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# Perturbed Companions of Ostrowski's Inequality for Absolutely Continuous Functions (I)

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**Abstract.** Perturbed companions of Ostrowski's inequality for absolutely continuous functions whose derivatives are either bounded or of bounded variation and applications are given.

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## 1 Introduction

In [16] we established the following companion of Ostrowski inequality [26] for Lebesgue *sup-norm*:

**Theorem 1.** *Let  $f : [a, b] \rightarrow \mathbb{R}$  be an absolutely continuous function on*

$[a, b]$ . If  $f' \in L_\infty[a, b]$ , then we have the inequalities

$$\begin{aligned} & \left| \frac{1}{2} [f(x) + f(a+b-x)] - \frac{1}{b-a} \int_a^b f(t) dt \right| \\ & \leq \frac{1}{b-a} \left[ \frac{(x-a)^2}{2} \|f'\|_{[a,x],\infty} + \left( \frac{a+b}{2} - x \right)^2 \|f'\|_{[x,a+b-x],\infty} \right. \\ & \quad \left. + \frac{(x-a)^2}{2} \|f'\|_{[a+b-x,b],\infty} \right] \quad (1.1) \end{aligned}$$

$$\leq \begin{cases} \left[ \frac{1}{8} + 2 \left( \frac{x - \frac{3a+b}{4}}{b-a} \right)^2 \right] (b-a) \|f'\|_{[a,b],\infty} \\ \left[ \frac{1}{2^{\alpha-1}} \left( \frac{x-a}{b-a} \right)^{2\alpha} + \left( \frac{x - \frac{a+b}{2}}{b-a} \right)^{2\alpha} \right]^{\frac{1}{\alpha}} \\ \quad \times \left[ \|f'\|_{[a,x],\infty}^\beta + \|f'\|_{[x,a+b-x],\infty}^\beta + \|f'\|_{[a+b-x,b],\infty}^\beta \right]^{\frac{1}{\beta}} (b-a) \\ \quad \text{if } \alpha > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1, \\ \max \left\{ \frac{1}{2} \left( \frac{x-a}{b-a} \right)^2, \left( \frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right\} \\ \quad \times \left[ \|f'\|_{[a,x],\infty} + \|f'\|_{[x,a+b-x],\infty} + \|f'\|_{[a+b-x,b],\infty} \right] (b-a) \end{cases}$$

for any  $x \in [a, \frac{a+b}{2}]$ , where

$$\|g\|_{[c,d],\infty} := \operatorname{ess\,sup}_{t \in [c,d]} |g(s)|.$$

The inequality (1.1), the first inequality in (1.1) and the constant  $\frac{1}{8}$  are sharp.

If in Theorem 1 we choose  $x = a$ , then we get

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{1}{4} (b-a) \|f'\|_{[a,b],\infty} \quad (1.2)$$

with  $\frac{1}{4}$  as a sharp constant (see for example [20, p. 25]).

If in the same theorem we now choose  $x = \frac{a+b}{2}$ , then we get

$$\begin{aligned} \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(t) dt \right| &\leq \frac{1}{8} (b-a) \left[ \|f'\|_{[a, \frac{a+b}{2}], \infty} + \|f'\|_{[\frac{a+b}{2}, b], \infty} \right] \\ &\leq \frac{1}{4} (b-a) \|f'\|_{[a, b], \infty} \end{aligned} \quad (1.3)$$

with the constants  $\frac{1}{8}$  and  $\frac{1}{4}$  being sharp. This result was obtained in [15] by a different argument.

It is natural to consider the following corollary.

**Corollary 2.** *With the assumptions in Theorem 1, one has the inequality:*

$$\left| \frac{1}{2} \left[ f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right) \right] - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{1}{8} (b-a) \|f'\|_{[a, b], \infty}. \quad (1.4)$$

The constant  $\frac{1}{8}$  is best possible in the sense that it cannot be replaced by a smaller quantity.

In the same paper [16] we established the corresponding inequalities for Lebesgue  $p$ -norms with  $p \geq 1$  as well as have provided some applications for cumulative distribution functions and some quadrature rules.

For a monograph devoted to Ostrowski type inequalities, see [20].

For research papers on Ostrowski's inequality see [1]-[19], [21]-[23] and [25].

Motivated by the above results, we investigate in this paper some perturbed versions of the inequality (1.1). Applications for cumulative distribution function are provided as well.

## 2 Some Identities

The following identity holds.

**Lemma 3.** *Assume that  $f : [a, b] \rightarrow \mathbb{C}$  is an absolutely continuous function*

on  $[a, b]$ . Then we have the equality

$$\begin{aligned}
 & \frac{1}{2} [f(x) + f(a+b-x)] + \frac{1}{2} (x-a)^2 \frac{\lambda_3(x) - \lambda_1(x)}{b-a} - \frac{1}{b-a} \int_a^b f(t) dt \\
 & = \frac{1}{b-a} \int_a^x (t-a) [f'(t) - \lambda_1(x)] dt \\
 & + \frac{1}{b-a} \int_x^{a+b-x} \left(t - \frac{a+b}{2}\right) [f'(t) - \lambda_2(x)] dt \\
 & + \frac{1}{b-a} \int_{a+b-x}^b (t-b) [f'(t) - \lambda_3(x)] dt,
 \end{aligned} \tag{2.1}$$

for any  $x \in [a, \frac{a+b}{2}]$  and  $\lambda_j(x)$ ,  $j = 1, 2, 3$  complex numbers.

