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ORIGINAL ARTICLE

# Some $q$ -analogues of Hermite–Hadamard inequality of functions of two variables on finite rectangles in the plane



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**Abstract** Preliminaries of  $q$ -calculus for functions of two variables over finite rectangles in the plane are introduced. Some  $q$ -analogues of the famous Hermite–Hadamard inequality of functions of two variables defined on finite rectangles in the plane are presented. A  $q_1q_2$ -Hölder inequality for functions of two variables over finite rectangles is also established to provide some quantum estimates of trapezoidal type inequality of functions of two variables whose  $q_1q_2$ -partial derivatives in absolute value with certain powers satisfy the criteria of convexity on co-ordinates.

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## 1. Introduction

Quantum calculus or  $q$ -calculus is the study of calculus without limits. In the eighteenth century, Euler initiated the study of  $q$ -calculus by introducing the number  $q$  in Newton's work of infinite series. Many remarkable results such as Jacobi's triple product identity and the theory of  $q$ -hypergeometric functions

were obtained in the nineteenth century. In early twentieth century, Jackson (1910) had started a symmetric study of  $q$ -calculus and introduced  $q$ -definite integrals. The subject of quantum calculus has numerous applications in different areas of mathematics and physics such as number theory, combinatorics, orthogonal polynomials, basic hypergeometric functions, quantum theory, mechanics and in theory of relativity. This subject has received exceptional consideration by many researchers and hence it has appeared as an interdisciplinary subject between mathematics and physics. Interested readers are referred to Ernst (2012), Gauchman (2004) and Kac and Cheung (2002) for some recent developments in the theory of quantum calculus and theory of inequalities in quantum calculus.

Theory of inequalities and theory of convex functions have been observed to be profoundly dependent on each other and consequently a vast literature on inequalities has been

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produced by a number of researchers using convex functions, see Dragomir and Pearce (2000), Dragomir and Agarwal (1998) and Ion (2007). The Hermite–Hadamard inequalities are extensively studied during past three decades and the following inequalities, known as Hermite–Hadamard inequalities, provide a necessary and sufficient condition for a continuous function  $f: I \subset \mathbb{R} \rightarrow \mathbb{R}$  to be convex on  $[a, b]$ , where  $a, b \in I$  with  $a < b$

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

For further reading on integral inequalities using classical convexity and other important inequalities we refer our reader to Sudsutad et al. (2015) and Tariboon and Ntouyas (2014). Most recently, Noor et al. (2015a,b,c) and Zhuang et al. (2016) have contributed to the ongoing research and have developed some integral inequalities which provide quantum estimates for the right part of the quantum analogue of Hermite–Hadamard inequality through  $q$ -differentiable convex and  $q$ -differentiable quasi-convex functions. Ghany (2009, 2012, 2013a,b) gave integral representations of basic completely monotone functions, integral representations of basic completely alternating functions,  $q$ -derivative of basic hypergeometric series with respect to parameters and discussed some properties of the derivatives of basic hypergeometric series with respect to parameters. Motivated by the recent progress in the field of quantum calculus, our aim is to further develop this theory for functions of two variables and to provide some quantum analogues of Hermite–Hadamard inequality of functions of two variables over finite rectangles. At the next step, we will also provide some quantum estimates for the right part of the  $q$ -analogue of Hermite–Hadamard inequality of functions of two variables using convexity and quasi-convexity on co-ordinates of the absolute value of the  $q_1q_2$ -partial derivatives.

## 2. Preliminaries

The readers are referred to Tariboon and Ntouyas (2013), Sudsutad et al. (2015), Kac and Cheung (2002) and Ernst (2012) for some  $q$ -calculus essentials and inequalities over finite intervals.

We will also use the following definite  $q$ -integrals to prove our results.

**Lemma 1** Sudsutad et al., 2015. *Let  $0 < q < 1$ , the following hold*

$$\Delta_q := \int_0^1 t|1 - (1+q)t| {}_0d_qt = \frac{q(1+4q+q^2)}{(1+q+q^2)(1+q)^3},$$

$$\Psi_q := \int_0^1 (1-t)|1 - (1+q)t| {}_0d_qt = \frac{q(1+3q^2+2q^3)}{(1+q+q^2)(1+q)^3}$$

and

$$\Phi_q := \int_0^1 |1 - (1+q)t| {}_0d_qt = \frac{2q}{(1+q)^2}.$$

In what follows we introduce  $q$ -partial derivatives and definite  $q$ -integrals for functions of two variables.

**Definition 1.** Let  $f: [a, b] \times [c, d] \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$  be a continuous function of two variables and  $0 < q_1 < 1, 0 < q_2 < 1$ , the partial  $q_1$ -derivatives,  $q_2$ -derivatives and  $q_1q_2$ -derivatives at  $(x, y) \in [a, b] \times [c, d]$  can be defined as follows

$$\frac{{}_a\partial_{q_1} f(x, y)}{{}_a\partial_{q_1} x} = \frac{f(q_1x + (1 - q_1)a, y) - f(x, y)}{(1 - q_1)(x - a)}, \quad x \neq a$$

$$\frac{{}_c\partial_{q_2} f(x, y)}{{}_c\partial_{q_2} y} = \frac{f(x, q_2y + (1 - q_2)c) - f(x, y)}{(1 - q_2)(y - c)}, \quad y \neq c$$

and

$$\begin{aligned} \frac{{}_{a,c}\partial_{q_1,q_2}^2 f(x, y)}{{}_a\partial_{q_1} x {}_c\partial_{q_2} y} &= \frac{1}{(1 - q_1)(1 - q_2)(y - c)(x - a)} \\ &\quad \times [f(q_1x + (1 - q_1)a, q_2y + (1 - q_2)c) \\ &\quad - f(q_1x + (1 - q_1)a, y) - f(x, q_2y + (1 - q_2)c) \\ &\quad + f(x, y)], \quad x \neq a, y \neq c. \end{aligned}$$

The function  $f: [a, b] \times [c, d] \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$  is said to be partially  $q_1$ -,  $q_2$ - and  $q_1q_2$ -differentiable on  $[a, b] \times [c, d]$  if  $\frac{{}_a\partial_{q_1} f(x, y)}{{}_a\partial_{q_1} x}, \frac{{}_c\partial_{q_2} f(x, y)}{{}_c\partial_{q_2} y}$  and  $\frac{{}_{a,c}\partial_{q_1,q_2}^2 f(x, y)}{{}_a\partial_{q_1} x {}_c\partial_{q_2} y}$  exist for all  $(x, y) \in [a, b] \times [c, d]$ . We can similarly define higher order partial derivatives.

**Definition 2.** Suppose that  $f: [a, b] \times [c, d] \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$  is a continuous function. Then the definite  $q_1q_2$ -integral on  $[a, b] \times [c, d]$  is defined by

$$\begin{aligned} \int_c^y \int_a^x f(x, y) {}_a d_{q_1} x {}_c d_{q_2} y &= (x - a)(y - c)(1 - q_1)(1 - q_2) \\ &\quad \times \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} q_1^n q_2^m f(q_1^n x + (1 - q_1^n)a, q_2^m y \\ &\quad + (1 - q_2^m)c) \end{aligned} \tag{2.1}$$

for  $(x, y) \in [a, b] \times [c, d]$ . It is clear from (2.1) that

$$\int_c^y \int_a^x f(x, y) {}_a d_{q_1} x {}_c d_{q_2} y = \int_a^x \int_c^y f(x, y) {}_c d_{q_2} y {}_a d_{q_1} x.$$

If  $(x_1, y_1) \in (a, x) \times (c, y)$ , then

$$\begin{aligned} \int_{y_1}^y \int_{x_1}^x f(x, y) {}_a d_{q_1} x {}_c d_{q_2} y &= \int_{y_1}^y \int_a^x f(x, y) {}_a d_{q_1} x {}_c d_{q_2} y - \int_{y_1}^y \int_a^{x_1} f(x, y) {}_a d_{q_1} x {}_c d_{q_2} y \\ &= \int_c^y \int_a^x f(x, y) {}_a d_{q_1} x {}_c d_{q_2} y - \int_c^{y_1} \int_a^x f(x, y) {}_a d_{q_1} x {}_c d_{q_2} y \\ &\quad - \int_c^y \int_a^{x_1} f(x, y) {}_a d_{q_1} x {}_c d_{q_2} y + \int_c^{y_1} \int_a^{x_1} f(x, y) {}_a d_{q_1} x {}_c d_{q_2} y. \end{aligned} \tag{2.2}$$

From (2.2), we also note that

$$\begin{aligned} \int_c^y \int_a^x f(x, y) {}_a d_{q_1} x {}_c d_{q_2} y &= \int_c^y \left( \int_a^x f(x, y) {}_a d_{q_1} x \right) {}_c d_{q_2} y \\ &= \int_a^x \left( \int_c^y f(x, y) {}_c d_{q_2} y \right) {}_a d_{q_1} x. \end{aligned}$$

**Remark 1.** It is easy to observe that the Definition 2 contains the Definition 2.3 in Tariboon and Ntouyas (2014) as special case when  $f$  is a function of single variable.

The following theorems hold for definite  $q_1q_2$ -double integrals.

