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SYMMETRIZED CONVEXITY AND HERMITE–HADAMARD TYPE INEQUALITIES

S. S. DRAGOMIR

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Abstract. In this paper we extend the Hermite-Hadamard inequality to the class of symmetrized convex functions. The corresponding version for h-convex functions is also investigated. Some examples of interest are provided as well.

1. Introduction

The following inequality holds for any convex function f defined on $\mathbb{R}$

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)dx \leqslant \frac{f(a)+f(b)}{2}, \quad a,b \in \mathbb{R}, \ a \neq b. \tag{1.1}$$

It was firstly discovered by Ch. Hermite in 1881 in the journal *Mathesis* (see [43]). But this result was nowhere mentioned in the mathematical literature and was not widely known as Hermite's result.

E. F. Beckenbach, a leading expert on the history and the theory of convex functions, wrote that this inequality was proven by J. Hadamard in 1893 [5]. In 1974, D. S. Mitrinović found Hermite's note in *Mathesis* [43]. Since (1.1) was known as Hadamard's inequality, the inequality is now commonly referred as the Hermite-Hadamard inequality.

For related results, see [10]–[19], [21]–[24], [31]–[34] and [46].

In this paper we show that the Hermite-Hadamard inequality can be extended to a larger class of functions containing the convex functions. The corresponding version for h-convex functions is also investigated. Some examples of interest are provided as well.

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2. Symmetrized convexity

For a function $f:[a,b]\to\mathbb{C}$ we consider the *symmetrical transform of* f on the interval $[a,b]$, denoted by $\breve{f}_{[a,b]}$ or simply $\breve{f}$, when the interval $[a,b]$ is implicit, which is defined by

$$\breve{f}(t):=\frac{1}{2}\left[f(t)+f(a+b-t)\right],\quad t\in[a,b].$$

The *anti-symmetrical transform of* f on the interval $[a,b]$ is denoted by $\tilde{f}_{[a,b]}$, or simply $\tilde{f}$ and is defined by

$$\tilde{f}(t):=\frac{1}{2}\left[f(t)-f(a+b-t)\right],t\in[a,b].$$

It is obvious that for any function f we have $\breve{f}+\tilde{f}=f$.

If f is convex on $[a,b]$, then for any $t_1,\ t_2\in[a,b]$ and $\alpha,\beta\geqslant 0$ with $\alpha+\beta=1$ we have

$$\begin{aligned}\breve{f}(\alpha t_1+\beta t_2)&=\frac{1}{2}\left[f(\alpha t_1+\beta t_2)+f(a+b-\alpha t_1-\beta t_2)\right]\\&=\frac{1}{2}\left[f(\alpha t_1+\beta t_2)+f(\alpha(a+b-t_1)+\beta(a+b-t_2))\right]\\&\leqslant\frac{1}{2}\left[\alpha f(t_1)+\beta f(t_2)+\alpha f(a+b-t_1)+\beta f(a+b-t_2)\right]\\&=\frac{1}{2}\alpha\left[f(t_1)+f(a+b-t_1)\right]+\frac{1}{2}\beta\left[f(t_2)+f(a+b-t_2)\right]\\&=\alpha\breve{f}(t_1)+\beta\breve{f}(t_2),\end{aligned}$$

which shows that $\breve{f}$ is convex on $[a,b]$.

Consider the real numbers $a<b$ and define the function $f_0:[a,b]\to\mathbb{R},\ f_0(t)=t^3$. We have

$$\breve{f}_0(t):=\frac{1}{2}\left[t^3+(a+b-t)^3\right]=\frac{3}{2}(a+b)t^2-\frac{3}{2}(a+b)^2t+\frac{1}{2}(a+b)^3$$

for any $t\in\mathbb{R}$.

Since the second derivative $\left(\breve{f}_0\right)''(t)=3(a+b),\ t\in\mathbb{R}$, then $\breve{f}_0$ is strictly convex on $[a,b]$ if $\frac{a+b}{2}>0$ and strictly concave on $[a,b]$ if $\frac{a+b}{2}<0$. Therefore if $a<0<b$ with $\frac{a+b}{2}>0$, then we can conclude that f_0 is not convex on $[a,b]$ while $\breve{f}_0$ is convex on $[a,b]$.

We can introduce the following concept of convexity.

DEFINITION 1. We say that the function $f:[a,b]\to\mathbb{R}$ is symmetrized convex (concave) on the interval $[a,b]$ if the *symmetrical transform* $\breve{f}$ is convex (concave) on $[a,b]$.

Now, if we denote by $Con[a,b]$ the closed convex cone of convex functions defined on $[a,b]$ and by $SCon[a,b]$ the class of symmetrized convex functions, then from the above remarks we can conclude that

$$Con[a,b] \subsetneq SCon[a,b]. \tag{2.1}$$

Also, if $[c,d] \subset [a,b]$ and $f \in SCon[a,b]$, then this does not imply in general that $f \in SCon[c,d]$.