*Proof.* Using the integration by parts formula for Lebesgue integral, we have

$$\begin{aligned}
 & \int_a^x (t-a) [f'(t) - \lambda_1(x)] dt \\
 & = \int_a^x (t-a) f'(t) dt - \lambda_1(x) \int_a^x (t-a) dt \\
 & = (x-a) f(x) - \int_a^x f(t) dt - \frac{1}{2} \lambda_1(x) (x-a)^2,
 \end{aligned}$$

$$\begin{aligned}
 & \int_x^{a+b-x} \left(t - \frac{a+b}{2}\right) [f'(t) - \lambda_2(x)] dt \\
 & = \int_x^{a+b-x} \left(t - \frac{a+b}{2}\right) f'(t) dt - \lambda_2(x) \int_x^{a+b-x} \left(t - \frac{a+b}{2}\right) dt \\
 & = f(a+b-x) \left(\frac{a+b}{2} - x\right) - f(x) \left(x - \frac{a+b}{2}\right) - \int_x^{a+b-x} f(t) dt \\
 & - \lambda_2(x) \int_x^{a+b-x} \left(t - \frac{a+b}{2}\right) dt \\
 & = f(a+b-x) \left(\frac{a+b}{2} - x\right) - f(x) \left(x - \frac{a+b}{2}\right) - \int_x^{a+b-x} f(t) dt,
 \end{aligned}$$

since, by symmetry

$$\int_x^{a+b-x} \left(t - \frac{a+b}{2}\right) dt = 0$$

and

$$\begin{aligned} & \int_{a+b-x}^b (t-b) [f'(t) - \lambda_3(x)] dt \\ &= \int_{a+b-x}^b (t-b) f'(t) dt - \lambda_3(x) \int_{a+b-x}^b (t-b) dt \\ &= (x-a) f(a+b-x) - \int_{a+b-x}^b f(t) dt + \frac{1}{2} \lambda_3(x) (x-a)^2. \end{aligned}$$

Summing the above equalities, we deduce

$$\begin{aligned} & \int_a^x (t-a) [f'(t) - \lambda_1(x)] dt + \int_x^{a+b-x} \left(t - \frac{a+b}{2}\right) [f'(t) - \lambda_2(x)] dt \\ &+ \int_{a+b-x}^b (t-b) [f'(t) - \lambda_3(x)] dt \\ &= (b-a) \frac{f(x) + f(a+b-x)}{2} - \int_a^b f(t) dt + \frac{1}{2} [\lambda_3(x) - \lambda_1(x)] (x-a)^2, \end{aligned}$$

which is equivalent with the desired identity (2.1). ■

The following particular cases are of interest:

**Corollary 4.** *With the assumption of Lemma 3 we have the equalities*

$$\frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(t) dt = \frac{1}{b-a} \int_a^b \left(t - \frac{a+b}{2}\right) [f'(t) - \lambda_2] dt, \quad (2.2)$$

$$\begin{aligned} & f\left(\frac{a+b}{2}\right) + \frac{1}{8} (b-a) (\lambda_3 - \lambda_1) - \frac{1}{b-a} \int_a^b f(t) dt \\ &= \frac{1}{b-a} \int_a^{\frac{a+b}{2}} (t-a) [f'(t) - \lambda_1] dt + \frac{1}{b-a} \int_{\frac{a+b}{2}}^b (t-b) [f'(t) - \lambda_3] dt, \end{aligned} \quad (2.3)$$

and

$$\begin{aligned}
& \frac{1}{2} \left[ f \left( \frac{3a+b}{4} \right) + f \left( \frac{a+3b}{4} \right) \right] + \frac{1}{32} (b-a) (\lambda_3 - \lambda_1) - \frac{1}{b-a} \int_a^b f(t) dt \\
& = \frac{1}{b-a} \int_a^{\frac{3a+b}{4}} (t-a) [f'(t) - \lambda_1] dt \\
& + \frac{1}{b-a} \int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} \left( t - \frac{a+b}{2} \right) [f'(t) - \lambda_2] dt \\
& + \frac{1}{b-a} \int_{\frac{a+3b}{4}}^b (t-b) [f'(t) - \lambda_3] dt
\end{aligned} \tag{2.4}$$

for any  $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}$ .

The following particular result with no parameter in the left hand term holds:

**Corollary 5.** Assume that  $f : [a, b] \rightarrow \mathbb{C}$  is absolutely continuous on  $[a, b]$ . Then we have the equality

$$\begin{aligned}
& \frac{1}{2} [f(x) + f(a+b-x)] - \frac{1}{b-a} \int_a^b f(t) dt \\
& = \frac{1}{b-a} \int_a^x (t-a) [f'(t) - \lambda_1(x)] dt \\
& + \frac{1}{b-a} \int_x^{a+b-x} \left( t - \frac{a+b}{2} \right) [f'(t) - \lambda_2(x)] dt \\
& + \frac{1}{b-a} \int_{a+b-x}^b (t-b) [f'(t) - \lambda_1(x)] dt,
\end{aligned} \tag{2.5}$$

for any  $x \in [a, \frac{a+b}{2}]$  and  $\lambda_i(x), i = 1, 2$  complex numbers.