**Theorem 1.** Let  $f: [a, b] \times [c, d] \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$  be a continuous function. Then

- (1)  $\frac{a,c \partial_{q_1, x_c}^2}{a \partial_{q_1, x_c} \partial_{q_2, y}^2} \int_c^y \int_a^x f(t, s) {}_a d_{q_1} t {}_c d_{q_2} s = f(x, y)$
- (2)  $\int_c^y \int_a^x \frac{a,c \partial_{q_1, q_2}^2 f(t, s)}{a \partial_{q_1, t_c} \partial_{q_2, s}^2} {}_a d_{q_1} t {}_c d_{q_2} s = f(x, y)$
- (3)  $\int_{y_1}^y \int_{x_1}^x \frac{a,c \partial_{q_1, q_2}^2 f(t, s)}{a \partial_{q_1, t_c} \partial_{q_2, s}^2} {}_a d_{q_1} t {}_c d_{q_2} s = f(x, y) - f(x, y_1) - f(x_1, y) + f(x_1, y_1), \quad (x_1, y_1) \in (a, x) \times (c, y).$

**Proof**

- (1) By Definition 2 and the definition of partial  $q_1q_2$ -derivatives, we have

$$\begin{aligned} & \frac{a,c \partial_{q_1, q_2}^2}{a \partial_{q_1, x_c} \partial_{q_2, y}^2} \left[ (x-a)(y-c)(1-q_1)(1-q_2) \right. \\ & \quad \left. \times \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} q_1^n q_2^m f(q_1^n x + (1-q_1^n)a, q_2^m y + (1-q_2^m)c) \right] \\ &= \frac{1}{(1-q_1)(1-q_2)(y-c)(x-a)} \\ & \quad \times \left[ q_1 q_2 (1-q_1)(1-q_2)(y-c)(x-a) \right. \\ & \quad \times \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} q_1^n q_2^m f(q_1^{n+1} x + (1-q_1^{n+1})a, q_2^{m+1} y + (1-q_2^{m+1})c) \\ & \quad - q_1(x-a)(y-c)(1-q_1)(1-q_2) \\ & \quad \times \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} q_1^n q_2^m f(q_1^{n+1} x + (1-q_1^{n+1})a, q_2^m y + (1-q_2^m)c) \\ & \quad - q_2(x-a)(y-c)(1-q_1)(1-q_2) \\ & \quad \times \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} q_1^n q_2^m f(q_1^n x + (1-q_1^n)a, q_2^{m+1} y + (1-q_2^{m+1})c) \\ & \quad \left. + (x-a)(y-c)(1-q_1)(1-q_2) \right. \\ & \quad \left. \times \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} q_1^n q_2^m f(q_1^n x + (1-q_1^n)a, q_2^m y + (1-q_2^m)c) \right] \\ &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} q_1^n q_2^m f(q_1^n x + (1-q_1^n)a, q_2^m y + (1-q_2^m)c) \\ & \quad - \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} q_1^n q_2^m f(q_1^n x + (1-q_1^n)a, q_2^m y + (1-q_2^m)c) \\ & \quad - \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} q_1^n q_2^m f(q_1^n x + (1-q_1^n)a, q_2^m y + (1-q_2^m)c) \\ & \quad + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} q_1^n q_2^m f(q_1^n x + (1-q_1^n)a, q_2^m y + (1-q_2^m)c) = f(x, y) \end{aligned}$$

- (2) By the definition of partial  $q_1q_2$ -derivatives and Definition 2, we have

$$\begin{aligned} & \int_c^y \int_a^x \frac{a,c \partial_{q_1, q_2}^2 f(t, s)}{a \partial_{q_1, t_c} \partial_{q_2, s}^2} {}_c d_{q_2} s {}_a d_{q_1} t \\ &= \int_c^y \int_a^x \frac{1}{(1-q_1)(1-q_2)(s-c)(t-a)} \\ & \quad \times [f(q_1 t + (1-q_1)a, q_2 s + (1-q_2)c) - f(q_1 t + (1-q_1)a, s) \\ & \quad - f(t, q_2 s + (1-q_2)c) + f(t, s)] {}_c d_{q_2} s {}_a d_{q_1} t \\ &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} [f(q_1^{n+1} x + (1-q_1^{n+1})a, q_2^{m+1} y + (1-q_2^{m+1})c) \\ & \quad - f(q_1^{n+1} x + (1-q_1^{n+1})a, q_2^m y + (1-q_2^m)c) \\ & \quad - f(q_1^n x + (1-q_1^n)a, q_2^{m+1} y + (1-q_2^{m+1})c) \\ & \quad + f((q_1^n x + (1-q_1^n)a), (q_2^m y + (1-q_2^m)c))] \\ &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} f(q_1^n x + (1-q_1^n)a, q_2^m y + (1-q_2^m)c) \\ & \quad - \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} f(q_1^n x + (1-q_1^n)a, q_2^m y + (1-q_2^m)c) \\ & \quad - \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} f(q_1^n x + (1-q_1^n)a, q_2^m y + (1-q_2^m)c) \\ & \quad + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} f((q_1^n x + (1-q_1^n)a), (q_2^m y + (1-q_2^m)c)) = f(x, y). \end{aligned}$$

- (3) Using (2.2) and applying the result (2), we obtain

$$\begin{aligned} & \int_{y_1}^y \int_{x_1}^x \frac{a,c \partial_{q_1, q_2}^2 f(t, s)}{a \partial_{q_1, x_c} \partial_{q_2, y}^2} {}_a d_{q_1} t {}_c d_{q_2} s \\ &= \int_c^y \int_a^x \frac{a,c \partial_{q_1, q_2}^2 f(t, s)}{a \partial_{q_1, x_c} \partial_{q_2, y}^2} {}_a d_{q_1} t {}_c d_{q_2} s \\ & \quad - \int_c^{y_1} \int_a^x \frac{a,c \partial_{q_1, q_2}^2 f(t, s)}{a \partial_{q_1, x_c} \partial_{q_2, y}^2} {}_a d_{q_1} t {}_c d_{q_2} s \\ & \quad - \int_c^y \int_a^{x_1} \frac{a,c \partial_{q_1, q_2}^2 f(t, s)}{a \partial_{q_1, x_c} \partial_{q_2, y}^2} {}_a d_{q_1} t {}_c d_{q_2} s \\ & \quad + \int_c^{y_1} \int_a^{x_1} \frac{a,c \partial_{q_1, q_2}^2 f(t, s)}{a \partial_{q_1, x_c} \partial_{q_2, y}^2} {}_a d_{q_1} t {}_c d_{q_2} s \\ &= f(x, y) - f(x, y_1) - f(x_1, y) + f(x_1, y_1). \quad \square \end{aligned}$$

**Theorem 2.** Suppose that  $f, g: [a, b] \times [c, d] \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$  are continuous functions,  $\alpha \in \mathbb{R}$ . Then, for  $(x, y) \in [a, b] \times [c, d]$ ,

- (1)  $\int_c^y \int_a^x [f(t, s) + g(t, s)] {}_a d_{q_1} t {}_c d_{q_2} s = \int_c^y \int_a^x f(t, s) {}_a d_{q_1} t + \int_c^y \int_a^x g(t, s) {}_a d_{q_1} t {}_c d_{q_2} s.$
- (2)  $\int_c^y \int_a^x \alpha f(t, s) {}_a d_{q_1} t {}_c d_{q_2} s = \alpha \int_c^y \int_a^x f(t, s) {}_a d_{q_1} t {}_c d_{q_2} s.$
- (3) The following integration by parts formula for iterated  $q_1q_2$ -double integrals holds:

$$\begin{aligned} & \int_{y_1}^y \int_{x_1}^x f(t,s) \frac{{}_a c \partial_{q_1 q_2}^2 g(t,s)}{{}_a \partial_{q_1 x} {}_c \partial_{q_2 y}} {}_a d_{q_1 t} {}_c d_{q_2 s} \\ &= f(x,y)g(x,y) - f(x,y_1)g(x,y_1) - f(x_1,y)g(x_1,y) \\ &+ f(x_1,y_1)g(x_1,y_1) - \int_{y_1}^y g(x,q_2s + (1-q_2)c) \frac{{}_c \partial_{q_2} f(x,s)}{{}_c \partial_{q_2} s} {}_c d_{q_2} s \\ &+ \int_{y_1}^y g(x_1,q_2s + (1-q_2)c) \frac{{}_c \partial_{q_2} f(x_1,s)}{{}_c \partial_{q_2} s} {}_c d_{q_2} s \\ &- \int_{x_1}^x g(q_1t + (1-q_1)a,y) \frac{{}_a \partial_{q_1} f(t,y)}{{}_a \partial_{q_1} t} {}_a d_{q_1} t \\ &+ \int_{x_1}^x g(q_1t + (1-q_1)a,y_1) \frac{{}_a \partial_{q_1} f(t,y_1)}{{}_a \partial_{q_1} t} {}_a d_{q_1} t \\ &+ \int_{y_1}^y \int_{x_1}^x g(q_1t + (1-q_1)a, q_2s + (1-q_2)c) \\ &\times \frac{{}_c a \partial_{q_2 q_1}^2 f(t,s)}{{}_c \partial_{q_2 s} {}_a \partial_{q_1 t}} {}_a d_{q_1 t} {}_c d_{q_2 s}, (x_1, y_1) \in (a, x) \times (c, y). \end{aligned}$$

**Proof.** The proof of (1) and (2) follows by definition of definite  $q_1 q_2$ -double integrals. (3) By applying (3) of Theorem 3.3 (Tariboon and Ntouyas, 2013), we have