THEOREM 1. *Assume that $f:[a,b] \to \mathbb{R}$ is symmetrized convex on the interval $[a,b]$. Then we have the Hermite-Hadamard inequalities*

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(t)dt \leqslant \frac{f(a)+f(b)}{2}. \tag{2.2}$$

Proof. Since $f:[a,b] \to \mathbb{R}$ is symmetrized convex on the interval $[a,b]$, then by writing the Hermite-Hadamard inequality for the function $\breve{f}$ we have

$$\breve{f}\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b \breve{f}(t)dt \leqslant \frac{\breve{f}(a)+\breve{f}(b)}{2}. \tag{2.3}$$

However

$$\breve{f}\left(\frac{a+b}{2}\right) = f\left(\frac{a+b}{2}\right),\ \frac{\breve{f}(a)+\breve{f}(b)}{2} = \frac{f(a)+f(b)}{2},$$

and

$$\int_a^b \breve{f}(t)dt = \frac{1}{2}\int_a^b [f(t)+f(a+b-t)]\,dt = \int_a^b f(t)\,dt.$$

Then by (2.3) we get (2.2). □

For similar results see [36].

The following result holds:

THEOREM 2. *Assume that $f:[a,b] \to \mathbb{R}$ is symmetrized convex on the interval $[a,b]$. Then for any $x \in [a,b]$ we have the bounds*

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{2}[f(x)+f(a+b-x)] \leqslant \frac{f(a)+f(b)}{2}. \tag{2.4}$$

Proof. Since $\breve{f}$ is convex on $[a,b]$ then for any $x \in [a,b]$ we have

$$\frac{\breve{f}(x)+\breve{f}(a+b-x)}{2} \geqslant \breve{f}\left(\frac{a+b}{2}\right)$$

and since

$$\frac{\breve{f}(x)+\breve{f}(a+b-x)}{2} = \frac{1}{2}[f(x)+f(a+b-x)]$$

while

$$\breve{f}\left(\frac{a+b}{2}\right)=f\left(\frac{a+b}{2}\right),$$

we get the first inequality in (2.4).

Also, by the convexity of $\breve{f}$ we have for any $x\in[a,b]$ that

$$\begin{aligned}\breve{f}(x)&\leqslant\frac{x-a}{b-a}\cdot\breve{f}(b)+\frac{b-x}{b-a}\cdot\breve{f}(a)\\&=\frac{x-a}{b-a}\cdot\frac{f(a)+f(b)}{2}+\frac{b-x}{b-a}\cdot\frac{f(a)+f(b)}{2}\\&=\frac{f(a)+f(b)}{2},\end{aligned}$$

which proves the second part of (2.4). □

REMARK 1. If $f:[a,b]\to\mathbb{R}$ is symmetrized convex on the interval $[a,b]$, then we have the bounds

$$\inf_{x\in[a,b]}\breve{f}(x)=\breve{f}\left(\frac{a+b}{2}\right)=f\left(\frac{a+b}{2}\right)$$

and

$$\sup_{x\in[a,b]}\breve{f}(x)=\breve{f}(a)=\breve{f}(b)=\frac{f(a)+f(b)}{2}.$$

COROLLARY 1. *If $f:[a,b]\to\mathbb{R}$ is symmetrized convex on the interval $[a,b]$ and $w:[a,b]\to[0,\infty)$ is integrable on $[a,b]$, then*

$$\begin{aligned}f\left(\frac{a+b}{2}\right)\int_a^b w(t)dt&\leqslant\frac{1}{2}\int_a^b w(t)\left[f(t)+f(a+b-t)\right]dt\\&\leqslant\frac{f(a)+f(b)}{2}\int_a^b w(t)dt.\end{aligned}\tag{2.5}$$

Moreover, if w is symmetric almost everywhere on $[a,b]$, i.e. $w(t)=w(a+b-t)$ for almost every $t\in[a,b]$, then

$$f\left(\frac{a+b}{2}\right)\int_a^b w(t)dt\leqslant\int_a^b w(t)f(t)dt\leqslant\frac{f(a)+f(b)}{2}\int_a^b w(t)dt.\tag{2.6}$$

Proof. The inequality (2.5) follows by (2.4) written for $x=t$, multiplying by $w(t)\geqslant 0$ and integrating over t on $[a,b]$.

By changing the variable, we have

$$\int_a^b w(t)f(a+b-t)dt=\int_a^b w(a+b-t)f(t)dt.$$

Since w is symmetric almost everywhere on $[a,b]$, then

$$\int_a^b w(a+b-t)f(t)dt = \int_a^b w(t)f(t)dt.$$

Therefore

$$\begin{aligned}&\frac{1}{2}\int_a^b w(t)\left[f(t)+f(a+b-t)\right]dt \\ &= \frac{1}{2}\left[\int_a^b w(t)f(t)dt + \int_a^b w(t)f(a+b-t)dt\right] \\ &= \frac{1}{2}\left[\int_a^b w(t)f(t)dt + \int_a^b w(t)f(t)dt\right] = \int_a^b w(t)f(t)dt\end{aligned}$$

and by (2.5) we get (2.6). □

REMARK 2. The inequality (2.6) was obtained by L. Fejér in 1906 for convex functions f and symmetric weights w. It has been shown now that this inequality remains valid for the larger class of symmetrized convex functions f on the interval $[a,b]$.

