ .

The general case for power functions is as follows.

**Corollary 3.** *Let  $f : [a, b] \rightarrow \mathbb{C}$  be an absolutely continuous function on the interval  $[a, b]$  with  $b > a > 0$ . If  $r \in \mathbb{R}, r \neq 0$ , then for any  $t, x \in [a, b]$  we have*

$$\begin{aligned}
& \left| \frac{f(x)}{x^r} - \frac{f(t)}{t^r} \right| \\
& \leq \begin{cases} \frac{1}{|r|} \|f'\ell - rf\|_\infty \left| \frac{1}{x^r} - \frac{1}{t^r} \right|, & \text{if } f'\ell - rf \in L_\infty[a, b], \\ \|f'\ell - rf\|_p \\ \quad \times \begin{cases} \frac{1}{|1-q(r+1)|^{1/q}} \left| \frac{1}{x^{1-q(r+1)}} - \frac{1}{t^{1-q(r+1)}} \right|^{1/q}, & \text{for } r \neq -\frac{1}{p} \\ |\ln x - \ln t|^{1/q}, & \text{for } r = -\frac{1}{p} \end{cases} \\ \text{if } f'\ell - rf \in L_p[a, b], \\ \|f'\ell - rf\|_1 \frac{1}{\min\{x^{r+1}, t^{r+1}\}}, \end{cases} \quad (13)
\end{aligned}$$

or, equivalently

$$\begin{aligned}
& |t^r f(x) - x^r f(t)| \\
& \leq \begin{cases} \frac{1}{|r|} \|f'\ell - rf\|_\infty |t^r - x^r|, & \text{if } f'\ell - rf \in L_\infty[a, b], \\ \|f'\ell - rf\|_p \\ \quad \times \begin{cases} \frac{t^r x^r}{|1-q(r+1)|^{1/q}} \left| \frac{1}{x^{1-q(r+1)}} - \frac{1}{t^{1-q(r+1)}} \right|^{1/q}, & \text{for } r \neq -\frac{1}{p} \\ t^r x^r |\ln x - \ln t|^{1/q}, & \text{for } r = -\frac{1}{p} \end{cases} \\ \text{if } f'\ell - rf \in L_p[a, b], \\ \|f'\ell - rf\|_1 \frac{t^r x^r}{\min\{x^{r+1}, t^{r+1}\}}, \end{cases} \quad (14)
\end{aligned}$$

where  $p > 1, \frac{1}{p} + \frac{1}{q} = 1$ .

The proof follows by (7) for  $g(t) = t^r, t \in [a, b]$ . The details for calculations are omitted.

We have the following result for exponential.

**Corollary 4.** *Let  $f : [a, b] \rightarrow \mathbb{C}$  be an absolutely continuous function on the*

interval  $[a, b]$  and  $\alpha \in \mathbb{R}$ ,  $\alpha \neq 0$ . Then for any  $t, x \in [a, b]$  we have

$$\left| \frac{f(x)}{\exp(i\alpha x)} - \frac{f(t)}{\exp(i\alpha t)} \right| \leq \begin{cases} \|f' - i\alpha f\|_{\infty} |x - t| & \text{if } f' - i\alpha f \in L_{\infty}[a, b], \\ \|f' - i\alpha f\|_p |x - t|^{1/q} & \text{if } f' - i\alpha f \in L_p[a, b] \\ & p > 1, \\ & \frac{1}{p} + \frac{1}{q} = 1, \\ \|f' - i\alpha f\|_1 & \end{cases} \quad (15)$$

or, equivalently

$$|\exp(i\alpha t) f(x) - f(t) \exp(i\alpha x)| \leq \begin{cases} \|f' - i\alpha f\|_{\infty} |x - t| & \text{if } f' - i\alpha f \in L_{\infty}[a, b], \\ \|f' - i\alpha f\|_p |x - t|^{1/q} & \text{if } f' - i\alpha f \in L_p[a, b] \\ & p > 1, \\ & \frac{1}{p} + \frac{1}{q} = 1, \\ \|f' - i\alpha f\|_1. & \end{cases} \quad (16)$$

### 3 An Inequality Generalizing Ostrowski's

The following result holds:

