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Research Article

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Double integral inequalities of Hermite–Hadamard type for *h*-convex functions on linear spaces

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Abstract: Some double integral inequalities of Hermite–Hadamard type for h-convex functions defined on convex subsets in real or complex linear spaces are given. Applications for norm inequalities are provided as well.

Keywords: Convex functions, integral inequalities, h-convex functions

MSC 2010: 26D15, 25D10

1 Introduction

The inequality

$$(b-a)f\Big(\frac{a+b}{2}\Big) \le \int_a^b f(x)\,dx \le (b-a)\frac{f(a)+f(b)}{2}, \quad a,b \in \mathbb{R},\ a<b, \tag{1.1}$$

holds for any convex function f defined on $\mathbb{R}$. It was first discovered by Hermite and was published in 1881 in the journal *Mathesis* (see [40]). But this result was nowhere mentioned in the mathematical literature and was not widely known as Hermite's result. Beckenbach, a leading expert on the history and theory of convex functions, wrote that this inequality was proved by Hadamard [5] in 1893. In 1974, Mitrinović found Hermite's note [40] in *Mathesis*. Since (1.1) was known as Hadamard's inequality, the inequality is now commonly referred to as the Hermite–Hadamard inequality. For related results, see [3, 4, 9–23, 30–34, 37, 43].

Let X be a vector space over the real or complex number field $\mathbb{K}$ and let $x, y \in X$, $x \neq y$. Define the segment

$$[x,y] := \{(1-t)x + ty,\ t \in]0,1]\}.$$

We consider the function $f : [x,y] \to \mathbb{R}$ and the associated function $g(x,y) : [0,1] \to \mathbb{R}$ defined by

$$g(x,y)(t) := f((1-t)x+ty), \quad t \in [0,1].$$

Note that f is convex on $[x,y]$ if and only if $g(x,y)$ is convex on $[0,1]$.

For any convex function defined on a segment $[x,y] \subset X$, we have the *Hermite–Hadamard integral inequality* (see [18, p. 2] and [19, p. 2])

$$f\Big(\frac{x+y}{2}\Big) \le \int_0^1 f((1-t)x+ty)\,dt \le \frac{f(x)+f(y)}{2}, \tag{1.2}$$

which can be derived from the classical Hermite–Hadamard inequality (1.1) for the convex function $g(x,y)$.

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Since $f(x) = \|x\|^p$ for $x \in X$ and $1 \le p < \infty$ is a convex function, then, for any $x, y \in X$, from (1.2) we have the norm inequality (see [44, p. 106])

$$\left\|\frac{x+y}{2}\right\|^p \le \int_0^1 \|(1-t)x + ty\|^p \, dt \le \frac{\|x\|^p + \|y\|^p}{2}.$$

Motivated by the above results, in this paper we obtain double integral inequalities of Hermite–Hadamard type in which upper and lower bounds for the quantity

$$\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f\left(\frac{\alpha x + \beta y}{\alpha + \beta}\right) d\beta \, d\alpha$$

are provided for some classes of h-convex functions defined on linear spaces. Applications for norm inequalities and for Godunova–Levin-type functions are also given.

2 A double integral inequality for convex functions

For $a, b, c, d \ge 0$ with $b > a$ and $d > c$, we define the positive quantity

$$I(a, b; c, d) := \int_a^b \left(\int_c^d \left(\frac{\alpha}{\alpha + \beta}\right) d\beta \right) d\alpha \tag{2.1}$$

and we have the following representation.