The following result also holds.

THEOREM 3. *Assume that $f:[a,b]\to\mathbb{R}$ is symmetrized convex on the interval $[a,b]$. Then for any $x,y\in[a,b]$ with $x\neq y$ we have the Hermite-Hadamard inequalities*

$$\begin{aligned}&\frac{1}{2}\left[f\left(\frac{x+y}{2}\right)+f\left(a+b-\frac{x+y}{2}\right)\right] \\ &\leqslant \frac{1}{2(y-x)}\left[\int_x^y f(t)dt + \int_{a+b-y}^{a+b-x} f(t)dt\right] \\ &\leqslant \frac{1}{4}\left[f(x)+f(a+b-x)+f(y)+f(a+b-y)\right].\end{aligned} \tag{2.7}$$

Proof. Since $\breve{f}_{[a,b]}$ is convex on $[a,b]$, then $\breve{f}_{[a,b]}$ is also convex on any subinterval $[x,y]$ (or $[y,x]$) where $x,y\in[a,b]$.

By Hermite-Hadamard inequalities for convex functions we have

$$\breve{f}_{[a,b]}\left(\frac{x+y}{2}\right) \leqslant \frac{1}{y-x}\int_x^y \breve{f}_{[a,b]}(t)dt \leqslant \frac{\breve{f}_{[a,b]}(x)+\breve{f}_{[a,b]}(y)}{2} \tag{2.8}$$

for any $x,y\in[a,b]$ with $x\neq y$.

We have

$$\breve{f}_{[a,b]}\left(\frac{x+y}{2}\right) = \frac{1}{2}\left[f\left(\frac{x+y}{2}\right)+f\left(a+b-\frac{x+y}{2}\right)\right],$$

$$\begin{aligned}\int_x^y \breve{f}_{[a,b]}(t)dt &= \frac{1}{2}\int_x^y [f(t)+f(a+b-t)]\,dt\\ &= \frac{1}{2}\int_x^y f(t)dt + \frac{1}{2}\int_x^y f(a+b-t)dt\\ &= \frac{1}{2}\int_x^y f(t)dt + \frac{1}{2}\int_{a+b-y}^{a+b-x} f(t)dt\end{aligned}$$

and

$$\frac{\breve{f}_{[a,b]}(x)+\breve{f}_{[a,b]}(y)}{2} = \frac{1}{4}[f(x)+f(a+b-x)+f(y)+f(a+b-y)].$$

Then by (2.8) we deduce the desired result (2.7). □

REMARK 3. If we take $x=a$ and $y=b$ in (2.7), then we get (2.2).

If, for a given $x\in[a,b]$, we take $y=a+b-x$, then from (2.7) we get

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{2\left(\frac{a+b}{2}-x\right)}\int_x^{a+b-x} f(t)dt \leqslant \frac{1}{2}[f(x)+f(a+b-x)], \tag{2.9}$$

where $x\neq\frac{a+b}{2}$, provided that $f:[a,b]\to\mathbb{R}$ is symmetrized convex on the interval $[a,b]$.

Integrating this inequality over x we get the following refinement of the first part of (2.2)

$$\begin{aligned}f\left(\frac{a+b}{2}\right) &\leqslant \frac{1}{2(b-a)}\int_a^b \left[\frac{1}{\left(\frac{a+b}{2}-x\right)}\int_x^{a+b-x} f(t)\,dt\right]dx \\ &\leqslant \frac{1}{b-a}\int_a^b f(t)dt,\end{aligned} \tag{2.10}$$

provided that $f:[a,b]\to\mathbb{R}$ is symmetrized convex on the interval $[a,b]$.

When the function is convex, we have the following inequalities as well:

REMARK 4. If $f:[a,b]\to\mathbb{R}$ is convex, then from (2.7) we have the inequalities

$$\begin{aligned}f\left(\frac{a+b}{2}\right) &\leqslant \frac{1}{2}\left[f\left(\frac{x+y}{2}\right)+f\left(a+b-\frac{x+y}{2}\right)\right]\\ &\leqslant \frac{1}{2(y-x)}\left[\int_x^y f(t)dt + \int_{a+b-y}^{a+b-x} f(t)dt\right]\\ &\leqslant \frac{1}{4}[f(x)+f(a+b-x)+f(y)+f(a+b-y)]\end{aligned} \tag{2.11}$$

for any $x,y\in[a,b]$, $x\neq y$.