**Theorem 7.** *Let  $f, g : [a, b] \rightarrow \mathbb{C}$  be absolutely continuous functions on the interval  $[a, b]$ . If  $0 < m \leq |g(t)| \leq M < \infty$  for any  $t \in [a, b]$ , then*

$$\begin{aligned}
& \left| f(x) \int_a^b g(t) dt - g(x) \int_a^b f(t) dt \right| \\
& \leq \left( \frac{M}{m} \right)^2 \begin{cases} \|f'g - fg'\|_\infty (b-a)^2 \left[ \frac{1}{4} + \left( \frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] & \text{if } f'g - fg' \in L_\infty[a, b], \\ \|f'g - fg'\|_p \left[ \frac{(b-x)^{1+1/q} + (x-a)^{1+1/q}}{1+1/q} \right] & \text{if } f'g - fg' \in L_p[a, b] \\ & p > 1, \\ & \frac{1}{p} + \frac{1}{q} = 1, \\ \|f'g - fg'\|_1 (b-a) & \end{cases}
\end{aligned} \tag{17}$$

for any  $x \in [a, b]$ .

*Proof.* Utilizing (8) we have

$$\begin{aligned}
& \left| f(x) \int_a^b g(t) dt - g(x) \int_a^b f(t) dt \right| \\
& \leq \int_a^b |g(t) f(x) - f(t) g(x)| dt \\
& \leq \begin{cases} \|f'g - fg'\|_\infty |g(x)| \int_a^b \left( |g(t)| \left| \int_t^x \frac{1}{|g(s)|^2} ds \right| \right) dt, \\ \|f'g - fg'\|_p |g(x)| \int_a^b \left( |g(t)| \left| \int_t^x \frac{1}{|g(s)|^{2q}} ds \right|^{1/q} \right) dt, \\ \|f'g - fg'\|_1 |g(x)| \int_a^b \left( |g(t)| \sup_{s \in [t, x] \cap [x, t]} \left\{ \frac{1}{|g(s)|^2} \right\} \right) dt \end{cases}
\end{aligned} \tag{18}$$

for any  $x \in [a, b]$ , which is of interest in itself.

Since  $0 < m \leq |g(t)| \leq M < \infty$  for any  $t \in [a, b]$ , then

$$\begin{aligned} |g(x)| \int_a^b \left( |g(t)| \left| \int_t^x \frac{1}{|g(s)|^2} ds \right| \right) dt &\leq \left( \frac{M}{m} \right)^2 \int_a^b |x-t| dt \\ &= \left( \frac{M}{m} \right)^2 \left[ \frac{1}{4} + \left( \frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right], \end{aligned}$$

$$\begin{aligned} |g(x)| \int_a^b \left( |g(t)| \left| \int_t^x \frac{1}{|g(s)|^{2q}} ds \right|^{1/q} \right) dt \\ \leq \left( \frac{M}{m} \right)^2 \int_a^b |x-t|^{1/q} dt = \left( \frac{M}{m} \right)^2 \frac{(b-x)^{1+1/q} + (x-a)^{1+1/q}}{1+1/q} \end{aligned}$$

and

$$|g(x)| \int_a^b \left( |g(t)| \sup_{s \in [t, x] \cap ([x, t])} \left\{ \frac{1}{|g(s)|^2} \right\} \right) dt \leq \left( \frac{M}{m} \right)^2 \int_a^b dt = \left( \frac{M}{m} \right)^2 (b-a)$$

for any  $x \in [a, b]$  and by (18) we get the desired result (17).  $\square$

*Remark 1.* If we take  $g(t) = 1$ ,  $t \in [a, b]$  in the first inequality (17) we recapture Ostrowski's inequality.

**Corollary 5.** *With the assumptions in Theorem 7 we have the midpoint inequalities*

$$\begin{aligned} &\left| f\left(\frac{a+b}{2}\right) \int_a^b g(t) dt - g\left(\frac{a+b}{2}\right) \int_a^b f(t) dt \right| \\ &\leq \left( \frac{M}{m} \right)^2 \begin{cases} \frac{1}{4} (b-a)^2 \|f'g - fg'\|_\infty & \text{if } f'g - fg' \in L_\infty[a, b], \\ \frac{1}{2^{1/q}(1+1/q)} (b-a)^{1+1/q} \|f'g - fg'\|_p & \text{if } f'g - fg' \in L_p[a, b] \\ & p > 1, \\ & \frac{1}{p} + \frac{1}{q} = 1. \end{cases} \end{aligned} \quad (19)$$

The following result also holds:

**Theorem 8.** Let  $f, g : [a, b] \rightarrow \mathbb{C}$  be absolutely continuous functions on the interval  $[a, b]$ ,  $g(x) \neq 0$  for  $x \in [a, b]$  and  $g^{-2} \in L_\infty[a, b]$ . Then

$$\left| \frac{f(x)}{g(x)} \int_a^b g(t) dt - \int_a^b f(t) dt \right| \leq \|g^{-2}\|_\infty \times \begin{cases} \|f'g - fg'\|_\infty \int_a^b |g(t)| |x-t| dt, & \text{if } f'g - fg' \in L_\infty[a, b], \\ \|f'g - fg'\|_p \int_a^b |g(t)| |x-t|^{1/q} dt & \text{if } f'g - fg' \in L_p[a, b] \\ & p > 1, \\ & \frac{1}{p} + \frac{1}{q} = 1, \\ \|f'g - fg'\|_1 \int_a^b |g(t)| dt \end{cases} \quad (20)$$

for any  $x \in [a, b]$ .