Lemma 2.1. *Let $a, b, c, d \ge 0$ with $b > a$ and $d > c$. We have the equality*

$$I(a, b; c, d) = I_d(a, b) - I_c(a, b), \tag{2.2}$$

where $I_z(x, y)$ is defined for $x, y, z \ge 0$ with $y > x$ by

$$I_z(x, y) := \frac{1}{2}\left((y^2 - z^2)\ln(y + z) + (z^2 - x^2)\ln(x + z) + (y - x)\left(z - \frac{x + y}{2}\right)\right).$$

In particular, we have

$$I(a, b; a, b) = I_b(a, b) - I_a(a, b) = \frac{1}{2}(b - a)^2. \tag{2.3}$$

Proof. We have

$$\begin{aligned} I(a, b; c, d) &= \int_a^b \left(\int_c^d \left(\frac{\alpha}{\alpha + \beta}\right) d\beta \right) d\alpha \\ &= \int_a^b \alpha \left(\int_c^d \frac{d\beta}{\alpha + \beta} \right) d\alpha = \int_a^b \alpha(\ln(\alpha + d) - \ln(\alpha + d))\, d\alpha \\ &= \int_a^b \alpha \ln(\alpha + d)\, d\alpha - \int_a^b \alpha \ln(\alpha + d)\, d\alpha \\ &= \int_{a+d}^{b+d} (u - d)\ln u \, du - \int_{a+c}^{b+c} (u - c)\ln u \, du. \end{aligned} \tag{2.4}$$

Utilising the integration by parts formula, we have

$$\int_{a+d}^{b+d} (u-d)\ln u\,du = \frac{(u-d)^2}{2}\ln u\Big|_{a+d}^{b+d} - \frac{1}{2}\int_{a+d}^{b+d} \frac{(u-d)^2}{u}\,du$$
$$= \frac{b^2}{2}\ln(b+d) - \frac{a^2}{2}\ln(a+d) - \frac{1}{2}\int_{a+d}^{b+d}\frac{(u-d)^2}{u}\,du. \tag{2.5}$$

A straightforward calculation gives

$$\int_{a+d}^{b+d}\frac{(u-d)^2}{u}\,du = (b-a)\Big(\frac{a+b}{2}-d\Big) + d^2\ln(b+d) - d^2\ln(a+d). \tag{2.6}$$

From (2.5) and (2.6) we have

$$\int_{a+d}^{b+d}(u-d)\ln u\,du = \frac{b^2}{2}\ln(b+d) - \frac{a^2}{2}\ln(a+d) - \frac{1}{2}\Big((b-a)\Big(\frac{a+b}{2}-d\Big) + d^2\ln(b+d) - d^2\ln(a+d)\Big)$$
$$= I_d(a,b).$$

Similarly, we have

$$\int_{a+c}^{b+c}(u-c)\ln u\,du = I_c(a,b)$$

and by (2.4) we get the desired identity (2.2).

Finally, one easily verifies that

$$I_b(a,b) = \frac{1}{2}(b^2-a^2)\ln(a+b) + \frac{1}{4}(b-a)^2$$

and

$$I_a(a,b) = \frac{1}{2}(b^2-a^2)\ln(a+b) - \frac{1}{4}(b-a)^2,$$

which gives the desired equality (2.3). □

We have the following double integral inequality for convex functions.

Theorem 2.2. *Let $f : C \subseteq X \to [0,\infty)$ be a convex function on the convex set C in a linear space X. Then, for any $x, y \in C$ and for any $a, b, c, d \geq 0$ with $b > a$ and $d > c$, we have*

$$f\Big(\frac{I(a,b;c,d)}{(b-a)(d-c)}x + \frac{I(c,d;a,b)}{(b-a)(d-c)}y\Big) \leq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f\Big(\frac{\alpha x+\beta y}{\alpha+\beta}\Big)\,d\beta\,d\alpha$$
$$\leq \frac{I(a,b;c,d)}{(b-a)(d-c)}f(x) + \frac{I(c,d;a,b)}{(b-a)(d-c)}f(y), \tag{2.7}$$

where

$$I(a,b;c,d) := \int_a^b\Big(\int_c^d\Big(\frac{\alpha}{\alpha+\beta}\Big)\,d\beta\Big)\,d\alpha$$

and

$$I(c,d;a,b) := \int_a^b\Big(\int_c^d\Big(\frac{\beta}{\alpha+\beta}\Big)\,d\beta\Big)\,d\alpha.$$