$$\begin{aligned} & \int_{y_1}^y \left( \int_{x_1}^x f(t,s) \frac{{}_a c \partial_{q_1 q_2}^2 g(t,s)}{{}_a \partial_{q_1 t} {}_c \partial_{q_2 s}} {}_a d_{q_1} t \right) {}_c d_{q_2} s \\ &= \int_{y_1}^y \left[ f(x,s) \frac{{}_c \partial_{q_2} g(x,s)}{{}_c \partial_{q_2} s} - f(x_1,s) \frac{{}_c \partial_{q_2} g(x_1,s)}{{}_c \partial_{q_2} s} \right. \\ &\quad \left. - \int_{x_1}^x \frac{{}_c \partial_{q_2} g(q_1t + (1-q_1)a, s)}{{}_c \partial_{q_2} s} \frac{{}_a \partial_{q_1} f(t,s)}{{}_a \partial_{q_1} t} {}_a d_{q_1} t \right] {}_c d_{q_2} s \\ &= \int_{y_1}^y f(x,s) \frac{{}_c \partial_{q_2} g(x,s)}{{}_c \partial_{q_2} s} {}_c d_{q_2} s - \int_{y_1}^y f(x_1,s) \frac{{}_c \partial_{q_2} g(x_1,s)}{{}_c \partial_{q_2} s} {}_c d_{q_2} s \\ &\quad - \int_{x_1}^x \left( \int_{y_1}^y \frac{{}_c \partial_{q_2} g(q_1t + (1-q_1)a, s)}{{}_c \partial_{q_2} s} \frac{{}_a \partial_{q_1} f(t,s)}{{}_a \partial_{q_1} t} {}_c d_{q_2} s \right) {}_a d_{q_1} t \\ &= f(x,y)g(x,y) - f(x,y_1)g(x,y_1) \\ &\quad - \int_{y_1}^y g(x,q_2s + (1-q_2)c) \frac{{}_c \partial_{q_2} f(x,s)}{{}_c \partial_{q_2} s} {}_c d_{q_2} s \\ &\quad - f(x_1,y)g(x_1,y) + f(x_1,y_1)g(x_1,y_1) \\ &\quad + \int_{y_1}^y g(x_1,q_2s + (1-q_2)c) \frac{{}_c \partial_{q_2} f(x_1,s)}{{}_c \partial_{q_2} s} {}_c d_{q_2} s \\ &\quad - \int_{x_1}^x g(q_1t + (1-q_1)a,y) \frac{{}_a \partial_{q_1} f(t,y)}{{}_a \partial_{q_1} t} {}_a d_{q_1} t \\ &\quad + \int_{x_1}^x g(q_1t + (1-q_1)a,y_1) \frac{{}_a \partial_{q_1} f(t,y_1)}{{}_a \partial_{q_1} t} {}_a d_{q_1} t \\ &\quad + \int_{y_1}^y \int_{x_1}^x g(q_1t + (1-q_1)a, q_2s) \\ &\quad + (1-q_2)c \frac{{}_c a \partial_{q_2 q_1}^2 f(t,s)}{{}_c \partial_{q_2 s} {}_a \partial_{q_1 t}} {}_a d_{q_1 t} {}_c d_{q_2 s} \end{aligned}$$

which is the expected result.  $\square$

### 3. Main results

Before we proceed to prove the main results of this section, we refer the readers to Dragomir (2001), Latif and Alomari (2009) and Özdemir et al. (2012) to study the basic properties of

convex and quasi-convex functions on the co-ordinates on  $[a, b] \times [c, d]$ . In this section, we first prove Hermite–Hadamard type inequality for functions of two variable which are convex on the co-ordinates on  $[a, b] \times [c, d]$ .

**Theorem 3.** Let  $f: [a, b] \times [c, d] \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$  be convex on co-ordinates on  $[a, b] \times [c, d]$ , the following inequalities holds

$$\begin{aligned} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) &\leq \frac{1}{2(b-a)} \int_a^b f\left(x, \frac{c+d}{2}\right) {}_a d_{q_1} x \\ &\quad + \frac{1}{2(d-c)} \int_c^d f\left(\frac{a+b}{2}, y\right) {}_c d_{q_2} y \\ &\leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y) {}_c d_{q_2} y {}_a d_{q_1} x \\ &\leq \frac{q_1}{2(1+q_1)(d-c)} \int_c^d f(a,y) {}_c d_{q_2} y \\ &\quad + \frac{1}{2(1+q_1)(d-c)} \int_c^d f(b,y) {}_c d_{q_2} y \\ &\quad + \frac{1}{2(1+q_2)(b-a)} \int_a^b f(x,d) {}_a d_{q_1} x \\ &\quad + \frac{q_2}{2(1+q_2)(b-a)} \int_a^b f(x,c) {}_a d_{q_1} x \\ &\leq \frac{q_1 q_2 f(a,c) + q_1 f(a,d) + q_2 f(b,c) + f(b,d)}{(1+q_1)(1+q_2)}. \end{aligned}$$

**Proof.** Since  $f: [a, b] \times [c, d] \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$  is convex on co-ordinates on  $[a, b] \times [c, d]$ , we have

$$\begin{aligned} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) &= f\left(\frac{ta + (1-t)b + tb + (1-t)a}{2}, \frac{c+d}{2}\right) \\ &\leq \frac{1}{2} f\left(ta + (1-t)b, \frac{c+d}{2}\right) \\ &\quad + \frac{1}{2} f\left(ta + (1-t)b, \frac{c+d}{2}\right) \end{aligned}$$

The  $q_1$ -integration with respect to  $t$  over  $[0, 1]$ ,  $q_2$ -integration with respect to  $y$  over  $[c, d]$  on both sides of the above inequality and by the change of variables, give

$$f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \leq \frac{1}{b-a} \int_a^b f\left(x, \frac{c+d}{2}\right) {}_a d_{q_1} x. \tag{3.1}$$

Now

$$\begin{aligned} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) &= f\left(\frac{a+b}{2}, \frac{cs + (1-s)d + sd + (1-s)c}{2}\right) \\ &\leq \frac{1}{2} f\left(\frac{a+b}{2}, cs + (1-s)d\right) \\ &\quad + \frac{1}{2} f\left(\frac{a+b}{2}, cd + (1-s)c\right) \end{aligned}$$

The  $q_1$ -integration with respect to  $x$  over  $[a, b]$ ,  $q_2$ -integration with respect to  $s$  over  $[0, 1]$  on both sides of the above inequality and by the change of variables, give

$$f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \leq \frac{1}{d-c} \int_c^d f\left(\frac{a+b}{2}, y\right) {}_c d_{q_2} y. \tag{3.2}$$

Adding (3.1) and (3.2) and dividing both sides by 2, we get

$$f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \leq \frac{1}{2(b-a)} \int_a^b f\left(x, \frac{c+d}{2}\right) {}_a d_{q_1} x + \frac{1}{2(d-c)} \int_c^d f\left(\frac{a+b}{2}, y\right) {}_c d_{q_2} y. \quad (3.3)$$

Consider now

$$\begin{aligned} & \frac{1}{2(b-a)} \int_a^b f\left(x, \frac{c+d}{2}\right) \\ &= \frac{1}{2(b-a)} \int_a^b f\left(x, \frac{cs + (1-s)d + sd + (1-s)c}{2}\right) {}_a d_{q_1} x \\ &\leq \frac{1}{4(b-a)} \int_a^b f(x, cs + (1-s)d) {}_a d_{q_1} x \\ &+ \frac{1}{4(b-a)} \int_a^b f(x, sd + (1-s)c) {}_a d_{q_1} x \end{aligned}$$

The  $q_2$ -integration with respect to  $s$  over  $[0, 1]$ , yields

$$\begin{aligned} \frac{1}{2(b-a)} \int_a^b f\left(x, \frac{c+d}{2}\right) &\leq \frac{1}{4(b-a)} \int_a^b \int_c^d f(x, y) {}_c d_{q_2} y {}_a d_{q_1} x \\ &+ \frac{1}{4(b-a)} \int_a^b \int_c^d f(x, y) {}_c d_{q_2} y {}_a d_{q_1} x \\ &= \frac{1}{2(b-a)(d-c)} \int_a^b \int_c^d f(x, y) {}_c d_{q_2} y {}_a d_{q_1} x. \end{aligned} \quad (3.4)$$

Similarly

$$\begin{aligned} & \frac{1}{2(d-c)} \int_c^d f\left(\frac{a+b}{2}, y\right) {}_c d_{q_2} y \\ &\leq \frac{1}{2(b-a)(d-c)} \int_a^b \int_c^d f(x, y) {}_c d_{q_2} y {}_a d_{q_1} x. \end{aligned} \quad (3.5)$$

Addition of (3.4) and (3.5), gives

$$\begin{aligned} & \frac{1}{2(b-a)} \int_a^b f\left(x, \frac{c+d}{2}\right) + \frac{1}{2(d-c)} \int_c^d f\left(\frac{a+b}{2}, y\right) {}_c d_{q_2} y \\ &\leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) {}_c d_{q_2} y {}_a d_{q_1} x. \end{aligned} \quad (3.6)$$

We also observe that

$$\begin{aligned} & \int_a^b \int_c^d f(x, y) {}_c d_{q_2} y {}_a d_{q_1} x \\ &= (b-a) \int_0^1 \int_c^d f(tb + (1-t)a, y) {}_c d_{q_2} y {}_0 d_{q_1} t \\ &\leq (b-a) \int_0^1 \int_c^d (1-t)f(a, y) {}_c d_{q_2} y {}_0 d_{q_1} t \\ &+ (b-a) \int_0^1 \int_c^d tf(b, y) {}_c d_{q_2} y {}_0 d_{q_1} t \\ &= \frac{q_1(b-a)}{1+q_1} \int_c^d f(a, y) {}_c d_{q_2} y + \frac{(b-a)}{1+q_1} \int_c^d f(b, y) {}_c d_{q_2} y. \end{aligned} \quad (3.7)$$

and

$$\begin{aligned} & \int_a^b \int_c^d f(x, y) {}_c d_{q_2} y {}_a d_{q_1} x \\ &= (d-c) \int_a^b \int_0^1 f(x, sd + (1-s)c) {}_c d_{q_2} s {}_0 d_{q_1} x \\ &\leq (d-c) \int_a^b \int_0^1 sf(x, d) {}_c d_{q_2} s {}_0 d_{q_1} x \\ &+ (d-c) \int_a^b \int_0^1 (1-s)f(x, c) {}_c d_{q_2} s {}_0 d_{q_1} x \\ &= \frac{(d-c)}{1+q_2} \int_a^b f(x, d) {}_a d_{q_1} x + \frac{q_2(d-c)}{1+q_2} \int_a^b f(x, c) {}_a d_{q_1} x. \end{aligned} \quad (3.8)$$

Adding (3.7) and (3.8) and multiplying the resulting inequality by  $\frac{1}{2(b-a)(d-c)}$ , we get

$$\begin{aligned} & \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) {}_c d_{q_2} y {}_a d_{q_1} x \\ &\leq \frac{q_1}{2(1+q_1)(d-c)} \int_c^d f(a, y) {}_c d_{q_2} y \\ &+ \frac{1}{2(1+q_1)(d-c)} \int_c^d f(b, y) {}_c d_{q_2} y \\ &+ \frac{1}{2(1+q_2)(b-a)} \int_a^b f(x, d) {}_a d_{q_1} x \\ &+ \frac{q_2}{2(1+q_2)(b-a)} \int_a^b f(x, c) {}_a d_{q_1} x. \end{aligned} \quad (3.9)$$

