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Inequalities of Hermite-Hadamard Type for HA-Convex Functions

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Abstract. Some new inequalities of Hermite-Hadamard type for HA-convex functions defined on positive intervals are given.

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1. Introduction

Following [4] (see also [40]) we say that the function $f : I \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is *HA-convex* or *harmonically convex* if

$$f\left(\frac{xy}{tx+(1-t)\,y}\right) \leq (1-t)\,f(x) + tf(y) \tag{1}$$

for all $x, y \in I$ and $t \in [0,1]$. If the inequality in (1) is reversed, then f is said to be *HA-concave* or *harmonically concave*.

In order to avoid any confusion with the class of *AH-convex* functions, namely the functions satisfying the condition

$$f((1-t)\,x + ty) \leq \frac{f(x)\,f(y)}{(1-t)\,f(y) + tf(x)}, \tag{2}$$

we call the class of functions satisfying (1) as *HA-convex functions.*

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If $I \subset (0, \infty)$ and f is convex and nondecreasing function then f is HA-convex and if f is HA-convex and nonincreasing function then f is convex.

If $[a, b] \subset I \subset (0, \infty)$ and if we consider the function $g : \left[\frac{1}{b}, \frac{1}{a}\right] \to \mathbb{R}$, defined by $g(t) = f\left(\frac{1}{t}\right)$, then f is HA-convex on $[a, b]$ if and only if g is convex in the usual sense on $\left[\frac{1}{b}, \frac{1}{a}\right]$. Therefore, as examples of HA-convex functions we can take $f(t) = g\left(\frac{1}{t}\right)$, where g is any convex function on $\left[\frac{1}{b}, \frac{1}{a}\right]$.

For a convex function $h : [c, d] \to \mathbb{R}$, the following inequality is well known in the literature as the *Hermite-Hadamard inequality*

$$h\left(\frac{c+d}{2}\right) \le \frac{1}{d-c}\int_c^d h(t)\,dt \le \frac{h(c)+h(d)}{2}. \tag{3}$$

For related results, see [1]-[20], [23]-[26], [27]-[36] and [37]-[48].

If we write the Hermite-Hadamard inequality for the convex function $g(t) = f\left(\frac{1}{t}\right)$ on the closed interval $\left[\frac{1}{b}, \frac{1}{a}\right]$, then we have

$$f\left(\frac{1}{\frac{\frac{1}{a}+\frac{1}{b}}{2}}\right) \le \frac{1}{\frac{1}{a}-\frac{1}{b}}\int_{\frac{1}{b}}^{\frac{1}{a}} f\left(\frac{1}{t}\right)dt \le \frac{f\left(\frac{1}{\frac{1}{b}}\right)+f\left(\frac{1}{\frac{1}{a}}\right)}{2}$$

that is equivalent to

$$f\left(\frac{2ab}{a+b}\right) \le \frac{ab}{b-a}\int_{\frac{1}{b}}^{\frac{1}{a}} f\left(\frac{1}{t}\right)dt \le \frac{f(b)+f(a)}{2}. \tag{4}$$

Using the change of variable $s = \frac{1}{t}$, then

$$\int_{\frac{1}{b}}^{\frac{1}{a}} f\left(\frac{1}{t}\right)dt = \int_a^b \frac{f(s)}{s^2}ds$$

and by (4) we get

$$f\left(\frac{2ab}{a+b}\right) \le \frac{ab}{b-a}\int_a^b \frac{f(s)}{s^2}ds \le \frac{f(b)+f(a)}{2}. \tag{5}$$

The inequality (5) has been obtained in a different manner in [40] by I. Işcan.

Motivated by the above results, we establish in this paper some new inequalities of Hermite-Hadamard type for HA-convex functions.

2. A Refinement

We have the following representation result, see [25]. For the sake of completeness we give here a simple proof.

Lemma 2.1. *Let $g : [x, y] \subset \mathbb{R} \to \mathbb{C}$ be a Lebesgue integrable function on $[x, y]$. Then for any $\lambda \in [0, 1]$ we have the representation*

$$\int_0^1 g[(1-t)x + ty]\,dt = (1-\lambda)\int_0^1 g[(1-t)((1-\lambda)x + \lambda y) + ty]\,dt + \lambda\int_0^1 g[(1-t)x + t((1-\lambda)x + \lambda y)]\,dt. \tag{6}$$

Proof. For $\lambda = 0$ and $\lambda = 1$ the equality (6) is obvious.

Let $\lambda \in (0,1)$. Observe that

$$\int_0^1 g\left[(1-t)(\lambda y + (1-\lambda)x) + ty\right] dt$$
$$= \int_0^1 g\left[((1-t)\lambda + t) y + (1-t)(1-\lambda) x\right] dt$$

and

$$\int_0^1 g\left[t(\lambda y + (1-\lambda)x) + (1-t)x\right] dt = \int_0^1 g\left[t\lambda y + (1-\lambda t)x\right] dt.$$

If we make the change of variable $u := (1-t)\lambda + t$ then we have $1-u = (1-t)(1-\lambda)$ and $du = (1-\lambda)\,du$. Then

$$\int_0^1 g\left[((1-t)\lambda + t) y + (1-t)(1-\lambda)x\right] dt = \frac{1}{1-\lambda}\int_\lambda^1 g\left[uy + (1-u)x\right] du.$$

If we make the change of variable $u := \lambda t$ then we have $du = \lambda dt$ and

$$\int_0^1 g\left[t\lambda y + (1-\lambda t)x\right] dt = \frac{1}{\lambda}\int_0^\lambda g\left[uy + (1-u)x\right] du.$$

Therefore

$$(1-\lambda)\int_0^1 g\left[(1-t)(\lambda y + (1-\lambda)x) + ty\right] dt$$
$$+ \lambda \int_0^1 g\left[t(\lambda y + (1-\lambda)x) + (1-t)x\right] dt$$
$$= \int_\lambda^1 g\left[uy + (1-u)x\right] du + \int_0^\lambda g\left[uy + (1-u)x\right] du$$
$$= \int_0^1 g\left[uy + (1-u)x\right] du$$

and the identity (6) is proved.