Proof. Consider the function $g_{x,y} : [0,1] \to \mathbb{R}$ defined by $g_{x,y}(s) = f(sx + (1-s)y)$. This function is convex on $[0,1]$ and by Jensen's double integral inequality for real functions of a real variable we have

$$g_{x,y}\left(\frac{\int_a^b\int_c^d\left(\frac{\alpha}{\alpha+\beta}\right)d\beta\,d\alpha}{(b-a)(d-c)}\right) \le \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d g_{x,y}\left(\frac{\alpha}{\alpha+\beta}\right)d\beta\,d\alpha,$$

which is equivalent to

$$f\left(\frac{\int_a^b\int_c^d\left(\frac{\alpha}{\alpha+\beta}\right)d\beta\,d\alpha}{(b-a)(d-c)}x + \left(1 - \frac{\int_a^b\int_c^d\left(\frac{\alpha}{\alpha+\beta}\right)d\beta\,d\alpha}{(b-a)(d-c)}\right)y\right)$$
$$\le \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f\left(\frac{\alpha}{\alpha+\beta}x + \left(1-\frac{\alpha}{\alpha+\beta}\right)y\right)d\beta\,d\alpha.$$

By a simple calculation we obtain

$$f\left(\frac{\int_a^b\int_c^d\left(\frac{\alpha}{\alpha+\beta}\right)d\beta\,d\alpha}{(b-a)(d-c)}x + \frac{\int_a^b\int_c^d\left(\frac{\beta}{\alpha+\beta}\right)d\beta d\alpha}{(b-a)(d-c)}y\right) \le \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f\left(\frac{\alpha}{\alpha+\beta}x + \frac{\beta}{\alpha+\beta}y\right)d\beta\,d\alpha$$

and the first part of (2.7) is proved.

By the convexity of f we have

$$f\left(\frac{\alpha x + \beta y}{\alpha+\beta}\right) \le \frac{\alpha}{\alpha+\beta}f(x) + \frac{\beta}{\alpha+\beta}f(y)$$

for any $x, y \in C$ and for all $\alpha, \beta \ge 0$ with $\alpha + \beta > 0$. Integrating on the rectangle $[a,b] \times [c,d]$ gives

$$\int_a^b\int_c^d f\left(\frac{\alpha x + \beta y}{\alpha+\beta}\right)d\beta\,d\alpha \le f(x)\int_a^b\int_c^d \frac{\alpha}{\alpha+\beta}\,d\beta\,d\alpha + f(y)\int_a^b\int_c^d \frac{\beta}{\alpha+\beta}\,d\beta\,d\alpha,$$

which proves the second part of (2.7). □

Corollary 2.3. *Let $f : C \subseteq X \to [0,\infty)$ be a convex function on the convex set C in a linear space X. Then, for any $x, y \in C$ and for any $b > a \ge 0$, we have*

$$f\left(\frac{x+y}{2}\right) \le \frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{\alpha x + \beta y}{\alpha+\beta}\right)d\beta\,d\alpha \le \frac{f(x)+f(y)}{2}.$$

The proof follows from (2.7) by noticing that

$$I(a,b;a,b) = \frac{1}{2}(b-a)^2.$$

Remark 2.4. Let $(X, \|\cdot\|)$ be a real or complex linear space and let $p \ge 1$. Then, for any $x, y \in X$, we have

$$\left\|\frac{I(a,b;c,d)}{(b-a)(d-c)}x + \frac{I(c,d;a,b)}{(b-a)(d-c)}y\right\|^p \le \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d \left\|\frac{\alpha x + \beta y}{\alpha+\beta}\right\|^p d\beta\,d\alpha$$
$$\le \frac{I(a,b;c,d)}{(b-a)(d-c)}\|x\|^p + \frac{I(c,d;a,b)}{(b-a)(d-c)}\|y\|^p$$

for any $a, b, c, d \ge 0$ with $b > a$ and $d > c$. In particular, we have

$$\left\|\frac{x+y}{2}\right\|^p \le \frac{1}{(b-a)^2}\int_a^b\int_a^b \left\|\frac{\alpha x + \beta y}{\alpha+\beta}\right\|^p d\beta\,d\alpha \le \frac{\|x\|^p + \|y\|^p}{2}$$

for any $b > a \ge 0$.