If we integrate (2.11) over (x,y) on the square $[a,b]^2$ and divide by $(b-a)^2$, then we get the following refinement of the first Hermite-Hadamard inequality for convex functions

$$
\begin{aligned}
&f\left(\frac{a+b}{2}\right) &(2.12)\\
&\leqslant \frac{1}{2(b-a)^2}\left[\int_a^b\int_a^b f\left(\frac{x+y}{2}\right)dxdy+\int_a^b\int_a^b f\left(a+b-\frac{x+y}{2}\right)dxdy\right]\\
&\leqslant \frac{1}{2(b-a)^2}\int_a^b\int_a^b \frac{1}{y-x}\left[\int_x^y f(t)dt+\int_{a+b-y}^{a+b-x} f(t)dt\right]dxdy\\
&\leqslant \frac{1}{b-a}\int_a^b f(t)dt.
\end{aligned}
$$

We notice that, the second and the third inequalities also hold for the more general case of symmetrized convex functions on the interval $[a,b]$.

A concept of weaker symmetrized convexity can be introduced as follows:

DEFINITION 2. We say that the function $f:[a,b]\to\mathbb{R}$ is weak symmetrized convex (concave) on the interval $[a,b]$ if the *symmetrical transform* $\breve{f}$ is convex (concave) on the interval $\left[a,\frac{a+b}{2}\right]$.

We denote this class by $WSCon[a,b]$.

It is clear that any symmetrized convex function on $[a,b]$ is weak symmetrized convex on that interval. Also, there are weak symmetrized convex functions on $[a,b]$ that are not symmetrized convex on $[a,b]$.

If we consider the function $f_0:[a,b]\to\mathbb{R}$ defined by

$$
f_0(t)=\begin{cases} t^2,\ t\in\left[a,\frac{a+b}{2}\right],\\ \\ (a+b-t)^2,\ t\in\left(\frac{a+b}{2},b\right],\end{cases}
$$

then we observe that f_0 is convex on $\left[a,\frac{a+b}{2}\right]$ and $\left[\frac{a+b}{2},b\right]$ but not convex on the whole interval $[a,b]$. We also observe that f_0 is a symmetrical function on $[a,b]$ and then $\breve{f}_0=f_0$. Therefore f_0 is weak symmetrized convex function on $[a,b]$ but not symmetrized convex on that interval.

We have the following strict inclusion

$$
SCon[a,b]\subsetneq WSCon[a,b]. \tag{2.13}
$$

We also notice that f is weak symmetrized convex function on $[a,b]$ if and only if $\breve{f}$ is convex on the second half of the interval $[a,b]$, namely $\left[\frac{a+b}{2},b\right]$.

THEOREM 4. *Assume that $f:[a,b]\to\mathbb{R}$ is weak symmetrized convex on the interval $[a,b]$. Then for any $x,y\in\left[a,\frac{a+b}{2}\right]$ $x\neq y$ we have the Hermite-Hadamard inequalities (2.7).*

In particular, we have

$$\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]\leqslant\frac{1}{b-a}\int_a^b f(t)dt \qquad (2.14)$$
$$\leqslant\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right].$$

Proof. The first part follows from the proof of Theorem 3 for $x,y\in\left[a,\frac{a+b}{2}\right]$. The second part follows from the inequality (2.7) by taking $x=a$ and $y=\frac{a+b}{2}$. □

REMARK 5. We observe that if $f:[a,b]\to\mathbb{R}$ is weak symmetrized convex on the interval $[a,b]$, then the inequality (2.9) holds for any $x\in\left[a,\frac{a+b}{2}\right)$ and integrating on $\left[a,\frac{a+b}{2}\right]$ we also have

$$f\left(\frac{a+b}{2}\right)\leqslant\frac{1}{2(b-a)}\int_a^{\frac{a+b}{2}}\left[\frac{1}{\left(\frac{a+b}{2}-x\right)}\int_x^{a+b-x}f(t)dt\right]dx \qquad (2.15)$$
$$\leqslant\frac{1}{b-a}\int_a^b f(t)dt.$$

We can state in general the following result for symmetrized convex functions.

PROPOSITION 1. *Any inequality that holds for convex functions f defined on the interval $[a,b]$ will hold for symmetrized convex functions by replacing f with $\breve{f}_{[a,b]}$ and performing the required calculations.*

We can illustrate this fact with two simple examples.

It is known that, see [19], if $f:[a,b]\to\mathbb{R}$ is differentiable convex on (a,b), then for any $x,y\in(a,b)$ with $x\neq y$ we have

$$0\leqslant\frac{1}{y-x}\int_x^y f(t)-f\left(\frac{x+y}{2}\right)\leqslant\frac{1}{8}\left(f'(y)-f'(x)\right)(y-x). \qquad (2.16)$$

Now, if $f:[a,b]\to\mathbb{R}$ is differentiable and symmetrized convex on (a,b), then by writing (2.16) for $\breve{f}_{[a,b]}$ we have

$$0\leqslant\frac{1}{y-x}\int_x^y \breve{f}_{[a,b]}(t)-\breve{f}_{[a,b]}\left(\frac{x+y}{2}\right) \qquad (2.17)$$
$$\leqslant\frac{1}{8}\left(\left(\breve{f}_{[a,b]}\right)'(y)-\left(\breve{f}_{[a,b]}\right)'(x)\right)(y-x).$$