Corollary 2.1. *Let $f : [a,b] \subset (0,\infty) \to \mathbb{C}$ be a Lebesgue integrable function on $[a,b]$ and $\lambda \in [0,1]$, then we have the representation*

$$\int_0^1 f\left(\frac{ab}{(1-t)a + tb}\right) dt = (1-\lambda)\int_0^1 f\left(\frac{ab}{(1-t)((1-\lambda)a + \lambda b) + tb}\right) dt \tag{7}$$
$$+ \lambda \int_0^1 f\left(\frac{ab}{(1-t)a + t((1-\lambda)a + \lambda b)}\right) dt.$$

Proof. Consider the function $g : \left[\frac{1}{b}, \frac{1}{a}\right]$, $g(s) = f\left(\frac{1}{s}\right)$, $s \in \left[\frac{1}{b}, \frac{1}{a}\right]$.

We have by (6) for g and $x = \frac{1}{b}$, $y = \frac{1}{a}$ that

$$\int_0^1 f\left(\frac{ab}{(1-t)a + tb}\right) dt \tag{8}$$
$$= \int_0^1 f\left(\frac{1}{(1-t)\frac{1}{b} + t\frac{1}{a}}\right) dt$$

$$
\begin{aligned}
&= \int_0^1 g\left((1-t)\frac{1}{b} + t\frac{1}{a}\right) dt \\
&= (1-\lambda)\int_0^1 g\left[(1-t)\left((1-\lambda)\frac{1}{b} + \lambda\frac{1}{a}\right) + t\frac{1}{a}\right] dt \\
&+ \lambda\int_0^1 g\left[(1-t)\frac{1}{b} + t\left((1-\lambda)\frac{1}{b} + \lambda\frac{1}{a}\right)\right] dt \\
&= (1-\lambda)\int_0^1 f\left(\frac{1}{(1-t)\left((1-\lambda)\frac{1}{b} + \lambda\frac{1}{a}\right) + t\frac{1}{a}}\right) dt \\
&+ \lambda\int_0^1 f\left(\frac{1}{(1-t)\frac{1}{b} + t\left((1-\lambda)\frac{1}{b} + \lambda\frac{1}{a}\right)}\right) dt \\
&= (1-\lambda)\int_0^1 f\left(\frac{ab}{(1-t)\left((1-\lambda)a + \lambda b\right) + tb}\right) dt \\
&+ \lambda\int_0^1 f\left(\frac{ab}{(1-t)a + t\left((1-\lambda)a + \lambda b\right)}\right) dt.
\end{aligned}
$$

The following result holds.

Theorem 2.1. *Let $f : [a,b] \subset (0,\infty) \to \mathbb{R}$ be an HA-convex function on the interval $[a,b]$. Then for any $\lambda \in [0,1]$ we have the inequalities*

$$
\begin{aligned}
f\left(\frac{2ab}{a+b}\right) &\le (1-\lambda) f\left(\frac{2ab}{(1-\lambda)a + (\lambda+1)b}\right) + \lambda f\left(\frac{2ab}{(2-\lambda)a + \lambda b}\right) \qquad (9)\\
&\le \frac{ab}{b-a}\int_a^b \frac{f(t)}{t^2} dt \\
&\le \frac{1}{2}\left[f\left(\frac{ab}{(1-\lambda)a + \lambda b}\right) + (1-\lambda) f(a) + \lambda f(b)\right] \\
&\le \frac{f(a)+f(b)}{2}.
\end{aligned}
$$

Proof. Consider the function $g : \left[\frac{1}{b}, \frac{1}{a}\right], \ g(s) = f\left(\frac{1}{s}\right), \ s \in \left[\frac{1}{b}, \frac{1}{a}\right].$

Since g is convex on $\left[\frac{1}{b}, \frac{1}{a}\right]$, then by Hermite-Hadamard inequality for convex functions we have for $\lambda \in [0,1]$

$$
\begin{aligned}
g\left(\frac{(1-\lambda)a + (\lambda+1)b}{2ab}\right) &= g\left(\frac{(1-\lambda)\frac{1}{b} + \lambda\frac{1}{a} + \frac{1}{a}}{2}\right) \qquad (10)\\
&\le \int_0^1 g\left((1-t)\left((1-\lambda)\frac{1}{b} + \lambda\frac{1}{a}\right) + t\frac{1}{a}\right) dt \\
&\le \frac{g\left((1-\lambda)\frac{1}{b} + \lambda\frac{1}{a}\right) + g\left(\frac{1}{a}\right)}{2} \\
&= \frac{g\left(\frac{(1-\lambda)a + \lambda b}{ab}\right) + g\left(\frac{1}{a}\right)}{2}
\end{aligned}
$$

and

$$g\left(\frac{(2-\lambda)a+\lambda b}{2ab}\right) = g\left(\frac{\frac{1}{b}+(1-\lambda)\frac{1}{b}+\lambda\frac{1}{a}}{2}\right) \tag{11}$$
$$\leq \int_0^1 g\left((1-t)\frac{1}{b}+t\left((1-\lambda)\frac{1}{b}+\lambda\frac{1}{a}\right)\right)dt$$
$$\leq \frac{g\left(\frac{1}{b}\right)+g\left((1-\lambda)\frac{1}{b}+\lambda\frac{1}{a}\right)}{2}$$
$$= \frac{g\left(\frac{1}{b}\right)+g\left(\frac{(1-\lambda)a+\lambda b}{ab}\right)}{2}.$$

If we multiply (10) by $(1-\lambda)$ and 11 by λ, add the obtained inequalities and use the first part of the equality (8) we get

$$(1-\lambda)f\left(\frac{2ab}{(1-\lambda)a+(\lambda+1)b}\right)+\lambda f\left(\frac{2ab}{(2-\lambda)a+\lambda b}\right) \tag{12}$$
$$\leq \int_0^1 f\left(\frac{ab}{(1-t)a+tb}\right)dt$$
$$\leq (1-\lambda)\frac{f\left(\frac{ab}{(1-\lambda)a+\lambda b}\right)+f(a)}{2}+\lambda\frac{f(b)+f\left(\frac{ab}{(1-\lambda)a+\lambda b}\right)}{2}$$
$$= \frac{1}{2}\left[f\left(\frac{ab}{(1-\lambda)a+\lambda b}\right)+(1-\lambda)f(a)+\lambda f(b)\right].$$