Lastly, we have

$$\begin{aligned} & \frac{q_1}{2(1+q_1)(d-c)} \int_c^d f(a, y) {}_c d_{q_2} y + \frac{1}{2(1+q_1)(d-c)} \int_c^d f(b, y) {}_c d_{q_2} y \\ &+ \frac{1}{2(1+q_2)(b-a)} \int_a^b f(x, d) {}_a d_{q_1} x \\ &+ \frac{q_2}{2(1+q_2)(b-a)} \int_a^b f(x, c) {}_a d_{q_1} x \\ &\leq \frac{q_1}{2(1+q_1)} \int_0^1 f(a, sd + (1-s)c) {}_0 d_{q_2} s \\ &+ \frac{1}{2(1+q_1)} \int_0^1 f(b, sd + (1-s)c) {}_0 d_{q_2} s \\ &+ \frac{1}{2(1+q_2)} \int_0^1 f(tb + (1-t)a, d) {}_0 d_{q_1} t \\ &+ \frac{q_2}{2(1+q_2)} \int_0^1 f(tb + (1-t)a, c) {}_0 d_{q_1} t \\ &\leq \frac{q_1 f(a, d)}{2(1+q_1)} \int_0^1 s {}_0 d_{q_2} s + \frac{q_1 f(a, c)}{2(1+q_1)} \int_0^1 (1-s) {}_0 d_{q_2} s \\ &+ \frac{f(b, d)}{2(1+q_1)} \int_0^1 s {}_0 d_{q_2} s + \frac{f(b, c)}{2(1+q_1)} \int_0^1 (1-s) {}_0 d_{q_2} s \\ &+ \frac{f(b, d)}{2(1+q_2)} \int_0^1 t {}_0 d_{q_1} t + \frac{f(a, d)}{2(1+q_2)} \int_0^1 (1-t) {}_0 d_{q_1} t \\ &+ \frac{q_2 f(b, c)}{2(1+q_2)} \int_0^1 t {}_0 d_{q_1} t + \frac{q_2 f(a, c)}{2(1+q_2)} \int_0^1 (1-t) {}_0 d_{q_1} t \\ &= \frac{q_1 q_2 f(a, c) + q_1 f(a, d) + q_2 f(b, c) + f(b, d)}{(1+q_1)(1+q_2)}. \quad \square \end{aligned} \quad (3.10)$$

**Remark 2.** When  $q_1 \rightarrow 1^-$  and  $q_2 \rightarrow 1^-$ , Theorem 3 becomes Theorem 1 from Dragomir (2001, page 778).

We need the following results to prove our main results.

**Theorem 4** (Hölder inequality for double sums). Suppose  $(a_{nm})_{n,m \in \mathbb{N}}, (b_{nm})_{n,m \in \mathbb{N}}$  with  $a_{nm}, b_{nm} \in \mathbb{R}$  or  $\mathbb{C}$  and  $\frac{1}{p} + \frac{1}{p'} = 1, p, p' > 1$ , the following Hölder inequality for double sums holds

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} |a_{nm} b_{nm}| \leq \left( \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} |a_{nm}|^p \right)^{\frac{1}{p}} \left( \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} |b_{nm}|^{p'} \right)^{\frac{1}{p'}}$$

where all the sums are assumed to be finite.

**Theorem 5** ( $q_1 q_2$ -Hölder inequality for functions of two variables). Let  $f$  and  $g$  be functions defined on  $[a, b] \times [c, d]$  and  $0 < q_1, q_2 < 1$ . If  $\frac{1}{r_1} + \frac{1}{r_2} = 1$  with  $r_1, r_2 > 1$ , the following  $q_1 q_2$ -Hölder inequality holds

$$\int_a^x \int_c^y |f(x, y)g(x, y)| {}_c d_{q_2} y {}_a d_{q_1} x \leq \left( \int_a^b \int_c^d |f(x, y)|^{r_1} {}_c d_{q_2} y {}_a d_{q_1} x \right)^{\frac{1}{r_1}} \left( \int_a^b \int_c^d |g(x, y)|^{r_2} {}_c d_{q_2} y {}_a d_{q_1} x \right)^{\frac{1}{r_2}} \tag{3.11}$$

**Proof.** By the definition of  $q_1 q_2$ -integral and applying Theorem 4, we have

$$\begin{aligned} & \int_a^b \int_c^d |f(x, y)g(x, y)| {}_c d_{q_2} y {}_a d_{q_1} x \\ &= (1 - q_1)(1 - q_2)(x - a)(y - c) \\ & \quad \times \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} q_1^n q_2^m |f(q_1^n x + (1 - q_1^n)a, q_2^m y + (1 - q_2^m)c)| \\ & \quad \times |g(q_1^n x + (1 - q_1^n)a, q_2^m y + (1 - q_2^m)c)| \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} [(1 - q_1)(1 - q_2)(x - a)(y - c)]^{\frac{1}{r_1}} (q_1^n q_2^m)^{\frac{1}{r_1}} \\ & \quad \times |f(q_1^n x + (1 - q_1^n)a, q_2^m y + (1 - q_2^m)c)| \\ & \quad \times [(1 - q_1)(1 - q_2)(x - a)(y - c)]^{\frac{1}{r_2}} (q_1^n q_2^m)^{\frac{1}{r_2}} \\ & \quad \times |g(q_1^n x + (1 - q_1^n)a, q_2^m y + (1 - q_2^m)c)| \\ &\leq \left( (1 - q_1)(1 - q_2)(x - a)(y - c) \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} q_1^n q_2^m \right. \\ & \quad \times |f(q_1^n x + (1 - q_1^n)a, q_2^m y + (1 - q_2^m)c)|^{r_1} \Big)^{\frac{1}{r_1}} \\ & \quad \times \left( (1 - q_1)(1 - q_2)(x - a)(y - c) \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} q_1^n q_2^m \right. \\ & \quad \times |g(q_1^n x + (1 - q_1^n)a, q_2^m y + (1 - q_2^m)c)|^{r_2} \Big)^{\frac{1}{r_2}} \\ &= \left( \int_a^b \int_c^d |f(x, y)|^{r_1} {}_c d_{q_2} y {}_a d_{q_1} x \right)^{\frac{1}{r_1}} \\ & \quad \times \left( \int_a^b \int_c^d |g(x, y)|^{r_2} {}_c d_{q_2} y {}_a d_{q_1} x \right)^{\frac{1}{r_2}}. \quad \square \end{aligned}$$

**Lemma 2.** Let  $f: \Lambda \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$  be a twice partially  $q_1 q_2$ -differentiable function on  $\Lambda^\circ$  for  $0 < q_1, q_2 < 1$ . If partial  $q_1 q_2$ -derivative  $\frac{{}_a c \partial_{q_1} \partial_{q_2} f(t, s)}{a \partial_{q_1} t {}_c \partial_{q_2} s}$  is continuous and integrable on  $[a, b] \times [c, d] \subseteq \Lambda^\circ$ , then the following equality holds

$$\begin{aligned} Y_{q_1, q_2}(a, b, c, d)(f) &:= \frac{q_1 q_2 f(a, c) + q_1 f(a, d) + q_2 f(b, c) + f(b, d)}{(1 + q_1)(1 + q_2)} \\ & \quad - \frac{q_2}{(1 + q_2)(b - a)} \int_a^b f(x, c) {}_a d_{q_1} x \\ & \quad - \frac{1}{(1 + q_2)(b - a)} \int_a^b f(x, d) {}_a d_{q_1} x \\ & \quad - \frac{q_1}{(1 + q_1)(d - c)} \int_c^d f(a, y) {}_c d_{q_2} y \\ & \quad - \frac{1}{(1 + q_1)(d - c)} \int_c^d f(b, y) {}_c d_{q_2} y \\ & \quad + \frac{1}{(b - a)(d - c)} \int_a^b \int_c^d f(x, y) {}_c d_{q_2} y {}_a d_{q_1} x \\ &= \frac{q_1 q_2 (b - a)(d - c)}{(1 + q_1)(1 + q_2)} \int_0^1 \int_0^1 (1 - (1 + q_1)t)(1 - (1 + q_2)s) \\ & \quad \times \frac{{}_a c \partial_{q_1, q_2}^2 f((1 - t)a + tb, (1 - s)c + sd)}{a \partial_{q_1} t {}_c \partial_{q_2} s} {}_0 d_{q_1} t {}_0 d_{q_2} s. \tag{3.12} \end{aligned}$$