3 Double integral inequalities for *h*-convex functions

Assume that I and J are intervals in $\mathbb{R}$ with $(0,1) \subseteq J$ and the functions f and h are real, non-negative and defined on I and J, respectively.

Definition 3.1 (see [50]). Let $h : J \to [0,\infty)$ with h not identical to 0. We say that $f : I \to [0,\infty)$ is an h-convex function if, for all $x, y \in I$, we have

$$f(tx + (1-t)y) \le h(t)f(x) + h(1-t)f(y) \tag{3.1}$$

for all $t \in (0,1)$.

This concept can be extended for functions defined on convex subsets of linear spaces, in the same way as above, by replacing the interval I by the corresponding convex subset C of the linear space X.

If we take $y = x$ in (3.1), we get $f(x) \le (h(t) + h(1-t))f(x)$, which implies that $1 \le h(t) + h(1-t)$ for all $t \in (0,1)$. By taking

$$t = \frac{1}{2}$$

we also get

$$h\Big(\frac{1}{2}\Big) \ge \frac{1}{2}.$$

For some results concerning this class of functions see [6, 39, 46, 47, 49, 50].

We recall below some concepts of convexity that are well known in the literature and can be seen as particular instances of h-convex functions. Here, I is an interval in $\mathbb{R}$.

Definition 3.2 (see [35]). We say that $f : I \to \mathbb{R}$ is a Godunova–Levin function or that f belongs to the class $Q(I)$ if f is non-negative and, for all $x, y \in I$ and $t \in (0,1)$, we have

$$f(tx + (1-t)y) \le \frac{1}{t}f(x) + \frac{1}{1-t}f(y). \tag{3.2}$$

Some further properties of this class of functions can be found in [26, 27, 29, 41, 44, 45]. Among others, it has been noted that non-negative monotone and non-negative convex functions belong to this class of functions.

The above concept can be extended for functions $f : C \subseteq X \to [0,\infty)$, where C is a convex subset of the real or complex linear space X, and the inequality (3.2) is satisfied for any vectors $x, y \in C$ and $t \in (0,1)$. If the function $f : C \subseteq X \to \mathbb{R}$ is non-negative and convex, then it is of Godunova–Levin type.

Definition 3.3 (see [29]). We say that a function $f : I \to \mathbb{R}$ belongs to the class $P(I)$ if it is non-negative and, for all $x, y \in I$ and $t \in [0,1]$, we have

$$f(tx + (1-t)y) \le f(x) + f(y). \tag{3.3}$$

Obviously, $Q(I)$ contains $P(I)$ and, for applications, it is important to note that $P(I)$ also contains all non-negative monotone, convex and *quasi-convex functions*, i.e., non-negative functions satisfying

$$f(tx + (1-t)y) \le \max\{f(x), f(y)\} \tag{3.4}$$

for all $x, y \in I$ and $t \in [0,1]$. For some results on P-functions, see [29, 42], while the interested reader can consult [28] for quasi-convex functions.

If $f : C \subseteq X \to [0,\infty)$, where C is a convex subset of the real or complex linear space X, then we say that it is of P type (or quasi-convex) if the inequality (3.3) (or (3.4)) holds true for $x, y \in C$ and $t \in [0,1]$.