**Remark 1.** We get from (2.3) the following particular case:

$$\begin{aligned}
& f \left( \frac{a+b}{2} \right) - \frac{1}{b-a} \int_a^b f(t) dt \\
& = \frac{1}{b-a} \int_a^{\frac{a+b}{2}} (t-a) [f'(t) - \lambda_1] dt + \frac{1}{b-a} \int_{\frac{a+b}{2}}^b (t-b) [f'(t) - \lambda_1] dt,
\end{aligned} \tag{2.6}$$

for any  $\lambda_1 \in \mathbb{C}$ , while from (2.4) we get

$$\begin{aligned} & \frac{1}{2} \left[ f \left( \frac{3a+b}{4} \right) + f \left( \frac{a+3b}{4} \right) \right] - \frac{1}{b-a} \int_a^b f(t) dt \\ &= \frac{1}{b-a} \int_a^{\frac{3a+b}{4}} (t-a) [f'(t) - \lambda_1] dt \\ &+ \frac{1}{b-a} \int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} \left( t - \frac{a+b}{2} \right) [f'(t) - \lambda_2] dt \\ &+ \frac{1}{b-a} \int_{\frac{a+3b}{4}}^b (t-b) [f'(t) - \lambda_1] dt \end{aligned} \quad (2.7)$$

for any  $\lambda_1, \lambda_2 \in \mathbb{C}$ .

### 3 Inequalities for Bounded Derivatives

Now, for  $\gamma, \Gamma \in \mathbb{C}$  and  $[a, b]$  an interval of real numbers, define the sets of complex-valued functions

$$\begin{aligned} & \bar{U}_{[a,b]}(\gamma, \Gamma) \\ &:= \left\{ f : [a, b] \rightarrow \mathbb{C} \mid \operatorname{Re} \left[ (\Gamma - f(t)) (\overline{f(t)} - \bar{\gamma}) \right] \geq 0 \text{ for almost every } t \in [a, b] \right\} \end{aligned}$$

and

$$\bar{\Delta}_{[a,b]}(\gamma, \Gamma) := \left\{ f : [a, b] \rightarrow \mathbb{C} \mid \left| f(t) - \frac{\gamma + \Gamma}{2} \right| \leq \frac{1}{2} |\Gamma - \gamma| \text{ for a.e. } t \in [a, b] \right\}.$$

The following representation result may be stated.

**Proposition 6.** For any  $\gamma, \Gamma \in \mathbb{C}$ ,  $\gamma \neq \Gamma$ , we have that  $\bar{U}_{[a,b]}(\gamma, \Gamma)$  and  $\bar{\Delta}_{[a,b]}(\gamma, \Gamma)$  are nonempty, convex and closed sets and

$$\bar{U}_{[a,b]}(\gamma, \Gamma) = \bar{\Delta}_{[a,b]}(\gamma, \Gamma). \quad (3.1)$$

*Proof.* We observe that for any  $z \in \mathbb{C}$  we have the equivalence

$$\left| z - \frac{\gamma + \Gamma}{2} \right| \leq \frac{1}{2} |\Gamma - \gamma|$$

if and only if

$$\operatorname{Re} [(\Gamma - z) (\bar{z} - \bar{\gamma})] \geq 0.$$



This follows by the equality

$$\frac{1}{4} |\Gamma - \gamma|^2 - \left| z - \frac{\gamma + \Gamma}{2} \right|^2 = \operatorname{Re} [(\Gamma - z)(\bar{z} - \bar{\gamma})]$$

that holds for any  $z \in \mathbb{C}$ .

The equality (3.1) is thus a simple consequence of this fact. ■

On making use of the complex numbers field properties we can also state that:

**Corollary 7.** *For any  $\gamma, \Gamma \in \mathbb{C}$ ,  $\gamma \neq \Gamma$ , we have that*

$$\begin{aligned} \bar{U}_{[a,b]}(\gamma, \Gamma) = \{f : [a, b] \rightarrow \mathbb{C} \mid & (\operatorname{Re} \Gamma - \operatorname{Re} f(t))(\operatorname{Re} f(t) - \operatorname{Re} \gamma) \\ & + (\operatorname{Im} \Gamma - \operatorname{Im} f(t))(\operatorname{Im} f(t) - \operatorname{Im} \gamma) \geq 0 \text{ for a.e. } t \in [a, b]\}. \end{aligned} \quad (3.2)$$

Now, if we assume that  $\operatorname{Re}(\Gamma) \geq \operatorname{Re}(\gamma)$  and  $\operatorname{Im}(\Gamma) \geq \operatorname{Im}(\gamma)$ , then we can define the following set of functions as well:

$$\begin{aligned} \bar{S}_{[a,b]}(\gamma, \Gamma) := \{f : [a, b] \rightarrow \mathbb{C} \mid & \operatorname{Re}(\Gamma) \geq \operatorname{Re} f(t) \geq \operatorname{Re}(\gamma) \\ & \text{and } \operatorname{Im}(\Gamma) \geq \operatorname{Im} f(t) \geq \operatorname{Im}(\gamma) \text{ for a.e. } t \in [a, b]\}. \end{aligned} \quad (3.3)$$

One can easily observe that  $\bar{S}_{[a,b]}(\gamma, \Gamma)$  is closed, convex and

$$\emptyset \neq \bar{S}_{[a,b]}(\gamma, \Gamma) \subseteq \bar{U}_{[a,b]}(\gamma, \Gamma). \quad (3.4)$$