**Proof.** By the definition of partial  $q_1 q_2$ -derivatives and definite  $q_1 q_2$ -integrals, we have

$$\begin{aligned} & \int_0^1 \int_0^1 (1 - (1 + q_1)t)(1 - (1 + q_2)s) \\ & \quad \times \frac{{}_a c \partial_{q_1, q_2}^2 f((1 - t)a + tb, (1 - s)c + sd)}{a \partial_{q_1} t {}_c \partial_{q_2} s} {}_0 d_{q_1} t {}_0 d_{q_2} s \\ &= \frac{1}{(1 - q_1)(1 - q_2)(b - a)(d - c)} \int_0^1 \int_0^1 \frac{(1 - (1 + q_1)t)(1 - (1 + q_2)s)}{st} \\ & \quad \times [f(tq_1 b + (1 - tq_1)a, sq_2 d + (1 - sq_2)c) - f(tq_1 b + (1 - tq_1)a, sd + (1 - s)c) \\ & \quad - f(tb + (1 - t)a, q_2 sd + (1 - q_2)c) + f(tb + (1 - t)a, sd + (1 - s)c)] {}_0 d_{q_1} t {}_0 d_{q_2} s \\ &= \frac{1}{(b - a)(d - c)} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (1 - (1 + q_1)q_1^n)(1 - (1 + q_2)q_2^m) \\ & \quad \times [f(q_1^{n+1} b + (1 - q_1^{n+1})a, q_2^{m+1} d + (1 - q_2^{m+1})c) \\ & \quad - f(q_1^{n+1} b + (1 - q_1^{n+1})a, q_2^m d + (1 - q_2^m)c) \\ & \quad - f(q_1^n b + (1 - q_1^n)a, q_2^{m+1} d + (1 - q_2^{m+1})c) + f(q_1^n b + (1 - q_1^n)a, q_2^m d + (1 - q_2^m)c)] \\ &= \frac{1}{(b - a)(d - c)} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} f(q_1^n b + (1 - q_1^n)a, q_2^m d + (1 - q_2^m)c) \\ & \quad - \frac{1}{(b - a)(d - c)} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} f(q_1^n b + (1 - q_1^n)a, (1 - q_2^m)c + q_2^m d) \\ & \quad - \frac{1}{(b - a)(d - c)} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} f(q_1^n b + (1 - q_1^n)a, q_2^m d + (1 - q_2^m)c) \\ & \quad + \frac{1}{(d - c)(b - a)} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} f(q_1^n b + (1 - q_1^n)a, q_2^m d + (1 - q_2^m)c) \\ & \quad - \frac{(1 + q_1)}{q_1 (b - a)(d - c)} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} q_1^n f(q_1^n b + (1 - q_1^n)a, q_2^m d + (1 - q_2^m)c) \\ & \quad + \frac{(1 + q_1)}{q_1 (b - a)(d - c)} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} q_1^n f(q_1^n b + (1 - q_1^n)a, (1 - q_2^m)c + q_2^m d) \\ & \quad + \frac{(1 + q_1)}{(b - a)(d - c)} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} q_1^n f(q_1^n b + (1 - q_1^n)a, q_2^m d + (1 - q_2^m)c) \\ & \quad - \frac{(1 + q_1)}{(b - a)(d - c)} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} q_1^n f(q_1^n b + (1 - q_1^n)a, q_2^m d) \\ & \quad + (1 - q_2^m)c - \frac{(1 + q_2)}{q_2 (b - a)(d - c)} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} q_2^m f(q_1^n b + (1 - q_1^n)a, q_2^m d + (1 - q_2^m)c) \\ & \quad + \frac{(1 + q_2)}{(b - a)(d - c)} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} q_2^m f(q_1^n b + (1 - q_1^n)a, (1 - q_2^m)c + q_2^m d) \\ & \quad + \frac{(1 + q_2)}{q_2 (b - a)(d - c)} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} q_2^m f(q_1^n b + (1 - q_1^n)a, q_2^m d + (1 - q_2^m)c) \\ & \quad - \frac{(1 + q_2)}{(b - a)(d - c)} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} q_2^m f(q_1^n b + (1 - q_1^n)a, q_2^m d + (1 - q_2^m)c) \\ & \quad + \frac{(1 + q_1)(1 + q_2)}{q_1 q_2 (b - a)(d - c)} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} q_1^n q_2^m f(q_1^n b + (1 - q_1^n)a, q_2^m d + (1 - q_2^m)c) \\ & \quad - \frac{(1 + q_1)(1 + q_2)}{q_1 (b - a)(d - c)} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} q_1^n q_2^m f(q_1^n b + (1 - q_1^n)a, (1 - q_2^m)c + q_2^m d) \\ & \quad - \frac{(1 + q_1)(1 + q_2)}{q_2 (b - a)(d - c)} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} q_1^n q_2^m f(q_1^n b + (1 - q_1^n)a, q_2^m d + (1 - q_2^m)c) \\ & \quad + \frac{(1 + q_1)(1 + q_2)}{(b - a)(d - c)} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} q_1^n q_2^m f(q_1^n b + (1 - q_1^n)a, q_2^m d + (1 - q_2^m)c). \tag{3.13} \end{aligned}$$

We observe that

$$\begin{aligned} & \frac{1}{(b-a)(d-c)} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} f(q_1^n b + (1-q_1^n)a, q_2^m d + (1-q_2^m)c) \\ &= \frac{f(a,c)}{(b-a)(d-c)} + \frac{1}{(b-a)(d-c)} \\ & \times \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} f(q_1^n b + (1-q_1^n)a, q_2^m d + (1-q_2^m)c), \end{aligned} \quad (3.14)$$

$$\begin{aligned} & - \frac{1}{(b-a)(d-c)} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} f(q_1^n b + (1-q_1^n)a, (1-q_2^m)c + q_2^m d) \\ &= - \frac{f(a,d)}{(b-a)(d-c)} - \frac{1}{(b-a)(d-c)} \\ & \times \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} f(q_1^n b + (1-q_1^n)a, (1-q_2^m)c + q_2^m d), \end{aligned} \quad (3.15)$$

$$\begin{aligned} & - \frac{1}{(b-a)(d-c)} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} f(q_1^n b + (1-q_1^n)a, q_2^m d + (1-q_2^m)c) \\ &= - \frac{f(b,c)}{(b-a)(d-c)} - \frac{1}{(b-a)(d-c)} \\ & \times \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} f(q_1^n b + (1-q_1^n)a, q_2^m d + (1-q_2^m)c), \end{aligned} \quad (3.16)$$

$$\begin{aligned} & \frac{1}{(b-a)(d-c)} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} f(q_1^n b + (1-q_1^n)a, q_2^m d + (1-q_2^m)c) \\ &= \frac{f(b,d)}{(b-a)(d-c)} + \frac{1}{(b-a)(d-c)} \\ & \times \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} f(q_1^n b + (1-q_1^n)a, q_2^m d + (1-q_2^m)c), \end{aligned} \quad (3.17)$$

$$\begin{aligned} & - \frac{(1+q_1)}{q_1(b-a)(d-c)} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} q_1^n f(q_1^n b + (1-q_1^n)a, q_2^m d + (1-q_2^m)c) \\ &= \frac{(1+q_1)f(b,c)}{q_1(b-a)(d-c)} - \frac{(1+q_1)}{q_1(b-a)(d-c)} \\ & \times \sum_{n=0}^{\infty} q_1^n f(q_1^n b + (1-q_1^n)a, c) - \frac{(1+q_1)}{q_1(b-a)(d-c)} \\ & \times \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} q_1^n f(q_1^n b + (1-q_1^n)a, q_2^m d + (1-q_2^m)c), \end{aligned} \quad (3.18)$$

$$\begin{aligned} & \frac{(1+q_1)}{q_1(b-a)(d-c)} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} q_1^n f(q_1^n b + (1-q_1^n)a, (1-q_2^m)c + q_2^m d) \\ &= - \frac{(1+q_1)f(b,d)}{q_1(b-a)(d-c)} + \frac{(1+q_1)}{q_1(b-a)(d-c)} \\ & \times \sum_{n=0}^{\infty} q_1^n f(q_1^n b + (1-q_1^n)a, d) + \frac{(1+q_1)}{q_1(b-a)(d-c)} \\ & \times \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} q_1^n f(q_1^n b + (1-q_1^n)a, (1-q_2^m)c + q_2^m d), \end{aligned} \quad (3.19)$$

$$\begin{aligned} & \frac{(1+q_1)}{(b-a)(d-c)} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} q_1^n f(q_1^n b + (1-q_1^n)a, q_2^m d + (1-q_2^m)c) \\ &= \frac{(1+q_1)}{(b-a)(d-c)} \\ & \times \left[ - \sum_{n=0}^{\infty} q_1^n f(q_1^n b + (1-q_1^n)a, d) + \sum_{n=0}^{\infty} q_1^n f(q_1^n b + (1-q_1^n)a, c) \right] \\ & + \frac{(1+q_1)}{(b-a)(d-c)} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} q_1^n f(q_1^n b + (1-q_1^n)a, q_2^m d + (1-q_2^m)c), \end{aligned} \quad (3.20)$$

$$\begin{aligned} & - \frac{(1+q_2)}{q_2(b-a)(d-c)} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} q_2^m f(q_1^n b + (1-q_1^n)a, q_2^m d + (1-q_2^m)c) \\ &= \frac{(1+q_2)f(a,d)}{q_2(b-a)(d-c)} - \frac{(1+q_2)}{q_2(b-a)(d-c)} \\ & \times \sum_{m=0}^{\infty} q_2^m f(a, q_2^m d + (1-q_2^m)c) - \frac{(1+q_2)}{q_2(b-a)(d-c)} \\ & \times \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} q_2^m f(q_1^n b + (1-q_1^n)a, q_2^m d + (1-q_2^m)c), \end{aligned} \quad (3.21)$$

$$\begin{aligned} & \frac{(1+q_2)}{(b-a)(d-c)} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} q_2^m f(q_1^n b + (1-q_1^n)a, (1-q_2^m)c + q_2^m d) \\ &= - \frac{(1+q_2)}{(b-a)(d-c)} \\ & \times \left[ - \sum_{m=0}^{\infty} q_2^m f(b, (1-q_2^m)c + q_2^m d) + \sum_{m=0}^{\infty} q_2^m f(a, (1-q_2^m)c + q_2^m d) \right] \\ & + \frac{(1+q_2)}{(b-a)(d-c)} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} q_2^m f(q_1^n b + (1-q_1^n)a, (1-q_2^m)c + q_2^m d), \end{aligned} \quad (3.22)$$

$$\begin{aligned} & \frac{(1+q_2)}{q_2(b-a)(d-c)} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} q_2^m f(q_1^n b + (1-q_1^n)a, q_2^m d + (1-q_2^m)c) \\ &= - \frac{(1+q_2)f(b,d)}{q_2(b-a)(d-c)} \\ & + \frac{(1+q_2)}{q_2(b-a)(d-c)} \sum_{m=0}^{\infty} q_2^m f(b, q_2^m d + (1-q_2^m)c) \\ & + \frac{(1+q_2)}{q_2(b-a)(d-c)} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} q_2^m f(q_1^n b + (1-q_1^n)a, q_2^m d + (1-q_2^m)c), \end{aligned} \quad (3.23)$$