*Proof.* Utilizing (8) we have

$$\left| \frac{f(x)}{g(x)} \int_a^b g(t) dt - \int_a^b f(t) dt \right| \leq \begin{cases} \|f'g - fg'\|_\infty \int_a^b \left( |g(t)| \left| \int_t^x \frac{1}{|g(s)|^2} ds \right| \right) dt, \\ \|f'g - fg'\|_p \int_a^b \left( |g(t)| \left| \int_t^x \frac{1}{|g(s)|^{2q}} ds \right|^{1/q} \right) dt, \\ \|f'g - fg'\|_1 \int_a^b \left( |g(t)| \sup_{s \in [t, x] \cap [x, t]} \left\{ \frac{1}{|g(s)|^2} \right\} \right) dt \end{cases} \quad (21)$$

for any  $x \in [a, b]$ .

Since

$$\left| \int_t^x \frac{1}{|g(s)|^2} ds \right| \leq \|g^{-2}\|_\infty |x-t|,$$

$$\left| \int_t^x \frac{1}{|g(s)|^{2q}} ds \right|^{1/q} \leq \|g^{-2}\|_\infty |x-t|^{1/q}$$

and

$$\sup_{s \in [t, x] \cap [x, t]} \left\{ \frac{1}{|g(s)|^2} \right\} \leq \|g^{-2}\|_\infty$$

for any  $x, t \in [a, b]$ , then on making use of (21) we get the desired result (20).  $\square$

We have the midpoint inequalities:

**Corollary 6.** *With the assumptions of Theorem 8 we have*

$$\begin{aligned} & \left| \frac{f\left(\frac{a+b}{2}\right)}{g\left(\frac{a+b}{2}\right)} \int_a^b g(t) dt - \int_a^b f(t) dt \right| \\ & \leq \|g^{-2}\|_{\infty} \times \begin{cases} \|f'g - fg'\|_{\infty} \int_a^b |g(t)| \left| \frac{a+b}{2} - t \right| dt, & \text{if } f'g - fg' \in L_{\infty}[a, b], \\ \|f'g - fg'\|_p \int_a^b |g(t)| \left| \frac{a+b}{2} - t \right|^{1/q} dt & \begin{array}{l} \text{if } f'g - fg' \in L_p[a, b] \\ p > 1, \\ \frac{1}{p} + \frac{1}{q} = 1. \end{array} \end{cases} \end{aligned} \quad (22)$$

We have the following exponential version of Ostrowski's inequality as well:

**Theorem 9.** *Let  $f : [a, b] \rightarrow \mathbb{C}$  be an absolutely continuous function on the interval  $[a, b]$  and  $\alpha \in \mathbb{R}$ ,  $\alpha \neq 0$ . Then for any  $x \in [a, b]$  we have*

$$\begin{aligned} & \left| \frac{\exp(i\alpha(b-x)) - \exp(-i\alpha(x-a))}{i\alpha} f(x) - \int_a^b f(t) dt \right| \\ & \leq \begin{cases} \|f' - i\alpha f\|_{\infty} (b-a)^2 \left[ \frac{1}{4} + \left( \frac{t - \frac{a+b}{2}}{b-a} \right)^2 \right], & \text{if } f' - i\alpha f \in L_{\infty}[a, b], \\ \|f' - i\alpha f\|_p \frac{(b-x)^{1+1/q} + (x-a)^{1+1/q}}{1+1/q}, & \begin{array}{l} \text{if } f' - i\alpha f \\ \in L_p[a, b] \\ p > 1, \\ \frac{1}{p} + \frac{1}{q} = 1, \end{array} \\ \|f' - i\alpha f\|_1. \end{cases} \end{aligned} \quad (23)$$

*Proof.* If we write the inequality (18) for  $g(t) = \exp(i\alpha t)$ ,  $t \in [a, b]$ , then we get

$$\left| f(x) \int_a^b \exp(i\alpha t) dt - \exp(i\alpha x) \int_a^b f(t) dt \right| \leq \begin{cases} \|f' - i\alpha f\|_\infty \int_a^b |x - t| dt, & \text{if } f' - i\alpha f \in L_\infty[a, b] \\ \|f' - i\alpha f\|_p |g(x)| \int_a^b |x - t|^{1/q} dt, & \begin{array}{l} \text{if } f' - i\alpha f \\ \in L_p[a, b] \\ p > 1, \\ \frac{1}{p} + \frac{1}{q} = 1 \end{array} \\ \|f' - i\alpha f\|_1, & \end{cases}$$

which, after simple calculation, is equivalent with (23).