By the convexity of g we have

$$(1-\lambda)f\left(\frac{2ab}{(1-\lambda)a+(\lambda+1)b}\right)+\lambda f\left(\frac{2ab}{(2-\lambda)a+\lambda b}\right)$$
$$= (1-\lambda)g\left(\frac{(1-\lambda)a+(\lambda+1)b}{2ab}\right)+\lambda g\left(\frac{(2-\lambda)a+\lambda b}{2ab}\right)$$
$$\geq g\left(\frac{(1-\lambda)[(1-\lambda)a+(\lambda+1)b]}{2ab}+\frac{\lambda[(2-\lambda)a+\lambda b]}{2ab}\right)$$
$$= g\left(\frac{a+b}{2ab}\right) = f\left(\frac{2ab}{a+b}\right)$$

and

$$f\left(\frac{ab}{(1-\lambda)a+\lambda b}\right)+(1-\lambda)f(a)+\lambda f(b)$$
$$= g\left((1-\lambda)\frac{1}{b}+\lambda\frac{1}{a}\right)+(1-\lambda)f(a)+\lambda f(b)$$
$$\leq (1-\lambda)f(b)+\lambda f(a)+(1-\lambda)f(a)+\lambda f(b)$$
$$= f(a)+f(b)$$

and the desired inequality (9) is proved.

Corollary 2.2. *With the assumptions of Theorem 2.1 we have*

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{1}{2}\left[f\left(\frac{4ab}{a+3b}\right)+f\left(\frac{4ab}{3a+b}\right)\right] \leq \frac{ab}{b-a}\int_a^b \frac{f(t)}{t^2}dt \tag{13}$$

$$\leq \frac{1}{2}\left[f\left(\frac{2ab}{a+b}\right)+\frac{f(a)+f(b)}{2}\right] \leq \frac{f(a)+f(b)}{2}.$$

3. New Results

We recall some facts on the lateral derivatives of a convex function.

Suppose that I is an interval of real numbers with interior $\mathring{I}$ and $f : I \to \mathbb{R}$ is a convex function on I. Then f is continuous on $\mathring{I}$ and has finite left and right derivatives at each point of $\mathring{I}$. Moreover, if $x, y \in \mathring{I}$ and $x < y$, then $f'_-(x) \leq f'_+(x) \leq f'_-(y) \leq f'_+(y)$ which shows that both f'_- and f'_+ are nondecreasing function on $\mathring{I}$. It is also known that a convex function must be differentiable except for at most countably many points.

For a convex function $f : I \to \mathbb{R}$, the subdifferential of f denoted by ∂f is the set of all functions $\varphi : I \to [-\infty, \infty]$ such that $\varphi\left(\mathring{I}\right) \subset \mathbb{R}$ and

$$f(x) \geq f(a) + (x-a)\varphi(a) \text{ for any } x, a \in I. \tag{14}$$

It is also well known that if f is convex on I, then ∂f is nonempty, f'_-, $f'_+ \in \partial f$ and if $\varphi \in \partial f$, then

$$f'_-(x) \leq \varphi(x) \leq f'_+(x) \text{ for any } x \in \mathring{I}.$$

In particular, φ is a nondecreasing function.

If f is differentiable and convex on $\mathring{I}$, then $\partial f = \{f'\}$.

The *identric mean* $I(a,b)$ is defined by

$$I(a,b) := \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}$$

while the *logarithmic mean* is defined by

$$L(a,b) := \frac{b-a}{\ln b - \ln a}.$$

Theorem 3.1. *Let $f : [a,b] \subset (0,\infty) \to \mathbb{R}$ be an HA-convex function on the interval $[a,b]$. Then*

$$f(L(a,b)) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{(L(a,b)-a)\,bf(b) + (b-L(a,b))\,af(a)}{(b-a)\,L(a,b)}. \tag{15}$$

Proof. Since $f : [a,b] \subset (0,\infty) \to \mathbb{R}$ is an HA-convex function on the interval $[a,b]$, then the function $g : \left[\frac{1}{b}, \frac{1}{a}\right]$, $g(s) = f\left(\frac{1}{s}\right)$, is convex on $\left[\frac{1}{b}, \frac{1}{a}\right]$. Therefore f has partial derivatives in each point of (a,b) and by the gradient inequality for g we have for any $x, y \in (a,b)$ that

$$\begin{aligned} f(x) - f(y) &= g\left(\frac{1}{x}\right) - g\left(\frac{1}{y}\right) \geq g'_+\left(\frac{1}{y}\right)\left(\frac{1}{x}-\frac{1}{y}\right) \\ &= g'_+\left(\frac{1}{y}\right)\frac{y-x}{xy}. \end{aligned} \tag{16}$$

Since

$$g'_+(s) = f'_-\left(\frac{1}{s}\right)\left(-\frac{1}{s^2}\right),\ s \in \left(\frac{1}{b}, \frac{1}{a}\right)$$

then

$$g'_+\left(\frac{1}{y}\right) = f'_-(y)\left(-y^2\right)$$

and by (16) we have

$$f(x)-f(y)\geq f'_{-}(y)\,\frac{y-x}{xy}\left(-y^{2}\right)=f'_{-}(y)\,y\left(1-\frac{y}{x}\right).$$

Therefore we have

$$f(x)-f(y)\geq f'_{-}(y)\,y\left(1-\frac{y}{x}\right) \tag{17}$$

for any $x,y\in(a,b)$.

If we take the integral mean over x in (17), then we have

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(x)\,dx-f(y) &\geq \left(1-y\frac{1}{b-a}\int_a^b\frac{1}{x}dx\right)f'_{-}(y)\,y \\ &= \left(1-\frac{y}{L(a,b)}\right)f'_{-}(y)\,y\end{aligned} \tag{18}$$

for any $y\in(a,b)$.

Now, if we take $y=L(a,b)$ in (18), then we get the first inequality in (15).