$$\begin{aligned} & \frac{(1+q_1)(1+q_2)}{q_1 q_2 (b-a)(d-c)} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} q_1^n q_2^m f(q_1^n b + (1-q_1^n)a, q_2^m d + (1-q_2^m)c) \\ &= - \frac{(1+q_1)(1+q_2)f(b,d)}{q_1 q_2 (b-a)(d-c)} \\ & - \frac{(1+q_1)(1+q_2)}{q_1 q_2 (b-a)(d-c)} \sum_{n=0}^{\infty} q_1^n f(q_1^n b + (1-q_1^n)a, d) \\ & - \frac{(1+q_1)(1+q_2)}{q_1 q_2 (b-a)(d-c)} \sum_{m=0}^{\infty} q_2^m f(b, q_2^m d + (1-q_2^m)c) \\ & + \frac{(1+q_1)(1+q_2)}{q_1 q_2 (b-a)(d-c)} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} q_1^n q_2^m f(q_1^n b + (1-q_1^n)a, q_2^m d + (1-q_2^m)c) \end{aligned} \quad (3.24)$$

$$\begin{aligned} & - \frac{(1+q_1)(1+q_2)}{q_1(b-a)(d-c)} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} q_1^n q_2^m f(q_1^n b + (1-q_1^n)a, (1-q_2^m)c + q_2^m d) \\ &= \frac{(1+q_1)(1+q_2)}{q_1(b-a)(d-c)} \sum_{m=0}^{\infty} q_2^m f(b, (1-q_2^m)c + q_2^m d) \\ & - \frac{(1+q_1)(1+q_2)}{q_1(b-a)(d-c)} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} q_1^n q_2^m f(q_1^n b + (1-q_1^n)a, (1-q_2^m)c + q_2^m d) \end{aligned} \quad (3.25)$$

and

$$\begin{aligned} & - \frac{(1+q_1)(1+q_2)}{q_2(b-a)(d-c)} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} q_1^n q_2^m f(q_1^n b + (1-q_1^n)a, q_2^m d + (1-q_2^m)c) \\ &= \frac{(1+q_1)(1+q_2)}{q_2(b-a)(d-c)} \sum_{n=0}^{\infty} q_1^n f(q_1^n b + (1-q_1^n)a, d) \\ & - \frac{(1+q_1)(1+q_2)}{q_2(b-a)(d-c)} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} q_1^n q_2^m f(q_1^n b + (1-q_1^n)a, q_2^m d + (1-q_2^m)c). \end{aligned} \quad (3.26)$$



Using (3.14)–(3.26) in (3.13) and simplifying, we get

$$\begin{aligned}
 & \int_0^1 \int_0^1 (1 - (1 + q_1)t)(1 - (1 + q_2)s) \\
 & \times \frac{{}_{a,c}\partial_{q_1,q_2}^2 f((1-t)a + tb, (1-s)c + sd)}{{}_a\partial_{q_1} x {}_c\partial_{q_2} y} {}_0d_{q_1} t {}_0d_{q_2} s \\
 & = \frac{q_1 q_2 f(a, c) + q_1 f(a, d) + q_2 f(b, c) + f(b, d)}{q_1 q_2 (b - a)(d - c)} \\
 & - \frac{(1 + q_1)(1 - q_1)}{q_1 (b - a)(d - c)} \sum_{n=0}^{\infty} q_1^n f(q_1^n b + (1 - q_1^n) a, c) \\
 & - \frac{(1 + q_1)(1 - q_1)}{q_1 q_2 (b - a)(d - c)} \sum_{n=0}^{\infty} q_1^n f(q_1^n b + (1 - q_1^n) a, d) \\
 & - \frac{(1 + q_2)(1 - q_2)}{q_2 (b - a)(d - c)} \sum_{m=0}^{\infty} q_2^m f(a, q_2^m d + (1 - q_2^m) c) \\
 & - \frac{(1 + q_2)(1 - q_2)}{q_1 q_2 (b - a)(d - c)} \sum_{m=0}^{\infty} q_2^m f(b, (1 - q_2^m) c + q_2^m d) \\
 & + \frac{(1 + q_1)(1 + q_2)(1 - q_1)(1 - q_2)}{q_1 q_2 (b - a)(d - c)} \\
 & \times \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} q_1^n q_2^m f(q_1^n b + (1 - q_1^n) a, q_2^m d + (1 - q_2^m) c) \\
 & = \frac{q_1 q_2 f(a, c) + q_1 f(a, d) + q_2 f(b, c) + f(b, d)}{q_1 q_2 (b - a)(d - c)} \\
 & - \frac{(1 + q_1)}{q_1 (b - a)^2 (d - c)} \int_a^b f(x, c) {}_a d_{q_1} x \\
 & - \frac{(1 + q_1)}{q_1 q_2 (b - a)^2 (d - c)} \int_a^b f(x, d) {}_a d_{q_1} x \\
 & - \frac{(1 + q_2)}{q_2 (b - a)(d - c)^2} \int_c^d f(a, y) {}_c d_{q_2} y \\
 & - \frac{(1 + q_2)}{q_1 q_2 (b - a)(d - c)^2} \int_c^d f(b, y) {}_c d_{q_2} y \\
 & + \frac{(1 + q_1)(1 + q_2)}{q_1 q_2 (b - a)^2 (d - c)^2} \int_a^b \int_c^d f(x, y) {}_c d_{q_2} y {}_a d_{q_1} x. \quad (3.27)
 \end{aligned}$$

Multiplying both sides of (3.27) by  $\frac{q_1 q_2 (b - a)(d - c)}{(1 + q_1)(1 + q_2)}$ , we get the desired equality.  $\square$

**Remark 3.** As  $q_1 \rightarrow 1^-$  and  $q_2 \rightarrow 1^-$ ,  $\Upsilon_{q_1, q_2}(a, b, c, d)(f) \rightarrow \Upsilon(a, b, c, d)(f)$ , where

$$\begin{aligned}
 \Upsilon(a, b, c, d)(f) & := \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4} \\
 & - \frac{1}{2(b - a)} \int_a^b f(x, c) dx - \frac{1}{2(b - a)} \int_a^b f(x, d) dx \\
 & - \frac{1}{2(d - c)} \int_c^d f(a, y) dy - \frac{1}{2(d - c)} \int_c^d f(b, y) dy \\
 & + \frac{1}{(b - a)(d - c)} \int_a^b \int_c^d f(x, y) dy dx
 \end{aligned}$$

and hence the result of Lemma 2 becomes the result of Lemma 1 proved in Sarikaya et al. (2012, page 139).

Now, we can present some integral inequalities for functions whose partial  $q_1 q_2$ -derivatives satisfy the assumptions of convexity on co-ordinates on  $[a, b] \times [c, d]$ .

**Theorem 6.** Let  $f: \Lambda \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$  be a twice partially  $q_1 q_2$ -differentiable function on  $\Lambda^\circ$  with  $0 < q_1 < 1$  and  $0 < q_2 < 1$ . If partial  $q_1 q_2$ -derivative  $\frac{{}_{a,c}\partial_{q_1,q_2}^2 f}{{}_a\partial_{q_1} t {}_c\partial_{q_2} s}$  is continuous and integrable on  $[a, b] \times [c, d] \subseteq \Lambda^\circ$  and  $\left| \frac{{}_{a,c}\partial_{q_1,q_2}^2 f}{{}_a\partial_{q_1} t {}_c\partial_{q_2} s} \right|^r$  is convex on co-ordinates on  $[a, b] \times [c, d]$  for  $r \geq 1$ , then the following inequality holds

$$\begin{aligned}
 |\Upsilon_{q_1, q_2}(a, b, c, d)(f)| & \leq \frac{q_1 q_2 (b - a)(d - c)}{(1 + q_1)(1 + q_2)} (\Phi_{q_1} \Phi_{q_2})^{1-\frac{1}{r}} \\
 & \times \left\{ \Psi_{q_1} \Psi_{q_2} \left| \frac{{}_{a,c}\partial_{q_1,q_2}^2 f(a, c)}{{}_a\partial_{q_1} t {}_c\partial_{q_2} s} \right|^r + \Delta_{q_1} \Psi_{q_2} \left| \frac{{}_{a,c}\partial_{q_1,q_2}^2 f(a, d)}{{}_a\partial_{q_1} t {}_c\partial_{q_2} s} \right|^r \right. \\
 & \left. + \Delta_{q_2} \Psi_{q_1} \left| \frac{{}_{a,c}\partial_{q_1,q_2}^2 f(b, c)}{{}_a\partial_{q_1} t {}_c\partial_{q_2} s} \right|^r + \Delta_{q_1} \Delta_{q_2} \left| \frac{{}_{a,c}\partial_{q_1,q_2}^2 f(b, d)}{{}_a\partial_{q_1} t {}_c\partial_{q_2} s} \right|^r \right\}^{\frac{1}{r}}. \quad (3.28)
 \end{aligned}$$

**Proof.** Taking the absolute value on both sides of the equality of Lemma 2, using the  $q_1 q_2$ -Hölder inequality for functions of two variables and convexity of  $\left| \frac{{}_{a,c}\partial_{q_1,q_2}^2 f}{{}_a\partial_{q_1} t {}_c\partial_{q_2} s} \right|^r$  on co-ordinates on  $[a, b] \times [c, d]$ , we have

$$\begin{aligned}
 |\Upsilon_{q_1, q_2}(a, b, c, d)(f)| & \leq \frac{q_1 q_2 (b - a)(d - c)}{(1 + q_1)(1 + q_2)} \\
 & \times \left( \int_0^1 \int_0^1 |(1 - (1 + q_1)t)(1 - (1 + q_2)s)| {}_0d_{q_1} t {}_0d_{q_2} s \right)^{1-\frac{1}{r}} \\
 & \times \left( \int_0^1 \int_0^1 |(1 - (1 + q_1)t)(1 - (1 + q_2)s)| \right. \\
 & \times \left[ \left( (1 - t)(1 - s) \left| \frac{{}_{a,c}\partial_{q_1,q_2}^2 f(a, c)}{{}_a\partial_{q_1} t {}_c\partial_{q_2} s} \right|^r + (1 - t)s \left| \frac{{}_{a,c}\partial_{q_1,q_2}^2 f(a, d)}{{}_a\partial_{q_1} t {}_c\partial_{q_2} s} \right|^r \right. \right. \\
 & \left. \left. + (1 - s)t \left| \frac{{}_{a,c}\partial_{q_1,q_2}^2 f(b, c)}{{}_a\partial_{q_1} t {}_c\partial_{q_2} s} \right|^r + st \left| \frac{{}_{a,c}\partial_{q_1,q_2}^2 f(b, d)}{{}_a\partial_{q_1} t {}_c\partial_{q_2} s} \right|^r \right] {}_0d_{q_1} t {}_0d_{q_2} s \right)^{\frac{1}{r}}. \quad (3.29)
 \end{aligned}$$