Definition 3.4 (see [7]). Let s be a real number with $s \in (0,1]$. A function $f : [0,\infty) \to [0,\infty)$ is said to be s-convex (in the second sense) or Breckner s-convex if

$$f(tx + (1-t)y) \le t^s f(x) + (1-t)^s f(y)$$

for all $x, y \in [0,\infty)$ and $t \in [0,1]$.

For some properties of this class of functions, see [1, 2, 7, 8, 24, 25, 36, 38, 48].

The concept of Breckner s-convexity can be similarly extended for functions defined on convex subsets of linear spaces. It is well known that if $(X, \|\cdot\|)$ is a normed linear space, then the function $f(x) = \|x\|^p$ for $p \geq 1$ is convex on X. Utilising the elementary inequality $(a+b)^s \leq a^s + b^s$, which holds for any $a, b \geq 0$ and $s \in (0, 1]$, for the function $g(x) = \|x\|^s$ we have

$$g(tx + (1-t)y) = \|tx + (1-t)y\|^s \leq (t\|x\| + (1-t)\|y\|)^s \leq (t\|x\|)^s + ((1-t)\|y\|)^s = t^s g(x) + (1-t)^s g(y)$$

for any $x, y \in X$ and $t \in [0, 1]$, which shows that g is Breckner s-convex on X.

We can now introduce another concept of function that incorporates the classes of P-functions and of Godunova–Levin functions.

Definition 3.5. We say that the function $f : C \subseteq X \to [0, \infty)$ is of s-Godunova–Levin type with $s \in [0, 1]$ if

$$f(tx + (1-t)y) \leq \frac{1}{t^s} f(x) + \frac{1}{(1-t)^s} f(y)$$

for all $t \in (0, 1)$ and $x, y \in C$.

We observe that, for $s = 0$, we obtain the class of P-functions, while, for $s = 1$, we obtain the class of Godunova–Levin functions. If we denote by $Q_s(C)$ the class of s-Godunova–Levin functions defined on C, then we obviously have

$$P(C) = Q_0(C) \subseteq Q_{s_1}(C) \subseteq Q_{s_2}(C) \subseteq Q_1(C) = Q(C)$$

for $0 \leq s_1 \leq s_2 \leq 1$.

We can now prove the following generalisation of the Hermite–Hadamard inequality for h-convex functions defined on convex subsets of linear spaces.

Theorem 3.6. *Assume that the function $f : C \subseteq X \to [0, \infty)$ is an h-convex function with h Lebesgue integrable on $[0, 1]$. Let $y, x \in C$ and assume that the mapping $[0, 1] \ni t \mapsto f((1-t)x + ty)$ is Lebesgue integrable on $[0, 1]$. Then, we have*

$$\begin{aligned} \frac{1}{2h(\frac{1}{2})} f\left(\frac{x+y}{2}\right) &\leq \frac{1}{2(b-a)(d-c)} \int_a^b \int_c^d \left(f\left(\frac{\alpha x + \beta y}{\alpha + \beta}\right) + f\left(\frac{\beta x + \alpha y}{\alpha + \beta}\right)\right) d\beta\, d\alpha \\ &\leq \frac{f(x) + f(y)}{2(b-a)(d-c)} \int_a^b \int_c^d \left(h\left(\frac{\alpha}{\alpha + \beta}\right) + h\left(\frac{\beta}{\alpha + \beta}\right)\right) d\beta\, d\alpha \end{aligned} \tag{3.5}$$

for any $a, b, c, d \geq 0$ with $b > a$ and $d > c$.