**Theorem 8.** *Assume that  $f : [a, b] \rightarrow \mathbb{C}$  is an absolutely continuous function on  $[a, b]$  and  $x \in [a, \frac{a+b}{2}]$ . If there exist the complex numbers  $\gamma_j(x) \neq \Gamma_j(x)$ ,  $j = 1, 2, 3$  such that*

$$\begin{aligned} f' \in \bar{\Delta}_{[a,x]}(\gamma_1(x), \Gamma_1(x)) \cap \bar{\Delta}_{[x,a+b-x]}(\gamma_2(x), \Gamma_2(x)) \\ \cap \bar{\Delta}_{[a+b-x,b]}(\gamma_3(x), \Gamma_3(x)), \end{aligned} \quad (3.5)$$

then we have the inequality

$$\begin{aligned} & \left| \frac{1}{2} [f(x) + f(a+b-x)] + \frac{1}{4} (x-a)^2 \frac{\gamma_3(x) + \Gamma_3(x) - \gamma_1(x) - \Gamma_1(x)}{b-a} \right. \\ & \left. - \frac{1}{b-a} \int_a^b f(t) dt \right| \\ & \leq \frac{1}{4(b-a)} \left[ |\Gamma_1(x) - \gamma_1(x)| (x-a)^2 + 2 |\Gamma_2(x) - \gamma_2(x)| \left( \frac{a+b}{2} - x \right)^2 \right. \\ & \left. + |\Gamma_3(x) - \gamma_3(x)| (x-a)^2 \right]. \end{aligned} \quad (3.6)$$

*Proof.* Taking the modulus in the equality (2.1) written for

$$\begin{aligned}\lambda_1(x) &= \frac{\gamma_1(x) + \Gamma_1(x)}{2}, \lambda_2(x) = \frac{\gamma_2(x) + \Gamma_2(x)}{2}, \\ \lambda_3(x) &= \frac{\gamma_3(x) + \Gamma_3(x)}{2}\end{aligned}$$

and utilizing the condition (3.5) we have

$$\begin{aligned}& \left| \frac{1}{2} [f(x) + f(a+b-x)] + \frac{1}{4} (x-a)^2 \frac{\gamma_3(x) + \Gamma_3(x) - \gamma_1(x) - \Gamma_1(x)}{b-a} \right. \\ & \left. - \frac{1}{b-a} \int_a^b f(t) dt \right| \\ & \leq \frac{1}{b-a} \int_a^x (t-a) \left| f'(t) - \frac{\gamma_1(x) + \Gamma_1(x)}{2} \right| dt \\ & + \frac{1}{b-a} \int_x^{a+b-x} \left| t - \frac{a+b}{2} \right| \left| f'(t) - \frac{\gamma_2(x) + \Gamma_2(x)}{2} \right| dt \\ & + \frac{1}{b-a} \int_{a+b-x}^b (b-t) \left| f'(t) - \frac{\gamma_3(x) + \Gamma_3(x)}{2} \right| dt \\ & \leq \frac{1}{4(b-a)} |\Gamma_1(x) - \gamma_1(x)| (x-a)^2 \\ & + \frac{2}{4(b-a)} |\Gamma_2(x) - \gamma_2(x)| \left( \frac{a+b}{2} - x \right)^2 \\ & + \frac{1}{4(b-a)} |\Gamma_3(x) - \gamma_3(x)| (x-a)^2\end{aligned}$$

and the inequality (3.6) is proved. ■

**Corollary 9.** Assume that  $f : [a, b] \rightarrow \mathbb{C}$  is an absolutely continuous function on  $[a, b]$  and  $x \in [a, \frac{a+b}{2}]$ . If there exist the complex numbers  $\gamma_j(x) \neq \Gamma_j(x)$ ,  $j = 1, 2$  such that

$$\begin{aligned}f' &\in \bar{\Delta}_{[a,x]}(\gamma_1(x), \Gamma_1(x)) \cap \bar{\Delta}_{[x,a+b-x]}(\gamma_2(x), \Gamma_2(x)) \\ &\cap \bar{\Delta}_{[a+b-x,b]}(\gamma_1(x), \Gamma_1(x)),\end{aligned}\quad (3.7)$$

then we have the inequality

$$\begin{aligned}& \left| \frac{1}{2} [f(x) + f(a+b-x)] - \frac{1}{b-a} \int_a^b f(t) dt \right| \\ & \leq \frac{1}{2(b-a)} \left[ |\Gamma_1(x) - \gamma_1(x)| (x-a)^2 + |\Gamma_2(x) - \gamma_2(x)| \left( \frac{a+b}{2} - x \right)^2 \right].\end{aligned}\quad (3.8)$$

**Remark 2.** Assume that  $f : [a, b] \rightarrow \mathbb{C}$  is an absolutely continuous function on  $[a, b]$ .

If there exist the complex numbers  $\gamma_2 \neq \Gamma_2$  such that  $f' \in \bar{\Delta}_{[a,b]}(\gamma_2, \Gamma_2)$ , then

$$\left| \frac{1}{2} [f(a) + f(b)] - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{1}{8} (b-a) |\Gamma_2 - \gamma_2|. \quad (3.9)$$

If there exist the complex numbers  $\gamma_j \neq \Gamma_j$ ,  $j = 1, 3$  such that

$$f' \in \bar{\Delta}_{[a, \frac{a+b}{2}]}(\gamma_1, \Gamma_1) \cap \bar{\Delta}_{[\frac{a+b}{2}, b]}(\gamma_3, \Gamma_3),$$

then we have the inequality

$$\begin{aligned} & \left| f\left(\frac{a+b}{2}\right) + \frac{1}{16} (b-a) (\Gamma_3 + \gamma_3 - \Gamma_1 - \gamma_1) - \frac{1}{b-a} \int_a^b f(t) dt \right| \quad (3.10) \\ & \leq \frac{1}{16} (b-a) [|\Gamma_1 - \gamma_1| + |\Gamma_3 - \gamma_3|]. \end{aligned}$$

In particular, if  $f' \in \bar{\Delta}_{[a,b]}(\gamma_1, \Gamma_1)$  then

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{1}{8} (b-a) |\Gamma_1 - \gamma_1|. \quad (3.11)$$