However

$$\frac{1}{y-x}\int_x^y \breve{f}_{[a,b]}(t)=\frac{1}{2(y-x)}\left[\int_x^y f(t)dt+\int_{a+b-y}^{a+b-x}f(t)dt\right],$$

$$\breve{f}_{[a,b]}\left(\frac{x+y}{2}\right)=\frac{1}{2}\left[f\left(\frac{x+y}{2}\right)+f\left(a+b-\frac{x+y}{2}\right)\right]$$

and

$$\left(\breve{f}_{[a,b]}\right)'(y)-\left(\breve{f}_{[a,b]}\right)'(x)=\frac{1}{2}\left(f'(y)-f'(a+b-y)-f'(x)+f'(a+b-x)\right).$$

Then by (2.17) we get

$$\begin{aligned}0&\leqslant\frac{1}{2(y-x)}\left[\int_x^y f(t)dt+\int_{a+b-y}^{a+b-x}f(t)dt\right]\\&\quad-\frac{1}{2}\left[f\left(\frac{x+y}{2}\right)+f\left(a+b-\frac{x+y}{2}\right)\right]\\&\leqslant\frac{1}{16}\left[f'(y)-f'(a+b-y)-f'(x)+f'(a+b-x)\right](y-x)\end{aligned}\tag{2.18}$$

that holds for any $x,y\in(a,b)$ with $x\neq y$.

From this inequality, by taking $y=a+b-x$, we get

$$\begin{aligned}0&\leqslant\frac{1}{2\left(\frac{a+b}{2}-x\right)}\int_x^{a+b-x}f(t)\,dt-f\left(\frac{a+b}{2}\right)\\&\leqslant\frac{1}{4}\left[f'(a+b-x)-f'(x)\right]\left(\frac{a+b}{2}-x\right)\end{aligned}\tag{2.19}$$

for any $x\in(a,b)$ with $x\neq\frac{a+b}{2}$.

If $f:[a,b]\to\mathbb{R}$ is differentiable convex on (a,b), then for any $x,y\in(a,b)$ with $x\neq y$ we also have [20]

$$0\leqslant\frac{f(x)+f(y)}{2}-\frac{1}{y-x}\int_x^y f(t)\leqslant\frac{1}{8}\left(f'(y)-f'(x)\right)(y-x).\tag{2.20}$$

Now, if $f:[a,b]\to\mathbb{R}$ is differentiable and symmetrized convex on (a,b), then by a similar argument as above we have

$$\begin{aligned}0&\leqslant\frac{1}{4}\left[f(x)+f(a+b-x)+f(y)+f(a+b-y)\right]\\&\quad-\frac{1}{2(y-x)}\left[\int_x^y f(t)dt+\int_{a+b-y}^{a+b-x}f(t)dt\right]\\&\leqslant\frac{1}{16}\left[f'(y)-f'(a+b-y)-f'(x)+f'(a+b-x)\right](y-x)\end{aligned}\tag{2.21}$$

for any $x,y\in(a,b)$ with $x\neq y$.

In particular, we have

$$\begin{aligned}0&\leqslant\frac{1}{2}\left[f(x)+f(a+b-x)\right]-\frac{1}{2\left(\frac{a+b}{2}-x\right)}\int_x^{a+b-x}f(t)dt\\&\leqslant\frac{1}{4}\left[f'(a+b-x)-f'(x)\right]\left(\frac{a+b}{2}-x\right)\end{aligned}\tag{2.22}$$

for any $x\in(a,b)$ with $x\neq\frac{a+b}{2}$.

3. Symmetrized h-convexity

We recall here some concepts of convexity that are well known in the literature. Let I be an interval in $\mathbb{R}$.

DEFINITION 3. ([38]) We say that $f : I \to \mathbb{R}$ is a Godunova-Levin function or that f belongs to the class $Q(I)$ if f is non-negative and for all $x,y \in I$ and $t \in (0,1)$ we have

$$f(tx+(1-t)y) \leqslant \frac{1}{t}f(x)+\frac{1}{1-t}f(y). \tag{3.1}$$

Some further properties of this class of functions can be found in [27], [28], [30], [44], [47] and [48]. Among others, its has been noted that non-negative monotone and non-negative convex functions belong to this class of functions.

DEFINITION 4. ([30]) We say that a function $f : I \to \mathbb{R}$ belongs to the class $P(I)$ if it is nonnegative and for all $x,y \in I$ and $t \in [0,1]$ we have

$$f(tx+(1-t)y) \leqslant f(x)+f(y). \tag{3.2}$$

Obviously $Q(I)$ contains $P(I)$ and for applications it is important to note that also $P(I)$ contain all nonnegative monotone, convex and *quasi convex functions,* i. e. nonnegative functions satisfying

$$f(tx+(1-t)y) \leqslant \max\{f(x),f(y)\} \tag{3.3}$$

for all $x,y \in I$ and $t \in [0,1]$.

For some results on P-functions see [30] and [45] while for quasi convex functions, the reader can consult [29].