Proof. By the h-convexity of f we have

$$f(tx + (1-t)y) \leq h(t) f(x) + h(1-t) f(y) \tag{3.6}$$

and

$$f((1-t)x + ty) \leq h(1-t) f(x) + h(t) f(y) \tag{3.7}$$

for any $t \in [0, 1]$. Summing the inequalities (3.6) and (3.7) and dividing by 2 gives

$$\frac{1}{2}(f(tx + (1-t)y) + f((1-t)x + ty)) \leq \frac{1}{2}(h(1-t) + h(t))(f(x) + f(y)) \tag{3.8}$$

for any $t \in [0, 1]$. Taking

$$t = \frac{\alpha}{\alpha + \beta}$$

in (3.8) gives

$$\frac{1}{2}\left(f\left(\frac{\alpha x + \beta y}{\alpha + \beta}\right) + f\left(\frac{\beta x + \alpha y}{\alpha + \beta}\right)\right) \leq \frac{1}{2}\left(h\left(\frac{\alpha}{\alpha + \beta}\right) + h\left(\frac{\beta}{\alpha + \beta}\right)\right)(f(x) + f(y))$$

for any $\alpha, \beta \geq 0$ with $\alpha + \beta > 0$. Since the mapping $[0,1] \ni t \mapsto f((1-t)x + ty)$ is Lebesgue integrable on $[0,1]$, then the double integrals

$$\int_a^b \int_c^d f\Big(\frac{\alpha x + \beta y}{\alpha + \beta}\Big)\, d\beta\, d\alpha \quad \text{and} \quad \int_a^b \int_c^d f\Big(\frac{\alpha y + \beta x}{\alpha + \beta}\Big)\, d\beta\, d\alpha$$

exist and we get the second inequality in (3.5) by integrating the inequality on the rectangle $[a,b] \times [c,d]$ over (α, β).

From the h-convexity of f we also have

$$f\Big(\frac{z + w}{2}\Big) \leq h\Big(\frac{1}{2}\Big)(f(z) + f(w)) \tag{3.9}$$

for any $z, w \in C$. If we take

$$z = \frac{\alpha x + \beta y}{\alpha + \beta} \quad \text{and} \quad w = \frac{\beta x + \alpha y}{\alpha + \beta}$$

in (3.9), then we get

$$f\Big(\frac{x + y}{2}\Big) \leq h\Big(\frac{1}{2}\Big)\Big(f\Big(\frac{\alpha x + \beta y}{\alpha + \beta}\Big) + f\Big(\frac{\beta x + \alpha y}{\alpha + \beta}\Big)\Big)$$

for any $\alpha, \beta \geq 0$ with $\alpha + \beta > 0$. Integrating the inequality on the rectangle $[a,b] \times [c,d]$ over (α, β), we get the first inequality in (3.5). □

Corollary 3.7. *With the assumptions of Theorem 3.6 we have*

$$\frac{1}{2h(\frac{1}{2})} f\Big(\frac{x + y}{2}\Big) \leq \frac{1}{(b-a)^2} \int_a^b \int_a^b f\Big(\frac{\alpha x + \beta y}{\alpha + \beta}\Big)\, d\beta\, d\alpha \leq \frac{f(x) + f(y)}{(b-a)^2} \int_a^b \int_a^b h\Big(\frac{\alpha}{\alpha + \beta}\Big)\, d\beta\, d\alpha$$

for any $b > a \geq 0$.

The following result holds for convex functions.

Corollary 3.8. *Let $f : C \subseteq X \to [0, \infty)$ be a convex function on the convex set C in a linear space X. Then, for any $x, y \in C$ and for any $a, b, c, d \geq 0$ with $b > a$ and $d > c$, we have*

$$f\Big(\frac{x + y}{2}\Big) \leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d \Big(\frac{f(\frac{\alpha x + \beta y}{\alpha + \beta}) + f(\frac{\beta x + \alpha y}{\alpha + \beta})}{2}\Big)\, d\beta\, d\alpha \leq \frac{I(a,b;c,d) + I(c,d;a,b)}{(b-a)(d-c)} \frac{f(x) + f(y)}{2},$$

where $I(a,b;c,d)$ and $I(c,d;a,b)$ are defined in (2.1).