If there exist the complex numbers  $\gamma_j \neq \Gamma_j$ ,  $j = 1, 2, 3$  such that

$$f' \in \bar{\Delta}_{[a, \frac{3a+b}{4}]}(\gamma_1, \Gamma_1) \cap \bar{\Delta}_{[\frac{3a+b}{4}, \frac{a+3b}{4}]}(\gamma_2, \Gamma_2) \cap \bar{\Delta}_{[\frac{a+3b}{4}, b]}(\gamma_3, \Gamma_3),$$

then

$$\begin{aligned} & \left| \frac{1}{2} \left[ f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right) \right] + \frac{1}{64} (b-a) (\Gamma_3 + \gamma_3 - \Gamma_1 - \gamma_1) \right. \quad (3.12) \\ & \left. - \frac{1}{b-a} \int_a^b f(t) dt \right| \\ & \leq \frac{1}{64} (b-a) [|\Gamma_1 - \gamma_1| + 2|\Gamma_2 - \gamma_2| + |\Gamma_3 - \gamma_3|]. \end{aligned}$$

In particular, if  $\gamma_3 = \Gamma_3$  and  $\Gamma_3 = \Gamma_1$ , then

$$\begin{aligned} & \left| \frac{1}{2} \left[ f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right) \right] - \frac{1}{b-a} \int_a^b f(t) dt \right| \quad (3.13) \\ & \leq \frac{1}{32} (b-a) [|\Gamma_1 - \gamma_1| + |\Gamma_2 - \gamma_2|], \end{aligned}$$

provided

$$f' \in \bar{\Delta}_{[a, \frac{3a+b}{4}]}(\gamma_1, \Gamma_1) \cap \bar{\Delta}_{[\frac{3a+b}{4}, \frac{a+3b}{4}]}(\gamma_2, \Gamma_2) \cap \bar{\Delta}_{[\frac{a+3b}{4}, b]}(\gamma_1, \Gamma_1).$$

Moreover, if  $f' \in \bar{\Delta}_{[a, b]}(\gamma_1, \Gamma_1)$  then

$$\begin{aligned} & \left| \frac{1}{2} \left[ f \left( \frac{3a+b}{4} \right) + f \left( \frac{a+3b}{4} \right) \right] - \frac{1}{b-a} \int_a^b f(t) dt \right| \\ & \leq \frac{1}{16} (b-a) |\Gamma_1 - \gamma_1|. \end{aligned} \quad (3.14)$$

The case of real-valued functions is of interest.

**Remark 3.** If the function  $f : [a, b] \rightarrow \mathbb{R}$  is absolutely continuous and if there exist the constants  $l < L$  such that  $l \leq f'(t) \leq L$  for almost every  $t \in [a, b]$ , then we have the inequalities

$$\left| \frac{1}{2} [f(a) + f(b)] - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{1}{8} (b-a) (L-l), \quad (3.15)$$

$$\left| f \left( \frac{a+b}{2} \right) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{1}{8} (b-a) (L-l) \quad (3.16)$$

and

$$\begin{aligned} & \left| \frac{1}{2} \left[ f \left( \frac{3a+b}{4} \right) + f \left( \frac{a+3b}{4} \right) \right] - \frac{1}{b-a} \int_a^b f(t) dt \right| \\ & \leq \frac{1}{16} (b-a) (L-l). \end{aligned} \quad (3.17)$$

These results improve the corresponding inequalities from Introduction.

## 4 Inequalities for Derivatives of Bounded Variation

Assume that  $f : I \rightarrow \mathbb{C}$  is an absolutely continuous function on  $[a, b] \subset \overset{\circ}{I}$ , the interior of  $I$ . Then from (2.1) we have for  $\lambda_1(x) = f'(a)$ ,  $\lambda_2(x) =$

$\frac{f'(x)+f'(a+b-x)}{2}$  and  $\lambda_3(x) = f'(b)$  the equality

$$\begin{aligned} & \frac{1}{2}[f(x) + f(a+b-x)] + \frac{1}{2}(x-a)^2 \frac{f'(b) - f'(a)}{b-a} - \frac{1}{b-a} \int_a^b f(t) dt \\ &= \frac{1}{b-a} \int_a^x (t-a)[f'(t) - f'(a)] dt \\ &+ \frac{1}{b-a} \int_x^{a+b-x} \left(t - \frac{a+b}{2}\right) \left[f'(t) - \frac{f'(x) + f'(a+b-x)}{2}\right] dt \\ &+ \frac{1}{b-a} \int_{a+b-x}^b (t-b)[f'(t) - f'(b)] dt, \end{aligned} \quad (4.1)$$

for any  $x \in [a, \frac{a+b}{2}]$ .

We can state the following result.