DEFINITION 5. ([7]) Let s be a real number, $s \in (0,1]$. A function $f : [0,\infty) \to [0,\infty)$ is said to be s-convex (in the second sense) or Breckner s-convex if

$$f(tx+(1-t)y) \leqslant t^s f(x)+(1-t)^s f(y)$$

for all $x,y \in [0,\infty)$ and $t \in [0,1]$.

For some properties of this class of functions see [1], [2], [7], [8], [25], [26], [39], [41] and [50].

In order to unify the above concepts for functions of real variable, S. Varošanec introduced the concept of h-convex functions as follows.

Assume that I and J are intervals in $\mathbb{R}$, $(0,1) \subseteq J$ and functions h and f are real non-negative functions defined in J and I, respectively.

DEFINITION 6. ([53]) Let $h : J \to [0,\infty)$ with h not identical to 0. We say that $f : I \to [0,\infty)$ is an h-convex function if for all $x,y \in I$ we have

$$f(tx+(1-t)y) \leqslant h(t)f(x)+h(1-t)f(y) \tag{3.4}$$

for all $t \in (0,1)$.

For some results concerning this class of functions see [53], [6], [42], [51], [49] and [52].

We can introduce now another class of functions.

DEFINITION 7. We say that the function $f: I \to [0,\infty)$ is of s-Godunova-Levin type, with $s \in [0,1]$, if

$$f(tx+(1-t)y) \leqslant \frac{1}{t^s} f(x) + \frac{1}{(1-t)^s} f(y), \tag{3.5}$$

for all $t \in (0,1)$ and $x,y \in I$.

We observe that for $s=0$ we obtain the class of P-functions while for $s=1$ we obtain the class of Godunova-Levin. If we denote by $Q_s(I)$ the class of s-Godunova-Levin functions defined on I, then we obviously have

$$P(I) = Q_0(I) \subseteq Q_{s_1}(I) \subseteq Q_{s_2}(I) \subseteq Q_1(I) = Q(I)$$

for $0 \leqslant s_1 \leqslant s_2 \leqslant 1$.

The following inequality of Hermite-Hadamard type holds [49].

THEOREM 5. *Assume that the function $f: I \to [0,\infty)$ is an h-convex function with $h \in L[0,1]$. Let $y,x \in I$ with $y \neq x$ and assume that the mapping $[0,1] \ni t \mapsto f[(1-t)x+ty]$ is Lebesgue integrable on $[0,1]$. Then*

$$\frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{x+y}{2}\right) \leqslant \frac{1}{y-x}\int_x^y f(u)\,du \leqslant [f(x)+f(y)]\int_0^1 h(t)dt. \tag{3.6}$$

If we write (3.6) for $h(t)=t$, then we get the classical Hermite-Hadamard inequality for convex functions

$$f\left(\frac{x+y}{2}\right) \leqslant \frac{1}{y-x}\int_x^y f(u)\,du \leqslant \frac{f(x)+f(y)}{2}. \tag{3.7}$$

If we write (3.6) for the case of P-type functions $f: I \to [0,\infty)$, i.e., $h(t)=1, t \in [0,1]$, then we get the inequality

$$\frac{1}{2} f\left(\frac{x+y}{2}\right) \leqslant \frac{1}{y-x}\int_x^y f(u)\,du \leqslant f(x)+f(y), \tag{3.8}$$

that has been obtained for functions of real variable in [30].

If f is Breckner s-convex on I, for $s \in (0,1)$, then by taking $h(t)=t^s$ in (3.6) we get

$$2^{s-1} f\left(\frac{x+y}{2}\right) \leqslant \frac{1}{y-x}\int_x^y f(u)\,du \leqslant \frac{f(x)+f(y)}{s+1}, \tag{3.9}$$

that was obtained for functions of a real variable in [25].

If $f : I \to [0,\infty)$ is of s-Godunova-Levin type, with $s \in [0,1)$, then

$$\frac{1}{2^{s+1}} f\left(\frac{x+y}{2}\right) \leqslant \frac{1}{y-x}\int_x^y f(u)\,du \leqslant \frac{f(x)+f(y)}{1-s}. \tag{3.10}$$

We notice that for $s = 1$ the first inequality in (3.10) still holds [30], i.e.

$$\frac{1}{4} f\left(\frac{x+y}{2}\right) \leqslant \frac{1}{y-x}\int_x^y f(u)\,du. \tag{3.11}$$

We can introduce the following concept generalizing the notion of h-convexity.

DEFINITION 8. Assume that h is as in Definition 6. We say that the function $f : [a,b] \to [0,\infty)$ is h-symmetrized convex (concave) on the interval $[a,b]$ if the *symmetrical transform* $\breve{f}$ is h-convex (concave) on $[a,b]$.

Now, if we denote by $Con_h[a,b]$ the closed convex cone of h-convex functions defined on $[a,b]$ and by $SCon_h[a,b]$ the class of h-symmetrized convex, then, as in the previous section, we can conclude in general that

$$Con_h[a,b] \subsetneq SCon_h[a,b]. \tag{3.12}$$

DEFINITION 9. Assume that h is as in Definition 6. We say that the function $f : [a,b] \to \mathbb{R}$ is h-weak symmetrized convex (concave) on the interval $[a,b]$ if the *symmetrical transform* $\breve{f}$ is h-convex (concave) on the interval $\left[a, \frac{a+b}{2}\right]$.