Observe that for any $x\in[a,b]$ we have

$$\frac{1}{x}=\frac{\left(\frac{1}{a}-\frac{1}{x}\right)\frac{1}{b}+\left(\frac{1}{x}-\frac{1}{b}\right)\frac{1}{a}}{\frac{1}{a}-\frac{1}{b}}.$$

By the convexity of g on $\left[\frac{1}{b},\frac{1}{a}\right]$ we then have

$$\begin{aligned}f(x)=g\left(\frac{1}{x}\right)&=g\left(\frac{\left(\frac{1}{a}-\frac{1}{x}\right)\frac{1}{b}+\left(\frac{1}{x}-\frac{1}{b}\right)\frac{1}{a}}{\frac{1}{a}-\frac{1}{b}}\right) \\ &\leq\frac{\left(\frac{1}{a}-\frac{1}{x}\right)g\left(\frac{1}{b}\right)+\left(\frac{1}{x}-\frac{1}{b}\right)g\left(\frac{1}{a}\right)}{\frac{1}{a}-\frac{1}{b}} \\ &=\frac{\left(\frac{1}{a}-\frac{1}{x}\right)f(b)+\left(\frac{1}{x}-\frac{1}{b}\right)f(a)}{\frac{1}{a}-\frac{1}{b}}\end{aligned} \tag{19}$$

for any $x\in[a,b]$.

Taking the integral mean in (19) we get

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(x)\,dx &\leq\frac{\left(\frac{1}{a}-\frac{1}{b-a}\int_a^b\frac{1}{x}dx\right)f(b)+\left(\frac{1}{b-a}\int_a^b\frac{1}{x}dx-\frac{1}{b}\right)f(a)}{\frac{1}{a}-\frac{1}{b}} \\ &=\frac{\left(\frac{1}{a}-\frac{1}{L(a,b)}\right)f(b)+\left(\frac{1}{L(a,b)}-\frac{1}{b}\right)f(a)}{\frac{1}{a}-\frac{1}{b}} \\ &=\frac{\frac{L(a,b)-a}{aL(a,b)}f(b)+\frac{b-L(a,b)}{L(a,b)b}f(a)}{\frac{b-a}{ab}}\end{aligned}$$

and the second inequality in (15) is also proved.

Remark 3.1. *If $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ is a differentiable HA-convex function on the interval (a,b), then from (18) we have the following inequality*

$$\frac{1}{b-a}\int_a^b f(x)\,dx-f(y)\geq\left(1-\frac{y}{L(a,b)}\right)f'(y)\,y \tag{20}$$

for any $y \in (a,b)$.

We have

$$\frac{1}{b-a}\int_a^b f(x)\,dx - f(A(a,b)) \geq \left(1-\frac{A(a,b)}{L(a,b)}\right) f'(A(a,b))\,A(a,b) \tag{21}$$

and if $f'(A(a,b)) \leq 0,$ *then*

$$\frac{1}{b-a}\int_a^b f(x)\,dx \geq f(A(a,b)). \tag{22}$$

We have

$$\frac{1}{b-a}\int_a^b f(x)\,dx - f(I(a,b)) \geq \left(1-\frac{I(a,b)}{L(a,b)}\right) f'(I(a,b))\,I(a,b) \tag{23}$$

and if $f'(I(a,b)) \leq 0,$ *then*

$$\frac{1}{b-a}\int_a^b f(x)\,dx \geq f(I(a,b)). \tag{24}$$

We have

$$\frac{1}{b-a}\int_a^b f(x)\,dx - f(G(a,b)) \geq \left(1-\frac{G(a,b)}{L(a,b)}\right) f'(G(a,b))\,G(a,b) \tag{25}$$

and if $f'(G(a,b)) \geq 0,$ *then*

$$\frac{1}{b-a}\int_a^b f(x)\,dx \geq f(G(a,b)). \tag{26}$$

We have:

Theorem 3.2. *Let* $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ *be a HA-convex function on the interval* $[a,b]$. *Then*

$$f\left(\frac{a+b}{2}\right)\frac{a+b}{2} \leq \frac{1}{b-a}\int_a^b xf(x)\,dx \leq \frac{bf(b)+af(a)}{2}. \tag{27}$$

Proof. From the inequality (17), by multiplying with $x>0$ we have

$$xf(x) - xf(y) \geq f'_-(y)\,y\,(x-y) \tag{28}$$

for any $x,y \in (a,b)$.

Taking the integral mean over $x \in [a,b]$ we have

$$\frac{1}{b-a}\int_a^b xf(x)\,dx - f(y)\frac{1}{b-a}\int_a^b xdx \geq \left(\frac{1}{b-a}\int_a^b xdx - y\right) f'_-(y)\,y,$$

that is equivalent to

$$\frac{1}{b-a}\int_a^b xf(x)\,dx - f(y)\frac{a+b}{2} \geq \left(\frac{a+b}{2}-y\right) f'_-(y)\,y, \tag{29}$$

for any $y \in (a,b)$.

If we take in (29) $y=\frac{a+b}{2}$, then we get the first inequality in (27).

From the inequality (19) we also have

$$xf(x) \leq \frac{\left(\frac{x}{a}-1\right)f(b)+\left(1-\frac{x}{b}\right)f(a)}{\frac{1}{a}-\frac{1}{b}} \tag{30}$$

for any $x \in [a,b]$.

Taking the integral mean on (30) we get

$$\frac{1}{b-a}\int_a^b xf(x)\,dx \le \frac{\left(\frac{a+b}{2a}-1\right)f(b)+\left(1-\frac{a+b}{2b}\right)f(a)}{\frac{1}{a}-\frac{1}{b}} \tag{31}$$
$$= \frac{\frac{b-a}{2a}f(b)+\frac{b-a}{2b}f(a)}{\frac{b-a}{ab}} = \frac{bf(b)+af(a)}{2}$$

and the second inequality in (27) is proved.

Remark 3.2. *If $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ is a differentiable HA-convex function on the interval (a,b), then from (29) we have*

$$f(y)A(a,b)-\frac{1}{b-a}\int_a^b xf(x)\,dx \le (y-A(a,b))f'(y)y, \tag{32}$$

for any $y\in(a,b)$.