The details are omitted.  $\square$

**Corollary 7.** *With the assumptions of Theorem 9 we have the midpoint inequalities*

$$\left| \frac{\exp(i\alpha(\frac{b-a}{2})) - \exp(-i\alpha(\frac{b-a}{2}))}{i\alpha} f\left(\frac{a+b}{2}\right) - \int_a^b f(t) dt \right| \leq \begin{cases} \frac{1}{4} \|f' - i\alpha f\|_\infty (b-a)^2, & \text{if } f' - i\alpha f \in L_\infty[a, b], \\ \frac{1}{2^{1/q}(1+1/q)} (b-a)^{1+1/q} \|f' - i\alpha f\|_p, & \begin{array}{l} \text{if } f' - i\alpha f \in L_p[a, b] \\ p > 1, \frac{1}{p} + \frac{1}{q} = 1, \end{array} \end{cases} \quad (24)$$

or, equivalently

$$\begin{aligned}
& \left| \frac{2 \sin \left( \alpha \left( \frac{b-a}{2} \right) \right)}{\alpha} f \left( \frac{a+b}{2} \right) - \int_a^b f(t) dt \right| \\
& \leq \begin{cases} \frac{1}{4} \|f' - i\alpha f\|_{\infty} (b-a)^2, & \text{if } f' - i\alpha f \in L_{\infty}[a, b], \\ \frac{1}{2^{1/q}(1+1/q)} (b-a)^{1+1/q} \|f' - i\alpha f\|_p, & \text{if } f' - i\alpha f \in L_p[a, b] \\ & p > 1, \frac{1}{p} + \frac{1}{q} = 1. \end{cases} \quad (25)
\end{aligned}$$

#### 4 An Application for CBS-Inequality

The following inequality is well known in the literature as the Cauchy-Bunyakovsky-Schwarz inequality, or the CBS-inequality, for short:

$$\left| \int_a^b f(t) g(t) dt \right|^2 \leq \int_a^b |f(t)|^2 dt \int_a^b |g(t)|^2 dt, \quad (26)$$

provided that  $f, g \in L_2[a, b]$ .

We have the following result concerning some reverses of the CBS-inequality:

**Theorem 10.** *Let  $f, g : [a, b] \rightarrow \mathbb{C}$  be absolutely continuous functions on the interval  $[a, b]$  with  $g(t) \neq 0$  for all  $t \in [a, b]$ . Then*

$$\begin{aligned}
& 0 \leq \int_a^b |g(t)|^2 dt \int_a^b |f(t)|^2 dt - \left| \int_a^b f(t) g(t) dt \right|^2 \\
& \leq \frac{1}{2} \times \begin{cases} \|f'\bar{g} - f\bar{g}'\|_{\infty}^2 \left( \int_a^b |g(t)|^2 dt \right)^2 \left( \int_a^b \frac{1}{|g(t)|^2} dt \right)^2, & \text{if } f'\bar{g} - f\bar{g}' \in L_{\infty}[a, b], \\ & \frac{1}{|g|^2} \in L[a, b] \\ \|f'\bar{g} - f\bar{g}'\|_p^2 \left( \int_a^b |g(t)|^2 dt \right)^2 \left( \int_a^b \frac{1}{|g(t)|^{2q}} dt \right)^{2/q}, & \text{if } f'\bar{g} - f\bar{g}' \in L_p[a, b], \\ & \frac{1}{|g|^{2q}} \in L[a, b] \\ & p > 1, \\ & \frac{1}{p} + \frac{1}{q} = 1, \\ \|f'\bar{g} - f\bar{g}'\|_1^2 \left( \int_a^b |g(t)|^2 dt \right)^2 \operatorname{ess\,sup}_{t \in [a, b]} \left\{ \frac{1}{|g(t)|^4} \right\}, & \text{if } \frac{1}{|g|} \in L_{\infty}[a, b]. \end{cases} \quad (27)
\end{aligned}$$