We denote this class by $WSCon_h[a,b]$. As in the previous section, we can conclude in general that

$$SCon_h[a,b] \subsetneq WSCon_h[a,b]. \tag{3.13}$$

Utilising Theorem 5 and a similar proof to that of Theorem 3, we can state the following result as well:

THEOREM 6. *Assume that the function $f : [a,b] \to [0,\infty)$ is h-symmetrized convex on the interval $[a,b]$ with h integrable on $[0,1]$ and f integrable on $[a,b]$. Then for any $x,y \in [a,b]$ we have the Hermite-Hadamard inequalities*

$$\begin{aligned} &\frac{1}{4h\left(\frac{1}{2}\right)}\left[f\left(\frac{x+y}{2}\right) + f\left(a+b-\frac{x+y}{2}\right)\right] \\ &\leqslant \frac{1}{2(y-x)}\left[\int_x^y f(t)dt + \int_{a+b-y}^{a+b-x} f(t)dt\right] \\ &\leqslant \frac{1}{2}\left[f(x)+f(a+b-x)+f(y)+f(a+b-y)\right]\int_0^1 h(t)dt. \end{aligned} \tag{3.14}$$

In particular, we have

$$\frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(t)dt \leqslant [f(a)+f(b)]\int_0^1 h(t)dt. \tag{3.15}$$

REMARK 6. If, for a given $x \in [a,b]$, we take $y = a+b-x$, then from (3.14) we get

$$\frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{2\left(\frac{a+b}{2}-x\right)} \int_x^{a+b-x} f(t)dt \qquad (3.16)$$
$$\leqslant [f(x) + f(a+b-x)] \int_0^1 h(t)dt,$$

where $x \neq \frac{a+b}{2}$, provided that $f : [a,b] \to \mathbb{R}$ is h-symmetrized convex and integrable on the interval $[a,b]$.

Integrating on $[a,b]$ over x we get

$$\frac{1}{4h\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}\right) \qquad (3.17)$$
$$\leqslant \frac{1}{4(b-a)} \int_a^b \left[\frac{1}{\left(\frac{a+b}{2}-x\right)} \int_x^{a+b-x} f(t)dt\right] dx$$
$$\leqslant \frac{1}{b-a} \int_a^b f(x)\,dx \int_0^1 h(t)\,dt.$$

We have the following result as well:

THEOREM 7. *Assume that h is as in Definition 6. If the function $f : [a,b] \to [0,\infty)$ is h-symmetrized convex on the interval $[a,b]$, then we have the bounds*

$$\frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}\right) \leqslant \frac{f(x) + f(a+b-x)}{2} \qquad (3.18)$$
$$\leqslant \left[h\left(\frac{b-x}{b-a}\right) + h\left(\frac{x-a}{b-a}\right)\right] \frac{f(a) + f(b)}{2}$$

for any $x \in [a,b]$.

Proof. Since $\breve{f}$ is h-convex on $[a,b]$ then for any $x \in [a,b]$ we have

$$h\left(\frac{1}{2}\right)\left[\breve{f}(x) + \breve{f}(a+b-x)\right] \geqslant \breve{f}\left(\frac{a+b}{2}\right)$$

and since

$$\frac{\breve{f}(x) + \breve{f}(a+b-x)}{2} = \frac{1}{2}[f(x) + f(a+b-x)]$$

while

$$\breve{f}\left(\frac{a+b}{2}\right) = f\left(\frac{a+b}{2}\right),$$

we get the first inequality in (2.4).

Also, by the convexity of $\breve{f}$ we have for any $x \in [a,b]$ that

$$\begin{aligned}\breve{f}(x) &\leqslant h\left(\frac{x-a}{b-a}\right) \cdot \breve{f}(b) + h\left(\frac{b-x}{b-a}\right) \cdot \breve{f}(a) \\ &= h\left(\frac{x-a}{b-a}\right) \cdot \frac{f(a)+f(b)}{2} + h\left(\frac{b-x}{b-a}\right) \cdot \frac{f(a)+f(b)}{2} \\ &= \left[h\left(\frac{b-x}{b-a}\right) + h\left(\frac{x-a}{b-a}\right)\right] \frac{f(a)+f(b)}{2},\end{aligned}$$

which proves the second part of (3.18). □

COROLLARY 2. *Assume that the function $f : [a,b] \to [0,\infty)$ is h-symmetrized convex on the interval $[a,b]$ with h integrable on $[0,1]$ and f integrable on $[a,b]$. If $w : [a,b] \to [0,\infty)$ is integrable on $[a,b]$, then*

$$\begin{aligned}&\frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}\right) \int_a^b w(t)dt \\ &\leqslant \frac{1}{2}\int_a^b w(t)\left[f(t) + f(a+b-t)\right]dt \\ &\leqslant \frac{f(a)+f(b)}{2} \int_a^b h\left(\frac{t-a}{b-a}\right)\left[w(t) + w(a+b-t)\right]dt.\end{aligned} \tag{3.19}$$