If we take in (32) $y=I(a,b)$, then we get

$$f(I(a,b))A(a,b)-\frac{1}{b-a}\int_a^b xf(x)\,dx \tag{33}$$
$$\le (I(a,b)-A(a,b))f'(I(a,b))I(a,b).$$

If $f'(I(a,b))\ge 0$, then

$$f(I(a,b))A(a,b)\le \frac{1}{b-a}\int_a^b xf(x)\,dx. \tag{34}$$

If we take in (32) $y=L(a,b)$, then we get

$$f(L(a,b))A(a,b)-\frac{1}{b-a}\int_a^b xf(x)\,dx \tag{35}$$
$$\le (L(a,b)-A(a,b))f'(L(a,b))L(a,b).$$

If $f'(L(a,b))\ge 0$, then

$$f(L(a,b))A(a,b)\le \frac{1}{b-a}\int_a^b xf(x)\,dx. \tag{36}$$

If we take in (32) $y=G(a,b)$, then we get

$$f(G(a,b))A(a,b)-\frac{1}{b-a}\int_a^b xf(x)\,dx \tag{37}$$
$$\le (G(a,b)-A(a,b))f'(G(a,b))G(a,b).$$

If $f'(G(a,b))\ge 0$, then

$$f(G(a,b))A(a,b)\le \frac{1}{b-a}\int_a^b xf(x)\,dx. \tag{38}$$

We use the following results obtained by the author in [21] and [22]

Lemma 3.1. *Let $h:[\alpha,\beta]\to\mathbb{R}$ be a convex function on $[\alpha,\beta]$. Then we have the inequalities*

$$\frac{1}{8}\left[h'_+\left(\frac{\alpha+\beta}{2}\right)-h'_-\left(\frac{\alpha+\beta}{2}\right)\right](\beta-\alpha) \tag{39}$$

$$\le \frac{h(\alpha)+h(\beta)}{2} - \frac{1}{\beta-\alpha}\int_\alpha^\beta h(t)\,dt$$
$$\le \frac{1}{8}\left[h'_-(\beta)-h'_+(\alpha)\right](\beta-\alpha)$$

and

$$\frac{1}{8}\left[h'_+\left(\frac{\alpha+\beta}{2}\right)-h'_-\left(\frac{\alpha+\beta}{2}\right)\right](\beta-\alpha) \tag{40}$$
$$\le \frac{1}{\beta-\alpha}\int_\alpha^\beta h(t)\,dt - h\left(\frac{\alpha+\beta}{2}\right)$$
$$\le \frac{1}{8}\left[h'_-(\beta)-h'_+(\alpha)\right](\beta-\alpha).$$

The constant $\frac{1}{8}$ *is best possible in (39) and (40).*

We have:

Theorem 3.3. *Let* $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ *be a HA-convex function on the interval* $[a,b]$. *Then*

$$\frac{1}{2}\left[f'_+\left(\frac{2ab}{a+b}\right)-f'_-\left(\frac{2ab}{a+b}\right)\right]\frac{ab}{(a+b)^2}(b-a) \tag{41}$$
$$\le \frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int_a^b\frac{f(t)}{t^2}dt$$
$$\le \frac{1}{8}\left[\frac{f'_-(b)\,b^2-f'_+(a)\,a^2}{ab}\right](b-a)$$

and

$$\frac{1}{2}\left[f'_+\left(\frac{2ab}{a+b}\right)-f'_-\left(\frac{2ab}{a+b}\right)\right]\frac{ab}{(a+b)^2}(b-a) \tag{42}$$
$$\le \frac{ab}{b-a}\int_a^b\frac{f(t)}{t^2}dt - f\left(\frac{2ab}{a+b}\right)$$
$$\le \frac{1}{8}\left[\frac{f'_-(b)\,b^2-f'_+(a)\,a^2}{ab}\right](b-a).$$

Proof. Since $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ is an *HA*-convex function on the interval $[a,b]$, then the function $g:\left[\frac{1}{b},\frac{1}{a}\right]$, $g(s)=f\left(\frac{1}{s}\right)$, is convex on $\left[\frac{1}{b},\frac{1}{a}\right]$.

We know that

$$g'_\pm(s)=f'_\mp\left(\frac{1}{s}\right)\left(-\frac{1}{s^2}\right),\ s\in\left(\frac{1}{b},\frac{1}{a}\right).$$

If we use the inequality (39) for the convex function $g:\left[\frac{1}{b},\frac{1}{a}\right]\to\mathbb{R}$, then we have

$$\frac{1}{8}\left[f'_-\left(\frac{1}{\frac{\frac{1}{b}+\frac{1}{a}}{2}}\right)\left(-\frac{1}{\left(\frac{\frac{1}{b}+\frac{1}{a}}{2}\right)^2}\right)-f'_+\left(\frac{1}{\frac{\frac{1}{b}+\frac{1}{a}}{2}}\right)\left(-\frac{1}{\left(\frac{\frac{1}{b}+\frac{1}{a}}{2}\right)^2}\right)\right]$$
$$\times\left(\frac{1}{a}-\frac{1}{b}\right)$$

$$\leq \frac{f(a)+f(b)}{2} - \frac{1}{\frac{1}{a}-\frac{1}{b}} \int_{\frac{1}{b}}^{\frac{1}{a}} f\left(\frac{1}{s}\right) ds$$
$$\leq \frac{1}{8}\left[f'_+\left(\frac{1}{\frac{1}{a}}\right)\left(-\frac{1}{\left(\frac{1}{a}\right)^2}\right) - f'_-\left(\frac{1}{\frac{1}{b}}\right)\left(-\frac{1}{\left(\frac{1}{b}\right)^2}\right)\right]\left(\frac{1}{a}-\frac{1}{b}\right)$$

that is equivalent to

$$\frac{1}{8}\left[f'_+\left(\frac{1}{\frac{\frac{1}{b}+\frac{1}{a}}{2}}\right)\left(\frac{1}{\left(\frac{\frac{1}{b}+\frac{1}{a}}{2}\right)^2}\right) - f'_-\left(\frac{1}{\frac{\frac{1}{b}+\frac{1}{a}}{2}}\right)\left(\frac{1}{\left(\frac{\frac{1}{b}+\frac{1}{a}}{2}\right)^2}\right)\right]$$
$$\times\left(\frac{1}{a}-\frac{1}{b}\right)$$
$$\leq \frac{f(a)+f(b)}{2} - \frac{1}{\frac{1}{a}-\frac{1}{b}} \int_{\frac{1}{b}}^{\frac{1}{a}} f\left(\frac{1}{s}\right) ds$$
$$\leq \frac{1}{8}\left[f'_-\left(\frac{1}{\frac{1}{b}}\right)\left(\frac{1}{\left(\frac{1}{b}\right)^2}\right) - f'_+\left(\frac{1}{\frac{1}{a}}\right)\left(\frac{1}{\left(\frac{1}{a}\right)^2}\right)\right]\left(\frac{1}{a}-\frac{1}{b}\right),$$