**Theorem 10.** Assume that  $f : I \rightarrow \mathbb{C}$  is an absolutely continuous function on  $[a, b] \subset \overset{\circ}{I}$ . If the derivative  $f'$  is of bounded variation on  $[a, b]$ , then

$$\begin{aligned} & \left| \frac{1}{2}[f(x) + f(a+b-x)] + \frac{1}{2}(x-a)^2 \frac{f'(b) - f'(a)}{b-a} - \frac{1}{b-a} \int_a^b f(t) dt \right| \\ & \leq \frac{1}{b-a} \int_a^x (t-a) \bigvee_a^t(f') dt + \frac{1}{2(b-a)} \left(x - \frac{a+b}{2}\right)^2 \bigvee_x^{a+b-x}(f') \\ & + \frac{1}{b-a} \int_{a+b-x}^b (b-t) \bigvee_t^b(f') dt \\ & \leq \frac{1}{2(b-a)} \left[ (x-a)^2 \bigvee_a^x(f') + \left(x - \frac{a+b}{2}\right)^2 \bigvee_x^{a+b-x}(f') + (x-a)^2 \bigvee_{a+b-x}^b(f') \right] \\ & \leq \begin{cases} \frac{1}{2(b-a)} \max \left\{ (x-a)^2, \left(x - \frac{a+b}{2}\right)^2 \right\} \bigvee_a^b(f') \\ \frac{1}{2(b-a)} \max \left\{ \bigvee_a^x(f'), \bigvee_x^{a+b-x}(f'), \bigvee_{a+b-x}^b(f') \right\} \left[ 2(x-a)^2 + \left(x - \frac{a+b}{2}\right)^2 \right] \end{cases} \end{aligned} \quad (4.2)$$

for any  $x \in [a, \frac{a+b}{2}]$ .

*Proof.* If we take the modulus in (4.1) we get

$$\begin{aligned}
 & \left| \frac{1}{2} [f(x) + f(a+b-x)] + \frac{1}{2} (x-a)^2 \frac{f'(b) - f'(a)}{b-a} - \frac{1}{b-a} \int_a^b f(t) dt \right| \\
 & \leq \frac{1}{b-a} \int_a^x (t-a) |f'(t) - f'(a)| dt \\
 & + \frac{1}{b-a} \int_x^{a+b-x} \left| t - \frac{a+b}{2} \right| \left| f'(t) - \frac{f'(x) + f'(a+b-x)}{2} \right| dt \\
 & + \frac{1}{b-a} \int_{a+b-x}^b (b-t) |f'(t) - f'(b)| dt := K,
 \end{aligned} \tag{4.3}$$

for any  $x \in [a, \frac{a+b}{2}]$ .

Let  $x \in (a, \frac{a+b}{2})$ . Since  $f'$  is of bounded variation on  $[a, b]$ , then

$$|f'(t) - f'(a)| \leq \bigvee_a^t(f'),$$

for any  $t \in [a, x]$  and

$$\begin{aligned}
 & \left| f'(t) - \frac{f'(x) + f'(a+b-x)}{2} \right| \\
 & = \left| \frac{f'(t) - f'(x) + f'(t) - f'(a+b-x)}{2} \right| \\
 & \leq \frac{1}{2} [|f'(t) - f'(x)| + |f'(a+b-x) - f'(t)|] \leq \frac{1}{2} \bigvee_x^{a+b-x}(f')
 \end{aligned}$$

for any  $t \in [x, a+b-x]$ .

We also have

$$|f'(t) - f'(b)| \leq \bigvee_t^b(f'), \quad t \in [a+b-x, b].$$

Then we get

$$\begin{aligned}
K &\leq \frac{1}{b-a} \int_a^x (t-a) \bigvee_a^t (f') dt + \frac{1}{2(b-a)} \bigvee_x^{a+b-x} (f') \int_x^{a+b-x} \left| t - \frac{a+b}{2} \right| dt \\
&\quad + \frac{1}{b-a} \int_{a+b-x}^b (b-t) \bigvee_t^b (f') dt \\
&\leq \frac{1}{b-a} \bigvee_a^x (f') \int_a^x (t-a) dt + \frac{1}{2(b-a)} \bigvee_x^{a+b-x} (f') \int_x^{a+b-x} \left| t - \frac{a+b}{2} \right| dt \\
&\quad + \frac{1}{b-a} \bigvee_{a+b-x}^b (f') \int_{a+b-x}^b (b-t) dt \\
&= \frac{1}{2(b-a)} (x-a)^2 \bigvee_a^x (f') + \frac{1}{2(b-a)} \left( x - \frac{a+b}{2} \right)^2 \bigvee_x^{a+b-x} (f') \\
&\quad + \frac{1}{2(b-a)} (x-a)^2 \bigvee_{a+b-x}^b (f'),
\end{aligned}$$

which proves the first two inequalities in (4.2).

The last part is obvious by the maximum properties. ■

**Corollary 11.** *With the assumptions of Theorem 10 we have*

$$\left| \frac{1}{2} [f(a) + f(b)] - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{1}{8} (b-a) \bigvee_a^b (f'), \quad (4.4)$$

$$\begin{aligned}
&\left| f\left(\frac{a+b}{2}\right) + \frac{1}{8} (b-a) [f'(b) - f'(a)] - \frac{1}{b-a} \int_a^b f(t) dt \right| \quad (4.5) \\
&\leq \frac{1}{b-a} \int_a^{\frac{a+b}{2}} (t-a) \bigvee_a^t (f') dt + \frac{1}{b-a} \int_{\frac{a+b}{2}}^b (b-t) \bigvee_t^b (f') dt \\
&\leq \frac{1}{8} (b-a) \bigvee_a^b (f')
\end{aligned}$$

and

$$\begin{aligned}
& \left| \frac{1}{2} \left[ f \left( \frac{3a+b}{4} \right) + f \left( \frac{a+3b}{4} \right) \right] + \frac{1}{32} (b-a) [f'(b) - f'(a)] \right. \\
& \left. - \frac{1}{b-a} \int_a^b f(t) dt \right| \\
& \leq \frac{1}{b-a} \int_a^{\frac{3a+b}{4}} (t-a) \bigvee_a^t(f') dt + \frac{1}{32} (b-a) \bigvee_{\frac{3a+b}{4}}^{\frac{a+3b}{4}}(f') \\
& + \frac{1}{b-a} \int_{\frac{a+3b}{4}}^b (b-t) \bigvee_t^b(f') dt \\
& \leq \frac{1}{32} (b-a) \bigvee_a^b(f').
\end{aligned} \tag{4.6}$$

## 5 Applications for PDF

Now, let  $X$  be a random variable taking values in the finite interval  $[a, b]$ , with the *probability density function* (PDF)  $f : [a, b] \rightarrow [0, \infty)$  and with the *cumulative distribution function* (CDF)  $F(x) = \Pr(X \leq x) = \int_a^x f(t) dt$ . We know that  $F$  is monotonic nondecreasing and absolutely continuous on  $[a, b]$ ,  $F' = f$  almost everywhere on  $[a, b]$  and  $F(a) = 0$ ,  $F(b) = \int_a^b f(t) dt = 1$ .