From Lemma 1, we observe that

$$\begin{aligned}
 & \int_0^1 \int_0^1 |(1 - (1 + q_1)t)(1 - (1 + q_2)s)| {}_0d_{q_1} t {}_0d_{q_2} s \\
 & = \left( \int_0^1 |(1 - (1 + q_1)t)| {}_0d_{q_1} t \right) \left( \int_0^1 |(1 - (1 + q_2)s)| {}_0d_{q_2} s \right) \\
 & = \Phi_{q_1} \Phi_{q_2}, \\
 & \int_0^1 \int_0^1 (1 - t)(1 - s)|(1 - (1 + q_1)t)(1 - (1 + q_2)s)| {}_0d_{q_1} t {}_0d_{q_2} s \\
 & = \left( \int_0^1 (1 - t)|1 - (1 + q_1)t| {}_0d_{q_1} t \right) \\
 & \times \left( \int_0^1 (1 - s)|1 - (1 + q_2)s| {}_0d_{q_2} s \right) = \Psi_{q_1} \Psi_{q_2}, \\
 & \int_0^1 \int_0^1 (1 - t)s|(1 - (1 + q_1)t)(1 - (1 + q_2)s)| {}_0d_{q_1} t {}_0d_{q_2} s \\
 & = \left( \int_0^1 (1 - t)|1 - (1 + q_1)t| {}_0d_{q_1} t \right) \\
 & \times \left( \int_0^1 s|1 - (1 + q_2)s| {}_0d_{q_2} s \right) = \Delta_{q_1} \Psi_{q_2},
 \end{aligned}$$

$$\begin{aligned} & \int_0^1 \int_0^1 t(1-s)|(1-(1+q_1)t)(1-(1+q_2)s)| {}_0d_{q_1}t {}_0d_{q_2}s \\ &= \left( \int_0^1 t|1-(1+q_1)t| {}_0d_{q_1}t \right) \\ & \quad \times \left( \int_0^1 (1-s)|1-(1+q_2)s| {}_0d_{q_2}s \right) = \Delta_{q_2} \Psi_{q_1} \end{aligned}$$

and

$$\begin{aligned} & \int_0^1 \int_0^1 ts|(1-(1+q_1)t)(1-(1+q_2)s)| {}_0d_{q_1}t {}_0d_{q_2}s \\ &= \left( \int_0^1 t|1-(1+q_1)t| {}_0d_{q_1}t \right) \\ & \quad \times \left( \int_0^1 (1-s)|1-(1+q_2)s| {}_0d_{q_2}s \right) = \Delta_{q_1} \Delta_{q_2}. \end{aligned}$$

Using the values of the above  $q_1q_2$ -integrals, we get the required inequality.  $\square$

**Remark 4.** When  $q_1 \rightarrow 1^-$  and  $q_2 \rightarrow 1^-$  in Theorem 6, we get the result proved in Theorem 4 in Sarikaya et al. (2012, page 146).

**Theorem 7.** Let  $f: \Lambda \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$  be a twice partially  $q_1q_2$ -differentiable function on  $\Lambda^\circ$  with  $0 < q_1 < 1$  and  $0 < q_2 < 1$ . If partial  $q_1q_2$ -derivative  $\frac{a.c.\partial_{q_1,q_2}^2 f}{a\partial_{q_1}t {}_c\partial_{q_2}s}$  is continuous and integrable on  $[a, b] \times [c, d] \subseteq \Lambda^\circ$  and  $\left| \frac{a.c.\partial_{q_1,q_2}^2 f}{a\partial_{q_1}t {}_c\partial_{q_2}s} \right|^{r_1}$  is convex on co-ordinates on  $[a, b] \times [c, d]$  for  $r_1 > 1$ , then the following inequality holds

$$\begin{aligned} |\Upsilon_{q_1,q_2}(a, b, c, d)(f)| &\leq \frac{q_1q_2(b-a)(d-c)}{[(1+q_1)(1+q_2)]^{1+\frac{1}{r_1}}} (A_{q_1}(r_2)A_{q_2}(r_2))^{\frac{1}{r_2}} \\ &\quad \times \left\{ q_1q_2 \left| \frac{a.c.\partial_{q_1,q_2}^2 f(a, c)}{a\partial_{q_1}t {}_c\partial_{q_2}s} \right|^{r_1} + q_1 \left| \frac{a.c.\partial_{q_1,q_2}^2 f(a, d)}{a\partial_{q_1}t {}_c\partial_{q_2}s} \right|^{r_1} \right. \\ &\quad \left. + q_2 \left| \frac{a.c.\partial_{q_1,q_2}^2 f(b, c)}{a\partial_{q_1}t {}_c\partial_{q_2}s} \right|^{r_1} + \left| \frac{a.c.\partial_{q_1,q_2}^2 f(b, d)}{a\partial_{q_1}t {}_c\partial_{q_2}s} \right|^{r_1} \right)^{\frac{1}{r_1}}, \end{aligned} \tag{3.30}$$

where  $\frac{1}{r_1} + \frac{1}{r_2} = 1$ .

**Proof.** Taking the absolute value on both sides of the equality of Lemma 2, using the  $q_1q_2$ -Hölder inequality for functions of two variables and convexity of  $\left| \frac{a.c.\partial_{q_1,q_2}^2 f}{a\partial_{q_1}t {}_c\partial_{q_2}s} \right|^r$  on co-ordinates on  $[a, b] \times [c, d]$ , we have

$$\begin{aligned} |\Upsilon_{q_1,q_2}(a, b, c, d)(f)| &\leq \frac{q_1q_2(b-a)(d-c)}{(1+q_1)(1+q_2)} \\ &\quad \times \left( \int_0^1 \int_0^1 |(1-(1+q_1)t)(1-(1+q_2)s)|^{r_2} {}_0d_{q_1}t {}_0d_{q_2}s \right)^{\frac{1}{r_2}} \\ &\quad \times \left( \int_0^1 \int_0^1 \left[ (1-t)(1-s) \left| \frac{a.c.\partial_{q_1,q_2}^2 f(a, c)}{a\partial_{q_1}t {}_c\partial_{q_2}s} \right|^{r_1} \right. \right. \\ &\quad \left. \left. + (1-t)s \left| \frac{a.c.\partial_{q_1,q_2}^2 f(a, d)}{a\partial_{q_1}t {}_c\partial_{q_2}s} \right|^{r_1} + (1-s)t \left| \frac{a.c.\partial_{q_1,q_2}^2 f(b, c)}{a\partial_{q_1}t {}_c\partial_{q_2}s} \right|^{r_1} \right. \right. \\ &\quad \left. \left. + st \left| \frac{a.c.\partial_{q_1,q_2}^2 f(b, d)}{a\partial_{q_1}t {}_c\partial_{q_2}s} \right|^{r_1} \right] {}_0d_{q_1}t {}_0d_{q_2}s \right)^{\frac{1}{r_1}}. \end{aligned} \tag{3.31}$$

We observe that

$$\begin{aligned} \int_0^1 |1-(1+q_1)t|^{r_2} {}_0d_{q_1}t &= \int_0^{\frac{1}{1+q_1}} (1-(1+q_1)t)^{r_2} {}_0d_{q_1}t \\ &\quad + \int_{\frac{1}{1+q_1}}^1 ((1+q_1)t-1)^{r_2} {}_0d_{q_1}t. \end{aligned} \tag{3.32}$$

Consider the first  $q_1$ -integral from (3.32) and making use of the substitution  $1-(1+q_1)t = s$ , we obtain

$$\begin{aligned} \int_0^{\frac{1}{1+q_1}} (1-(1+q_1)t)^{r_2} {}_0d_{q_1}t &= -\frac{1}{1+q_1} \int_1^0 s^{r_2} {}_0d_{q_1}s \\ &= \frac{1}{1+q_1} \int_0^1 s^{r_2} {}_0d_{q_1}s \\ &= \frac{1-q_1}{(1+q_1)(1-q_1^{r_2+1})}. \end{aligned} \tag{3.33}$$

Consider the second  $q_1$ -integral from (3.32) and making use of the substitution  $(1+q_1)t-1 = s$ , we get

$$\begin{aligned} \int_{\frac{1}{1+q_1}}^1 ((1+q_1)t-1)^{r_2} {}_0d_{q_1}t &= \frac{1}{1+q_1} \int_0^{q_1} s^{r_2} {}_0d_{q_1}s \\ &= \frac{(1-q_1)q_1^{r_2+1}}{(1+q_1)(1-q_1^{r_2+1})}. \end{aligned} \tag{3.34}$$

Substitution of (3.33) and (3.34) in (3.32) gives

$$\begin{aligned} \int_0^1 |1-(1+q_1)t|^{r_2} {}_0d_{q_1}t &= \frac{(1-q_1)(1+q_1^{r_2+1})}{(1+q_1)(1-q_1^{r_2+1})} \\ &= A_{q_1}(r_2). \end{aligned} \tag{3.35}$$

Similarly, one can have

$$\begin{aligned} \int_0^1 |1-(1+q_2)s|^{r_2} {}_0d_{q_2}s &= \frac{(1-q_2)(1+q_2^{r_2+1})}{(1+q_2)(1-q_2^{r_2+1})} \\ &= A_{q_2}(r_2). \end{aligned} \tag{3.36}$$