namely

$$\frac{1}{8}\left[f'_+\left(\frac{2ab}{a+b}\right) - f'_-\left(\frac{2ab}{a+b}\right)\right]\frac{4a^2b^2}{(a+b)^2}\left(\frac{b-a}{ab}\right)$$
$$\leq \frac{f(a)+f(b)}{2} - \frac{1}{\frac{1}{a}-\frac{1}{b}} \int_{\frac{1}{b}}^{\frac{1}{a}} f\left(\frac{1}{s}\right) ds$$
$$\leq \frac{1}{8}\left[f'_-(b)\,b^2 - f'_+(a)\,a^2\right]\left(\frac{b-a}{ab}\right).$$

Observe that

$$\int_{\frac{1}{b}}^{\frac{1}{a}} f\left(\frac{1}{s}\right) ds = \int_a^b \frac{f(t)}{t^2} dt$$

and the inequality (41) is proved.

The inequality (42) follows by (40).

Corollary 3.1. *If $f : [a,b] \subset (0,\infty) \to \mathbb{R}$ is a differentiable HA-convex function on the interval (a,b), then*

$$0 \leq \frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(t)}{t^2}dt \leq \frac{1}{8}\left[\frac{f'_-(b)\,b^2 - f'_+(a)\,a^2}{ab}\right](b-a) \tag{43}$$

and

$$0 \leq \frac{ab}{b-a}\int_a^b \frac{f(t)}{t^2}dt - f\left(\frac{2ab}{a+b}\right) \leq \frac{1}{8}\left[\frac{f'_-(b)\,b^2 - f'_+(a)\,a^2}{ab}\right](b-a). \tag{44}$$

4. Related Results

We have the following result:

Theorem 4.1. *Let $f : [a, b] \subset (0, \infty) \to \mathbb{R}$ be an HA-convex function on the interval $[a, b]$. Then*

$$\frac{1}{2}\left[xf(x) + \frac{(b-x)\,bf(b) + (x-a)\,af(a)}{b-a}\right] \geq \frac{1}{b-a}\int_a^b yf(y)\,dy \tag{45}$$

for any $x \in [a, b]$.

Proof. From (17) we have

$$f(x) - f(y) \geq f'_-(y)\,y\left(1 - \frac{y}{x}\right) \tag{46}$$

for any $x, y \in (a, b)$.

If we take the integral mean over y in (46), then we have

$$f(x) - \frac{1}{b-a}\int_a^b f(y)\,dy \geq \frac{1}{b-a}\int_a^b f'_-(y)\,ydy - \frac{1}{x}\frac{1}{b-a}\int_a^b f'_-(y)\,y^2dy \tag{47}$$

for any $x \in (a, b)$.

Integrating by parts in the Lebesgue integral, we have

$$\int_a^b f'_-(y)\,ydy = bf(b) - af(a) - \int_a^b f(y)\,dy$$

and

$$\int_a^b f'_-(y)\,y^2dy = b^2f(b) - a^2f(a) - 2\int_a^b yf(y)\,dy.$$

Utilising (47) we obtain

$$\begin{aligned} & f(x) - \frac{1}{b-a}\int_a^b f(y)\,dy \\ & \geq \frac{1}{b-a}\left(bf(b) - af(a) - \int_a^b f(y)\,dy\right) \\ & - \frac{1}{x}\frac{1}{b-a}\left(b^2f(b) - a^2f(a) - 2\int_a^b yf(y)\,dy\right) \\ & = \frac{bf(b) - af(a)}{b-a} - \frac{1}{b-a}\int_a^b f(y)\,dy \\ & - \frac{1}{x}\frac{b^2f(b) - a^2f(a)}{b-a} + \frac{2}{x}\frac{1}{b-a}\int_a^b yf(y)\,dy \end{aligned} \tag{48}$$

that is equivalent to

$$f(x) + \frac{1}{x}\frac{b^2f(b) - a^2f(a)}{b-a} - \frac{bf(b) - af(a)}{b-a} \geq \frac{2}{x}\frac{1}{b-a}\int_a^b yf(y)\,dy.$$

If we multiply this inequality by $\frac{x}{2}$, then we get

$$\frac{1}{2}\left[xf(x) + \frac{b^2f(b) - a^2f(a) - xbf(b) + xaf(a)}{b-a}\right] \geq \frac{1}{b-a}\int_a^b yf(y)\,dy,$$

and the inequality (45) is proved.

Remark 4.1. *If we take in (45) $x = \frac{a+b}{2}$, then we get*

$$\frac{1}{2}\left[\frac{a+b}{2}f\left(\frac{a+b}{2}\right)+\frac{bf(b)+af(a)}{2}\right]\geq\frac{1}{b-a}\int_a^b yf(y)\,dy. \tag{49}$$

If we take in (45) $x = \frac{2ab}{a+b}$, then we get

$$\frac{1}{2}\left[\frac{2ab}{a+b}f\left(\frac{2ab}{a+b}\right)+\frac{b^2f(b)+a^2f(a)}{a+b}\right]\geq\frac{1}{b-a}\int_a^b yf(y)\,dy. \tag{50}$$

We have:

Theorem 4.2. *Let $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ be a HA-convex function on the interval $[a,b]$. Then*

$$\begin{aligned}&\frac{1}{x}\left[\frac{(b-x)f(b)+(x-a)f(a)}{b-a}-\frac{1}{b-a}\int_a^b f(y)\,dy\right] \\ &\geq\frac{1}{L(a,b)}\left[\frac{1}{\ln b-\ln a}\int_a^b\frac{f(y)}{y}dy-f(x)\right]\end{aligned} \tag{51}$$

for any $x\in[a,b]$.