Assume that  $g : [a, b] \rightarrow \mathbb{R}$  is an absolutely continuous function on  $[a, b]$  and there exist the constants  $m < M$  such that

$$m \leq g'(t) \leq M \text{ for almost every } t \in [a, b]$$

then, by Corollary 9, we have the inequality

$$\begin{aligned}
& \left| \frac{1}{2} [g(x) + g(a+b-x)] - \frac{1}{b-a} \int_a^b g(t) dt \right| \\
& \leq \frac{1}{2} (M-m) \left[ \frac{1}{8} + 2 \left( \frac{x - \frac{3a+b}{4}}{b-a} \right)^2 \right] (b-a)
\end{aligned} \tag{5.1}$$

for any  $x \in [a, \frac{a+b}{2}]$ .



**Proposition 12.** Let  $X$  be a random variable taking values in the finite interval  $[a, b]$ , with PDF  $f : [a, b] \rightarrow [0, \infty)$  and with CDF  $F(x) = \Pr(X \leq x) = \int_a^x f(t) dt$ . If there exist the constants  $m < M$  such that

$$m \leq f(t) \leq M \text{ for almost every } t \in [a, b]$$

then,

$$\begin{aligned} & \left| \frac{1}{2} [F(x) + F(a+b-x)] - \frac{b - E(X)}{b-a} \right| \\ & \leq \frac{1}{2} (M - m) \left[ \frac{1}{8} + 2 \left( \frac{x - \frac{3a+b}{4}}{b-a} \right)^2 \right] (b-a) \end{aligned} \quad (5.2)$$

for any  $x \in [a, \frac{a+b}{2}]$ , where  $E(X) = \int_a^b t dF(t)$  is the expectation of  $X$ .

*Proof.* Follows from (5.1) for  $g = F$  and by taking into account that

$$\int_a^b F(t) dt = b - E(X).$$

■

**Corollary 13.** With the assumptions in Proposition 12, we have

$$\left| \frac{1}{2} \left[ F\left(\frac{3a+b}{4}\right) + F\left(\frac{a+3b}{4}\right) \right] - \frac{b - E(X)}{b-a} \right| \leq \frac{1}{16} (M - m) (b - a) \quad (5.3)$$

Utilising Theorem 10 we can also state:

**Proposition 14.** If PDF  $f : [a, b] \rightarrow [0, \infty)$  is of bounded variation on  $[a, b]$ ,

then

$$\begin{aligned}
& \left| \frac{1}{2} [F(x) + F(a+b-x)] + \frac{1}{2} (x-a)^2 \frac{f(b) - f(a)}{b-a} - \frac{b - E(X)}{b-a} \right| \quad (5.4) \\
& \leq \frac{1}{b-a} \int_a^x (t-a) \mathop{\mathbb{V}}_a^t(f) dt + \frac{1}{2(b-a)} \left(x - \frac{a+b}{2}\right)^{2a+b-x} \mathop{\mathbb{V}}_x^{a+b-x}(f) \\
& + \frac{1}{b-a} \int_{a+b-x}^b (b-t) \mathop{\mathbb{V}}_t^b(f) dt \\
& \leq \frac{1}{2(b-a)} \\
& \times \left[ (x-a)^2 \mathop{\mathbb{V}}_a^x(f) + \left(x - \frac{a+b}{2}\right)^{2a+b-x} \mathop{\mathbb{V}}_x^{a+b-x}(f) + (x-a)^2 \mathop{\mathbb{V}}_{a+b-x}^b(f) \right] \\
& \leq \frac{1}{2(b-a)} \\
& \times \begin{cases} \max \left\{ (x-a)^2, \left(x - \frac{a+b}{2}\right)^2 \right\} \mathop{\mathbb{V}}_a^b(f), \\ \max \left\{ \mathop{\mathbb{V}}_a^x(f), \mathop{\mathbb{V}}_x^{a+b-x}(f), \mathop{\mathbb{V}}_{a+b-x}^b(f) \right\} \left[ 2(x-a)^2 + \left(x - \frac{a+b}{2}\right)^2 \right] \end{cases}
\end{aligned}$$

for any  $x \in [a, \frac{a+b}{2}]$ .