Finally, we also have

$$\begin{aligned} \int_0^1 (1-t) {}_0d_{q_1}t &= \frac{q_1}{1+q_1}, \quad \int_0^1 (1-s) {}_0d_{q_2}s = \frac{q_2}{1+q_2}, \\ \int_0^1 t {}_0d_{q_1}t &= \frac{1}{1+q_1} \quad \text{and} \quad \int_0^1 s {}_0d_{q_2}s = \frac{1}{1+q_2}. \quad \square \end{aligned}$$

**Remark 5.** When  $q_1 \rightarrow 1^-$  and  $q_2 \rightarrow 1^-$  in Theorem 7, we get the following inequality proved in Sarikaya et al. (2012, page 144).

$$\begin{aligned} |\Upsilon(a, b, c, d)(f)| &\leq \frac{(b-a)(d-c)}{4} \left( \frac{1}{r_2+1} \right)^{\frac{r_2}{2}} \\ &\quad \times \left\{ \frac{\left| \frac{\partial^2 f(a,c)}{\partial t \partial s} \right|^{r_1} + \left| \frac{\partial^2 f(a,d)}{\partial t \partial s} \right|^{r_1} + \left| \frac{\partial^2 f(b,c)}{\partial t \partial s} \right|^{r_1} + \left| \frac{\partial^2 f(b,d)}{\partial t \partial s} \right|^{r_1}}{4} \right)^{\frac{1}{r_1}}. \end{aligned} \tag{3.37}$$

Indeed, the inequality (3.37) follows by applying L'Hospital rule to the limits

$$\lim_{q_1 \rightarrow 1^-} \frac{1-q_1}{1-q_1^{r_2+1}} \quad \text{and} \quad \lim_{q_2 \rightarrow 1^-} \frac{1-q_2}{1-q_2^{r_2+1}}.$$

The next two results are for quasi-convex functions on co-ordinates on  $[a, b] \times [c, d]$ .

**Theorem 8.** Let  $f: \Lambda \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$  be a twice partially  $q_1q_2$ -differentiable function on  $\Lambda^\circ$  with  $0 < q_1 < 1$  and  $0 < q_2 < 1$ . If partial  $q_1q_2$ -derivative  $\frac{{}_{a,c}\partial_{q_1,q_2}^2 f}{{}_a\partial_{q_1}t {}_c\partial_{q_2}s}$  is continuous and integrable on  $[a, b] \times [c, d] \subseteq \Lambda^\circ$  and  $\left| \frac{{}_{a,c}\partial_{q_1,q_2}^2 f}{{}_a\partial_{q_1}t {}_c\partial_{q_2}s} \right|^r$  is quasi-convex on co-ordinates on  $[a, b] \times [c, d]$  for  $r \geq 1$ , then the following inequality holds

$$|\Upsilon_{q_1,q_2}(a, b, c, d)(f)| \leq \frac{q_1q_2(b-a)(d-c)}{(1+q_1)(1+q_2)} \left( \frac{4q_1q_2}{(1+q_1)^2(1+q_2)^2} \right) \times \sup \left\{ \left| \frac{{}_{a,c}\partial_{q_1,q_2}^2 f(a,c)}{{}_a\partial_{q_1}t {}_c\partial_{q_2}s} \right|, \left| \frac{{}_{a,c}\partial_{q_1,q_2}^2 f(b,c)}{{}_a\partial_{q_1}t {}_c\partial_{q_2}s} \right|, \left| \frac{{}_{a,c}\partial_{q_1,q_2}^2 f(a,d)}{{}_a\partial_{q_1}t {}_c\partial_{q_2}s} \right|, \left| \frac{{}_{a,c}\partial_{q_1,q_2}^2 f(b,d)}{{}_a\partial_{q_1}t {}_c\partial_{q_2}s} \right| \right\}. \tag{3.38}$$

**Proof.** Lemma 2, an application of the  $q_1q_2$ -Hölder inequality and quasi-convexity of  $\left| \frac{{}_{a,c}\partial_{q_1,q_2}^2 f}{{}_a\partial_{q_1}t {}_c\partial_{q_2}s} \right|^r$  on  $[a, b] \times [c, d]$ , yield

$$|\Upsilon_{q_1,q_2}(a, b, c, d)(f)| \leq \frac{q_1q_2(b-a)(d-c)}{(1+q_1)(1+q_2)} \times \left( \int_0^1 \int_0^1 |(1-(1+q_1)t)(1-(1+q_2)s)| {}_0d_{q_1}t {}_0d_{q_2}s \right)^{1-\frac{1}{r}} \times \left( \int_0^1 \int_0^1 |(1-(1+q_1)t)(1-(1+q_2)s)| \times \left| \frac{{}_{a,c}\partial_{q_1,q_2}^2 f((1-t)a+tb, (1-s)c+sd)}{{}_a\partial_{q_1}t {}_c\partial_{q_2}s} \right|^r {}_0d_{q_1}t {}_0d_{q_2}s \right)^{\frac{1}{r}} = \frac{q_1q_2(b-a)(d-c)}{(1+q_1)(1+q_2)} \times \left( \int_0^1 \int_0^1 |(1-(1+q_1)t)(1-(1+q_2)s)| {}_0d_{q_1}t {}_0d_{q_2}s \right) \times \left( \sup \left\{ \left| \frac{{}_{a,c}\partial_{q_1,q_2}^2 f(a,c)}{{}_a\partial_{q_1}t {}_c\partial_{q_2}s} \right|^r, \left| \frac{{}_{a,c}\partial_{q_1,q_2}^2 f(b,c)}{{}_a\partial_{q_1}t {}_c\partial_{q_2}s} \right|^r, \left| \frac{{}_{a,c}\partial_{q_1,q_2}^2 f(a,d)}{{}_a\partial_{q_1}t {}_c\partial_{q_2}s} \right|^r, \left| \frac{{}_{a,c}\partial_{q_1,q_2}^2 f(b,d)}{{}_a\partial_{q_1}t {}_c\partial_{q_2}s} \right|^r \right\} \right)^{\frac{1}{r}}. \tag{3.39}$$

Now using the properties of supremum and Lemma 1, we get the required result from (3.39).  $\square$

**Theorem 9.** Let  $f: \Lambda \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$  be a twice partially  $q_1q_2$ -differentiable function on  $\Lambda^\circ$  with  $0 < q_1 < 1$  and  $0 < q_2 < 1$ . If partial  $q_1q_2$ -derivative  $\frac{{}_{a,c}\partial_{q_1,q_2}^2 f}{{}_a\partial_{q_1}t {}_c\partial_{q_2}s}$  is continuous and integrable on  $[a, b] \times [c, d] \subseteq \Lambda^\circ$  and  $\left| \frac{{}_{a,c}\partial_{q_1,q_2}^2 f}{{}_a\partial_{q_1}t {}_c\partial_{q_2}s} \right|^{r_1}$  is convex on co-ordinates on  $[a, b] \times [c, d]$  for  $r_1 > 1$ , then the following inequality holds

$$|\Upsilon_{q_1,q_2}(a, b, c, d)(f)| \leq \frac{q_1q_2(b-a)(d-c)}{(1+q_1)(1+q_2)} (A_{q_1}(r_2)A_{q_2}(r_2))^{\frac{1}{2}} \times \sup \left\{ \left| \frac{{}_{a,c}\partial_{q_1,q_2}^2 f(a,c)}{{}_a\partial_{q_1}t {}_c\partial_{q_2}s} \right|, \left| \frac{{}_{a,c}\partial_{q_1,q_2}^2 f(b,c)}{{}_a\partial_{q_1}t {}_c\partial_{q_2}s} \right|, \left| \frac{{}_{a,c}\partial_{q_1,q_2}^2 f(a,d)}{{}_a\partial_{q_1}t {}_c\partial_{q_2}s} \right|, \left| \frac{{}_{a,c}\partial_{q_1,q_2}^2 f(b,d)}{{}_a\partial_{q_1}t {}_c\partial_{q_2}s} \right| \right\}. \tag{3.40}$$

**Proof.** With the similar reasoning as in proving (3.38), we notice that

$$|\Upsilon_{q_1,q_2}(a, b, c, d)(f)| \leq \frac{q_1q_2(b-a)(d-c)}{(1+q_1)(1+q_2)} \times \left( \int_0^1 \int_0^1 |(1-(1+q_1)t)(1-(1+q_2)s)| {}_0d_{q_1}t {}_0d_{q_2}s \right)^{\frac{1}{r_1}} \times \left( \int_0^1 \int_0^1 \left| \frac{{}_{a,c}\partial_{q_1,q_2}^2 f((1-t)a+tb, (1-s)c+sd)}{{}_a\partial_{q_1}t {}_c\partial_{q_2}s} \right|^{r_2} {}_0d_{q_1}t {}_0d_{q_2}s \right)^{\frac{1}{r_2}}. \tag{3.41}$$

Using the properties of the supremum, the quasi-convexity of  $\left| \frac{{}_{a,c}\partial_{q_1,q_2}^2 f}{{}_a\partial_{q_1}t {}_c\partial_{q_2}s} \right|^r$  on  $[a, b] \times [c, d]$ , (3.35) and (3.36), we get (3.40).  $\square$

**Remark 6.** As  $q_1 \rightarrow 1^-$  and  $q_2 \rightarrow 1^-$  in Theorems 8 and 9, we get the corresponding results of classical calculus of functions of two variables.

### 4. Conclusion

In this manuscript partial  $q_1q_2$ -derivative and definite  $q_1q_2$ -integrals over the finite rectangles are discussed for the first time. Some  $q$ -analogues of integral inequalities for functions of two variables are presented using the notion of  $q$ -calculus of functions of two variables over the finite rectangles and the concept of two types of convexity on co-ordinates. The results of this paper have very clear physical understanding of minimizing the error bounds in the two variable trapezoidal rule.

### Competing interests

The author declares that he has no competing interests.

### Authors' Contributions

All the authors have contributed equality in preparing the manuscript. All the authors have approved the final version of the manuscript.

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### Appendix A. Supplementary data

Supplementary data associated with this article can be found, in the online version, at <http://dx.doi.org/10.1016/j.jksus.2016.07.001>.

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