Proof. By dividing with $y>0$ in (17) we have

$$\frac{1}{y}f(x)-\frac{f(y)}{y}\geq f'_-(y)\left(1-\frac{y}{x}\right) \tag{52}$$

for any $x,y\in(a,b)$.

By taking the integral mean over y in (52) we obtain

$$\begin{aligned}&\frac{\ln b-\ln a}{b-a}f(x)-\frac{1}{b-a}\int_a^b\frac{f(y)}{y}dy \\ &\geq\frac{1}{b-a}\int_a^b f'_-(y)\,dy-\frac{1}{x}\frac{1}{b-a}\int_a^b yf'_-(y)\,dy \\ &=\frac{f(b)-f(a)}{b-a}-\frac{1}{x}\frac{bf(b)-af(a)-\int_a^b f(y)\,dy}{b-a} \\ &=\frac{f(b)-f(a)}{b-a}-\frac{1}{x}\frac{bf(b)-af(a)\,y}{b-a}+\frac{1}{x}\frac{1}{b-a}\int_a^b f(y)\,dy\end{aligned}$$

that is equivalent to

$$\begin{aligned}&\frac{1}{x}\frac{bf(b)-af(a)\,y}{b-a}-\frac{f(b)-f(a)}{b-a}-\frac{1}{b-a}\int_a^b\frac{f(y)}{y}dy \\ &\geq\frac{1}{x}\frac{1}{b-a}\int_a^b f(y)\,dy-\frac{\ln b-\ln a}{b-a}f(x)\end{aligned}$$

or, to

$$\begin{aligned}&\frac{1}{b-a}\left[\frac{b-x}{x}f(b)+\frac{x-a}{x}f(a)\right]-\frac{1}{b-a}\int_a^b\frac{f(y)}{y}dy \\ &\geq\frac{1}{x}\frac{1}{b-a}\int_a^b f(y)\,dy-\frac{\ln b-\ln a}{b-a}f(x)\end{aligned} \tag{53}$$

for any $x\in(a,b)$.

Rearranging the terms in (53) produces the desired result (51).

Remark 4.2. *If we take $x = L(a,b)$ in (51), then we get*

$$\frac{(b-L(a,b))f(b)+(L(a,b)-a)f(a)}{b-a}-\frac{1}{b-a}\int_a^b f(y)\,dy \tag{54}$$
$$\geq \frac{1}{\ln b-\ln a}\int_a^b \frac{f(y)}{y}dy - f(L(a,b)).$$

If we take $x = A(a,b)$ in (51), then we get

$$\frac{f(b)+f(a)}{2}-\frac{1}{b-a}\int_a^b f(y)\,dy \tag{55}$$
$$\geq \frac{A(a,b)}{L(a,b)}\left[\frac{1}{\ln b-\ln a}\int_a^b \frac{f(y)}{y}dy - f(A(a,b))\right].$$

If we take $x = H(a,b) := \frac{2ab}{a+b}$ in (51), then we get

$$\frac{bf(b)+af(a)}{b+a}-\frac{1}{b-a}\int_a^b f(y)\,dy \tag{56}$$
$$\geq \frac{H(a,b)}{L(a,b)}\left[\frac{1}{\ln b-\ln a}\int_a^b \frac{f(y)}{y}dy - f(H(a,b))\right].$$

If we take $x = G(a,b)$ in (51)), then we get

$$\frac{(b-G(a,b))f(b)+(G(a,b)-a)f(a)}{b-a}-\frac{1}{b-a}\int_a^b f(y)\,dy \tag{57}$$
$$\geq \frac{G(a,b)}{L(a,b)}\left[\frac{1}{\ln b-\ln a}\int_a^b \frac{f(y)}{y}dy - f(G(a,b))\right].$$

If the function $f : [a,b] \subset (0,\infty) \to \mathbb{R}$ is convex, then by Jensen's inequality we have

$$\begin{aligned}\frac{1}{\ln b-\ln a}\int_a^b \frac{f(y)}{y}dy &= \frac{1}{\int_a^b \frac{dy}{y}}\int_a^b \frac{f(y)}{y}dy \geq f\left(\frac{\int_a^b y\frac{dy}{y}}{\int_a^b \frac{dy}{y}}\right)\\ &= f\left(\frac{b-a}{\ln b-\ln a}\right) = f(L(a,b)).\end{aligned}$$

Therefore, for any function $f : [a,b] \subset (0,\infty) \to \mathbb{R}$ that is *convex and HA-convex*, by (54) we have

$$\frac{(b-L(a,b))f(b)+(L(a,b)-a)f(a)}{b-a}-\frac{1}{b-a}\int_a^b f(y)\,dy \tag{58}$$
$$\geq \frac{1}{\ln b-\ln a}\int_a^b \frac{f(y)}{y}dy - f(L(a,b)) \geq 0.$$

It is known that, if a function $f : [a,b] \subset (0,\infty) \to \mathbb{R}$ is *GA-convex*, namely

$$f\left(x^{1-\lambda}y^{\lambda}\right) \leq (\geq)(1-\lambda)f(x)+\lambda f(y)$$

for all $x,y \in [a,b]$ and $\lambda \in [0,1]$, then [25]

$$\frac{1}{\ln b-\ln a}\int_a^b \frac{f(y)}{y}dy \geq f(G(a,b)). \tag{59}$$

Therefore, for any function $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ that is *GA-convex and HA-convex*, by (57) we have

$$\frac{(b-G(a,b))f(b)+(G(a,b)-a)f(a)}{b-a}-\frac{1}{b-a}\int_a^b f(y)\,dy \tag{60}$$
$$\geq\frac{G(a,b)}{L(a,b)}\left[\frac{1}{\ln b-\ln a}\int_a^b\frac{f(y)}{y}dy-f(G(a,b))\right]\geq 0.$$

Theorem 4.3. *Let $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ be a HA-convex function on the interval $[a,b]$. Then*

$$\frac{1}{2x}\left(\frac{f(b)a(b-x)+f(a)b(x-a)}{b-a}+xf(x)\right)\geq\frac{ab}{b-a}\int_a^b\frac{f(y)}{y^2}dy \tag{61}$$

for any $x\in[a,b]$.