For two distinct positive numbers p and q, we consider the *logarithmic mean*

$$L(p,q) := \frac{p - q}{\ln p - \ln q}.$$

Corollary 3.9. *Assume that the function $f : C \subseteq X \to [0, \infty)$ is of Godunova–Levin type on C. Let $y, x \in C$ and assume that the mapping $[0,1] \ni t \mapsto f((1-t)x + ty)$ is Lebesgue integrable on $[0,1]$. Then, for any $a, b, c, d > 0$ with $b > a$ and $d > c$, we have*

$$\begin{aligned} \frac{1}{4} f\Big(\frac{x + y}{2}\Big) &\leq \frac{1}{2(b-a)(d-c)} \int_a^b \int_c^d \Big(f\Big(\frac{\alpha x + \beta y}{\alpha + \beta}\Big) + f\Big(\frac{\beta x + \alpha y}{\alpha + \beta}\Big)\Big)\, d\beta\, d\alpha \\ &\leq \frac{f(x) + f(y)}{2}\Big(2 + \frac{A(c,d)}{L(a,b)} + \frac{A(a,b)}{L(c,d)}\Big), \end{aligned} \tag{3.10}$$

where L is the logarithmic mean and A is the arithmetic mean of the numbers involved.

Proof. We take

$$h(t) = \frac{1}{t}, \quad t \in (0,1),$$

in (3.5) and we have to integrate the double integral

$$\int_a^b \int_c^d \left(\frac{\alpha + \beta}{\alpha} + \frac{\alpha + \beta}{\beta} \right) d\beta \, d\alpha.$$

Observe that

$$\begin{aligned}\int_a^b \int_c^d \frac{\alpha + \beta}{\alpha} d\beta \, d\alpha &= \int_a^b \int_c^d \left(1 + \frac{\beta}{\alpha}\right) d\beta \, d\alpha \\ &= (b-a)(d-c) + (\ln b - \ln a)\frac{d^2 - c^2}{2} \\ &= (b-a)(d-c)\left(1 + \frac{\ln b - \ln a}{b-a}\frac{c+d}{2}\right) \\ &= (b-a)(d-c)\left(1 + \frac{A(c,d)}{L(a,b)}\right)\end{aligned}$$

and

$$\int_a^b \int_c^d \frac{\alpha + \beta}{\beta} d\beta \, d\alpha = (b-a)(d-c)\left(1 + \frac{A(a,b)}{L(c,d)}\right),$$

which produce the second part of (3.10). □

Remark 3.10. With the assumptions of Corollary 3.9 we have the inequalities

$$\frac{1}{4} f\left(\frac{x+y}{2}\right) \le \frac{1}{(b-a)^2} \int_a^b \int_a^b f\left(\frac{\alpha x + \beta y}{\alpha + \beta}\right) d\beta \, d\alpha \le \left(1 + \frac{A(a,b)}{L(a,b)}\right)(f(x) + f(y))$$

for any $b > a > 0$.

Corollary 3.11. *Assume that the function $f : C \subseteq X \to [0, \infty)$ is of P type on C. Let $y, x \in C$ and assume that the mapping $[0,1] \ni t \mapsto f((1-t)x + ty)$ is Lebesgue integrable on $[0,1]$. Then, for any a, b, c, d with $b > a \ge 0$ and $d > c \ge 0$, we have*

$$\frac{1}{2} f\left(\frac{x+y}{2}\right) \le \frac{1}{2(b-a)(d-c)} \int_a^b \int_c^d \left(f\left(\frac{\alpha x + \beta y}{\alpha + \beta}\right) + f\left(\frac{\beta x + \alpha y}{\alpha + \beta}\right)\right) d\beta \, d\alpha \le f(x) + f(y)$$

and, in particular,

$$\frac{1}{2} f\left(\frac{x+y}{2}\right) \le \frac{1}{(b-a)^2} \int_a^b \int_a^b f\left(\frac{\alpha x + \beta y}{\alpha + \beta}\right) d\beta \, d\alpha \le f(x) + f(y).$$

The interested reader may obtain similar results for other h-convex functions as provided above. The details are omitted.

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