Moreover, if w is symmetric almost everywhere on $[a,b]$, then

$$\begin{aligned}\frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}\right) \int_a^b w(t)dt &\leqslant \int_a^b w(t) f(t)\, dt \\ &\leqslant \left[f(a) + f(b)\right] \int_a^b h\left(\frac{t-a}{b-a}\right) w(t)dt.\end{aligned} \tag{3.20}$$

Proof. From (3.18) we have

$$\begin{aligned}\frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}\right) &\leqslant \frac{f(t) + f(a+b-t)}{2} \\ &\leqslant \left[h\left(\frac{b-t}{b-a}\right) + h\left(\frac{t-a}{b-a}\right)\right] \frac{f(a)+f(b)}{2}\end{aligned}$$

for any $t \in [a,b]$.

Multiplying with $w(t) \geqslant 0$ and integrating over $t \in [a,b]$ we get

$$\begin{aligned}&\frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}\right) \int_a^b w(t)dt \\ &\leqslant \frac{1}{2}\int_a^b w(t)\left[f(t) + f(a+b-t)\right]dt \\ &\leqslant \frac{f(a)+f(b)}{2} \int_a^b \left[h\left(\frac{b-t}{b-a}\right) + h\left(\frac{t-a}{b-a}\right)\right] w(t)dt.\end{aligned} \tag{3.21}$$

Observe that, by changing the variable $t = a+b-s,\ s \in [a,b]$, we have

$$\int_a^b h\left(\frac{b-t}{b-a}\right) w(t)dt = \int_a^b h\left(\frac{s-a}{b-a}\right) w(a+b-s)ds,$$

then we get

$$\begin{aligned}&\int_a^b \left[h\left(\frac{b-t}{b-a}\right) + h\left(\frac{t-a}{b-a}\right)\right] w(t)dt \\ &= \int_a^b h\left(\frac{t-a}{b-a}\right) [w(t) + w(a+b-t)]\, dt\end{aligned}$$

and by (3.21) we obtain the second part of (3.19). □

Utilising the previous examples of h-convex functions the reader may state various inequalities of Hermite-Hadamard type.

For instance, if we assume that the functions $f : [a,b] \to [0,\infty)$ is integrable and of symmetrized Godunova-Levin type, then for the symmetric weight

$$w : [a,b] \to [0,\infty), \quad w(t) = (t-a)(b-t)$$

we have from (3.20) that

$$\begin{aligned}\frac{1}{4} f\left(\frac{a+b}{2}\right) \int_a^b (t-a)(b-t)\, dt &\leqslant \int_a^b (t-a)(b-t) f(t)\, dt \\ &\leqslant [f(a) + f(b)](b-a) \int_a^b (b-t)\, dt\end{aligned}$$

and since

$$\int_a^b (t-a)(b-t)\, dt = \frac{1}{6}(b-a)^3, \quad \int_a^b (b-t)\, dt = \frac{1}{2}(b-a)^2,$$

then we get the following inequality of interest:

$$\frac{1}{24} f\left(\frac{a+b}{2}\right)(b-a)^3 \leqslant \int_a^b (t-a)(b-t) f(t) dt \leqslant \frac{f(a)+f(b)}{2}(b-a)^3. \tag{3.22}$$

Moreover, if we assume that the function $f : [a,b] \to [0,\infty)$ is integrable and symmetrized Breckner s-convex with $s \in (0,1)$, then for the symmetric weight

$$w : [a,b] \to [0,\infty), \quad w(t) = (t-a)(b-t)$$

we have from (3.20) that

$$\begin{aligned}&\frac{1}{2^{1-s}} f\left(\frac{a+b}{2}\right) \int_a^b (t-a)(b-t)\, dt \\ &\leqslant \int_a^b (t-a)(b-t) f(t) dt \\ &\leqslant \frac{f(a)+f(b)}{(b-a)^s} \int_a^b (t-a)^{s+1}(b-t)\, dt\end{aligned}$$

and since

$$\int_a^b (t-a)^{s+1}(b-t)\,dt = \frac{(b-a)^{s+3}}{(s+2)(s+3)}$$

then we get the following inequality of interest:

$$\frac{1}{2^{2-s}3} f\left(\frac{a+b}{2}\right)(b-a)^3 \leqslant \int_a^b (t-a)(b-t)f(t)\,dt \leqslant \frac{f(a)+f(b)}{(s+2)(s+3)}(b-a)^3. \tag{3.23}$$

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S. S. Dragomir
Mathematics, College of Engineering & Science, Victoria University
PO Box 14428, Melbourne City, MC 8001, Australia
and
School of Computer Science & Applied Mathematics
University of the Witwatersrand
Private Bag 3, Johannesburg 2050, South Africa
e-mail: sever.dragomir@vu.edu.au
http://rgmia.org/dragomir