*Proof.* Utilising the inequality (8) we have

$$\begin{aligned} & \left| \overline{g(t)} f(x) - f(t) \overline{g(x)} \right| \\ & \leq \begin{cases} \|f'\bar{g} - f\bar{g}'\|_{\infty} |g(t)g(x)| \left| \int_t^x \frac{1}{|g(s)|^2} ds \right| & \text{if } f'\bar{g} - f\bar{g}' \\ & \in L_{\infty}[a, b], \\ \|f'\bar{g} - f\bar{g}'\|_p |g(t)g(x)| \left| \int_t^x \frac{1}{|g(s)|^{2q}} ds \right|^{1/q} & \text{if } f'\bar{g} - f\bar{g}' \\ & \in L_p[a, b] \\ & p > 1, \\ & \frac{1}{p} + \frac{1}{q} = 1, \\ \|f'\bar{g} - f\bar{g}'\|_1 |g(t)g(x)| \sup_{s \in [t, x] \setminus ([x, t])} \left\{ \frac{1}{|g(s)|^2} \right\}. \end{cases} \end{aligned} \quad (28)$$

for any  $t, x \in [a, b]$ .

Taking the square in (28) and integrating over  $(t, x) \in [a, b]^2$  we have

$$\begin{aligned} & \int_a^b \int_a^b \left| \overline{g(t)} f(x) - f(t) \overline{g(x)} \right|^2 dt dx \\ & \leq \begin{cases} \|f'\bar{g} - f\bar{g}'\|_{\infty}^2 \int_a^b \int_a^b |g(t)g(x)|^2 \left| \int_t^x \frac{1}{|g(s)|^2} ds \right|^2 dt dx, \\ \|f'\bar{g} - f\bar{g}'\|_p^2 \int_a^b \int_a^b |g(t)g(x)|^2 \left| \int_t^x \frac{1}{|g(s)|^{2q}} ds \right|^{2/q} dt dx, \\ \|f'\bar{g} - f\bar{g}'\|_1^2 \int_a^b \int_a^b |g(t)g(x)|^2 \sup_{s \in [t, x] \setminus ([x, t])} \left\{ \frac{1}{|g(s)|^4} \right\} dt dx. \end{cases} \end{aligned} \quad (29)$$

Observe that

$$\begin{aligned}
& \int_a^b \int_a^b \left| \overline{g(t)} f(x) - f(t) \overline{g(x)} \right|^2 dt dx \\
&= \int_a^b \int_a^b \left( |g(t)|^2 |f(x)|^2 - 2 \operatorname{Re} \left[ \overline{g(t)} f(x) \overline{f(t) g(x)} \right] + |g(x)|^2 |f(t)|^2 \right) dt dx \\
&= \int_a^b |g(t)|^2 dt \int_a^b |f(x)|^2 dx - 2 \operatorname{Re} \left[ \int_a^b \overline{f(t) g(t)} dt \int_a^b f(x) g(x) dx \right] \\
&+ \int_a^b |g(x)|^2 dx \int_a^b |f(t)|^2 dt \\
&= 2 \left[ \int_a^b |g(t)|^2 dt \int_a^b |f(t)|^2 dt - \left| \int_a^b f(t) g(t) dt \right|^2 \right],
\end{aligned}$$

$$\begin{aligned}
& \int_a^b \int_a^b \left[ |g(t) g(x)|^2 \left| \int_t^x \frac{1}{|g(s)|^2} ds \right|^2 \right] dt dx \\
&\leq \left( \int_a^b |g(t)|^2 dt \right)^2 \left( \int_a^b \frac{1}{|g(t)|^2} dt \right)^2,
\end{aligned}$$

$$\begin{aligned}
& \int_a^b \int_a^b \left[ |g(t) g(x)|^2 \left| \int_t^x \frac{1}{|g(s)|^{2q}} ds \right|^{2/q} \right] dt dx \\
&\leq \left( \int_a^b |g(t)|^2 dt \right)^2 \left( \int_a^b \frac{1}{|g(t)|^{2q}} dt \right)^{2/q}
\end{aligned}$$

and

$$\begin{aligned}
& \int_a^b \int_a^b \left[ |g(t) g(x)|^2 \sup_{s \in [t, x] \setminus ([x, t])} \left\{ \frac{1}{|g(s)|^4} \right\} \right] dt dx \\
&\leq \left( \int_a^b |g(t)|^2 dt \right)^2 \operatorname{ess\,sup}_{t \in [a, b]} \left\{ \frac{1}{|g(t)|^4} \right\},
\end{aligned}$$

then by (29) we get the desired result (27).  $\square$

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