Finally we have:

**Corollary 15.** *With the assumptions in Proposition 14, we have*

$$\begin{aligned}
 & \left| \frac{1}{2} \left[ F \left( \frac{3a+b}{4} \right) + F \left( \frac{a+3b}{4} \right) \right] + \frac{1}{32} (b-a) [f(b) - f(a)] \right. \\
 & \quad \left. - \frac{b - E(X)}{b-a} \right| \\
 & \leq \frac{1}{b-a} \int_a^{\frac{3a+b}{4}} (t-a) \bigvee_a^t(f) dt + \frac{1}{32} (b-a) \bigvee_{\frac{3a+b}{4}}^{\frac{a+3b}{4}}(f) \\
 & \quad + \frac{1}{b-a} \int_{\frac{a+3b}{4}}^b (b-t) \bigvee_t^b(f) dt \\
 & \leq \frac{1}{32} (b-a) \bigvee_a^b(f).
 \end{aligned} \tag{5.5}$$

## References

- [1] **G. A. Anastassiou**, Univariate Ostrowski inequalities, revisited, *Monatsh. Math.*, (2002), 175–189.
- [2] **P. Cerone and S. S. Dragomir**, *Midpoint-type rules from an inequalities point of view*, Ed. G. A. Anastassiou, Handbook of Analytic-Computational Methods in Applied Mathematics, CRC Press, New York, 135-200, 2000.
- [3] **P. Cerone and S. S. Dragomir**, *New bounds for the three-point rule involving the Riemann-Stieltjes integrals*, Advances in Statistics Combinatorics and Related Areas, C. Gulati, et al., World Science Publishing, 53-62, 2002.
- [4] **P. Cerone, S. S. Dragomir, and C. E. M. Pearce**, A generalised trapezoid inequality for functions of bounded variation, *Turkish J. Math.*, (2000), 147-163.
- [5] **P. Cerone, S. S. Dragomir, and J. Roumeliotis**, Some Ostrowski type inequalities for time differentiable mappings and applications, *Demonstratio Mathematica*, (1999), 697–712.
- [6] **S. S. Dragomir**, Ostrowski's inequality for monotonous mappings and applications, *J. Korean Soc. Ind. Appl. Math.*, (1999), 127-135.
- [7] **S. S. Dragomir**, The Ostrowski's integral inequality for Lipschitzian mappings and applications, *Comp. Math. Appl.*, (1999), 33-37.
- [8] **S. S. Dragomir**, The Ostrowski integral inequality for mappings of bounded variation, *Bull. Austral. Math. Soc.*, (1999), 495-508.
- [9] **S. S. Dragomir**, On the midpoint quadrature formula for mappings with bounded variation and applications, *Kragujevac J. Math.*, (2000), 13-18.

- [10] **S. S. Dragomir**, On the Ostrowski's inequality for Riemann-Stieltjes integral, *Korean J. Appl. Math.*, (2000), 477-485.
- [11] **S. S. Dragomir**, On the Ostrowski's integral inequality for mappings with bounded variation and applications, *Math. Ineq. Appl.*, (2001), 33-40.
- [12] **S. S. Dragomir**, On the Ostrowski inequality for Riemann-Stieltjes integral where is of Holder type and is of bounded variation and applications, *J. Korean Soc. Ind. Appl. Math.*, (2001), 35-45.
- [13] **S. S. Dragomir**, Ostrowski type inequalities for isotonic linear functionals, *J. Inequal. Pure Appl. Math.*, **Art. 68.**, (2002)
- [14] **S. S. Dragomir**, An inequality improving the first Hermite-Hadamard inequality for convex functions defined on linear spaces and applications for semi-inner products, *J. Inequal. Pure Appl. Math.*, **no. 2** (Article 31), (2002), 8 pp.
- [15] **S. S. Dragomir**, A refinement of Ostrowski's inequality for absolutely continuous functions whose derivatives belong to and applications, *Libertas Math.*, (2002), 49-63.
- [16] **S. S. Dragomir**, Some companions of Ostrowski's inequality for absolutely continuous functions and applications. Preprint RGMIA Res. Rep. Coll., *Bull. Korean Math. Soc.*, **no. 2** (Suppl. Art. 29.), (2005), 213-230.
- [17] **S. S. Dragomir**, An Ostrowski like inequality for convex functions and applications, *Revista Math. Complutense*, (2003), 373-382.
- [18] **S. S. Dragomir**, *Operator Inequalities of Ostrowski and Trapezoidal Type*, Springer Briefs in Mathematics, Springer, New York, 2012
- [19] **S. S. Dragomir, P. Cerone, J. Roumeliotis, and S. Wang**, A weighted version of Ostrowski inequality for mappings of Holder type and applications in numerical analysis, *Bull. Math. Soc. Sci. Math.*, (1999), 301-314.
- [20] **S. S. Dragomir and Th. M. Rassias**, *Ostrowski Type Inequalities and Applications in Numerical Integration*, Kluwer Academic Publishers, Dordrecht/Boston/London, 2002.
- [21] **S. S. Dragomir and S. Wang**, A new inequality of Ostrowski's type in norm and applications to some special means and to some numerical quadrature rules, *Tamkang J. of Math.*, (1997), 239-244.
- [22] **S. S. Dragomir and S. Wang**, Applications of Ostrowski's inequality to the estimation of error bounds for some special means and some numerical quadrature rules, *Appl. Math. Lett.*, (1998), 105-109.
- [23] **S. S. Dragomir and S. Wang**, A new inequality of Ostrowski's type in norm and applications to some special means and to some numerical quadrature rules, *Indian J. of Math.*, (1998), 245-304.
- [24] **A. Guessab and G. Schmeisser**, Sharp integral inequalities of the Hermite-Hadamard type, *J. Approx. Th.*, (2002), 260-288.
- [25] **A. M. Fink**, Bounds on the deviation of a function from its averages, *Czechoslovak Math. J.*, **No. 2**, (1992), 298-310.
- [26] **A. Ostrowski**, Uber die Absolutabweichung einer differentienbaren Funktionen von ihren Integralmittelwert, *Comment. Math. Hel.*, (1938), 226-227.

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