Proof. From (17) we have, by division with $y^2>0$, that

$$\frac{1}{y^2}f(x)-\frac{1}{y^2}f(y)\geq\frac{f'_-(y)}{y}\left(1-\frac{y}{x}\right)$$

for any $x,y\in(a,b)$.

Taking the integral mean over y we have

$$f(x)\frac{1}{b-a}\int_a^b\frac{1}{y^2}dy-\frac{1}{b-a}\int_a^b\frac{f(y)}{y^2}dy$$
$$\geq\frac{1}{b-a}\int_a^b\frac{f'_-(y)}{y}dy-\frac{1}{x}\frac{1}{b-a}\int_a^b f'_-(y)\,dy$$

that is equivalent to

$$\frac{f(x)}{ab}-\frac{1}{b-a}\int_a^b\frac{f(y)}{y^2}dy$$
$$\geq\frac{1}{b-a}\left[\frac{f(b)}{b}-\frac{f(a)}{a}+\int_a^b\frac{f(y)}{y^2}dy\right]-\frac{1}{x}\frac{f(b)-f(a)}{b-a}$$
$$=\frac{1}{b-a}\left(\frac{f(b)}{b}-\frac{f(a)}{a}\right)+\frac{1}{b-a}\int_a^b\frac{f(y)}{y^2}dy-\frac{1}{x}\frac{f(b)-f(a)}{b-a},$$

for any $x\in(a,b)$. This can be written as

$$\frac{1}{x}\frac{f(b)-f(a)}{b-a}-\frac{1}{b-a}\left(\frac{f(b)}{b}-\frac{f(a)}{a}\right)\geq\frac{2}{b-a}\int_a^b\frac{f(y)}{y^2}dy-\frac{f(x)}{ab}$$

or as

$$\frac{1}{2}\left(\frac{1}{b-a}\left[f(b)\frac{b-x}{xb}+f(a)\frac{x-a}{ax}\right]+\frac{f(x)}{ab}\right)\geq\frac{1}{b-a}\int_a^b\frac{f(y)}{y^2}dy.$$

This is equivalent to the desired result (61).

Remark 4.3. *If we take in (61) $x=\frac{a+b}{2}$, then we get*

$$\frac{1}{2}\left(\frac{f(b)a+f(a)b}{a+b}+f\left(\frac{a+b}{2}\right)\right)\geq\frac{ab}{b-a}\int_a^b\frac{f(y)}{y^2}dy. \tag{62}$$

If we take in (32) $x=\frac{2ab}{a+b}$, *then we get*

$$\frac{1}{2}\left[\frac{f(b)+f(a)}{2}+f\left(\frac{2ab}{a+b}\right)\right]\geq\frac{ab}{b-a}\int_a^b\frac{f(y)}{y^2}dy. \tag{63}$$

5. Applications

We consider the *arithmetic mean* $A(a,b)=\frac{a+b}{2}$, the *geometric mean* $G(a,b)=\sqrt{ab}$ and *harmonic mean* $H(a,b)=\frac{2ab}{a+b}$ for the positive numbers $a,b>0$.

The following well know order between these means, including logarithmic and identric means defined above, holds

$$H(a,b)\leq G(a,b)\leq L(a,b)\leq I(a,b)\leq A(a,b). \tag{64}$$

If we consider the HA-convex function $f:(0,\infty)\to\mathbb{R}$, $f(t)=t$ and we use the inequalities (9), then we have

$$\begin{aligned} H(a,b) &\leq (1-\lambda)\frac{2ab}{(1-\lambda)a+(\lambda+1)b}+\lambda\frac{2ab}{(2-\lambda)a+\lambda b} \\ &\leq \frac{G^2(a,b)}{L(a,b)}\leq\frac{1}{2}\left[\frac{ab}{(1-\lambda)a+\lambda b}+(1-\lambda)a+\lambda b\right]\leq A(a,b), \end{aligned} \tag{65}$$

for any $\lambda\in[0,1]$.

If we use the inequalities (43) and (44) we get

$$0\leq A(a,b)-\frac{G^2(a,b)}{L(a,b)}\leq\frac{1}{4}\frac{A(a,b)}{G^2(a,b)}(b-a)^2 \tag{66}$$

and

$$0\leq\frac{G^2(a,b)}{L(a,b)}-H(a,b)\leq\frac{1}{4}\frac{A(a,b)}{G^2(a,b)}(b-a)^2. \tag{67}$$

The first inequality in (66) also follows by (64).

Consider the function $f:(0,\infty)\to\mathbb{R}$, $f(t)=\frac{\ln t}{t}$. Observe that

$$g(t)=f\left(\frac{1}{t}\right)=-t\ln t,$$

which shows that f is HA-concave on $(0,\infty)$.

If we use the inequality (15) for HA-concave functions we have

$$\frac{\ln(L(a,b))}{L(a,b)}\geq\frac{1}{b-a}\int_a^b\frac{\ln t}{t}dt\geq\frac{(L(a,b)-a)\ln b+(b-L(a,b))\ln a}{(b-a)L(a,b)},$$

which is equivalent to

$$\frac{\ln(L(a,b))}{L(a,b)}\geq\frac{\ln G(a,b)}{L(a,b)}\geq\frac{(L(a,b)-a)\ln b+(b-L(a,b))\ln a}{(b-a)L(a,b)}. \tag{68}$$

The first inequality in (68) also follows (64).

From the second inequality we have

$$G(a,b)\geq b^{\frac{L(a,b)-a}{b-a}}a^{\frac{b-L(a,b)}{b-a}}. \tag{69}$$

If we write the inequality (63) for the HA-convex function $f:(0,\infty)\to\mathbb{R}$, $f(t)=t$, then we have

$$\frac{A(a,b)+H(a,b)}{2}\ge\frac{G^2(a,b)}{L(a,b)}. \tag{70}$$

If we write the inequalities (49) and (50) for the HA-concave function $f:(0,\infty)\to\mathbb{R}$, $f(t)=\frac{\ln t}{t}$, then we get

$$\sqrt{A(a,b)\,G(a,b)}\le I(a,b) \tag{71}$$

and

$$\frac{1}{2}\left[\ln\left(\frac{2ab}{a+b}\right)+\frac{b\ln b+a\ln a}{a+b}\right]\le\ln I(a,b). \tag{72}$$

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