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BOUNDING THE ČEBYŠEV FUNCTIONAL FOR A FUNCTION THAT IS CONVEX IN ABSOLUTE VALUE AND APPLICATIONS

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Abstract. Some sharp bounds for the Čebyšev functional of a function that is convex in absolute value and applications for functions of self-adjoint operators in Hilbert spaces via the spectral representation theorem are given.

Keywords: Čebyšev functional, self-adjoint operator, spectral representation theorem, integrable function, integral mean.

1. Introduction

For two Lebesgue integrable functions $f, g : [a, b] \rightarrow \mathbb{C}$, in order to compare the integral mean of the product with the product of the integral means, we consider the Čebyšev functional defined by

$$C(f, g) := \frac{1}{b-a} \int_a^b f(t)g(t) dt - \frac{1}{b-a} \int_a^b f(t) dt \cdot \frac{1}{b-a} \int_a^b g(t) dt.$$

In 1934, G. Grüss [14] showed that

$$(1.1) \quad |C(f, g)| \leq \frac{1}{4} (M - m)(N - n),$$

provided m, M, n, N are real numbers with the property that

$$(1.2) \quad -\infty < m \leq f \leq M < \infty, \quad -\infty < n \leq g \leq N < \infty \quad \text{a.e. on } [a, b].$$

The constant $\frac{1}{4}$ is best possible in (1.1) in the sense that it cannot be replaced by a smaller one.

Another lesser known inequality for $C(f, g)$ was derived in 1882 by Čebyšev [4] under the assumption that f', g' exist and are continuous on $[a, b]$, and is given by

$$(1.3) \quad |C(f, g)| \leq \frac{1}{12} \|f'\|_\infty \|g'\|_\infty (b - a)^2,$$

where $\|f'\|_\infty := \sup_{t \in [a,b]} |f'(t)| < \infty$.

The constant $\frac{1}{12}$ cannot be improved in general in (1.3).

Čebyšev's inequality (1.3) also holds if $f, g : [a, b] \rightarrow \mathbb{R}$ are assumed to be absolutely continuous and $f', g' \in L_\infty[a, b]$.

In 1970, A.M. Ostrowski [18] proved, amongst others, the following result that is in a sense a combination of the Čebyšev and Grüss results:

$$(1.4) \quad |C(f, g)| \leq \frac{1}{8} (b-a)(M-m) \|g'\|_\infty,$$

provided f is Lebesgue integrable on $[a, b]$ and satisfying (1.2) while $g : [a, b] \rightarrow \mathbb{R}$ is absolutely continuous and $g' \in L_\infty[a, b]$. Here the constant $\frac{1}{8}$ is also sharp.

In 1973, A. Lupuş [16] (see also [17, p. 210]) obtained the following result as well:

$$(1.5) \quad |C(f, g)| \leq \frac{1}{\pi^2} \|f'\|_2 \|g'\|_2 (b-a),$$

provided f, g are absolutely continuous and $f', g' \in L_2[a, b]$.

Here the constant $\frac{1}{\pi^2}$ is the best possible as well.

In [2], P. Cerone and S.S. Dragomir proved the following inequalities:

$$(1.6) \quad |C(f, g)| \leq \begin{cases} \inf_{\gamma \in \mathbb{R}} \|g - \gamma\|_\infty \cdot \frac{1}{b-a} \int_a^b \left| f(t) - \frac{1}{b-a} \int_a^b f(s) ds \right| dt, \\ \inf_{\gamma \in \mathbb{R}} \|g - \gamma\|_q \cdot \frac{1}{b-a} \left(\int_a^b \left| f(t) - \frac{1}{b-a} \int_a^b f(s) ds \right|^p dt \right)^{\frac{1}{p}} \\ \text{where } p > 1, 1/p + 1/q = 1. \end{cases}$$

For $\gamma = 0$, we get from the first inequality in (1.6)

$$(1.7) \quad |C(f, g)| \leq \|g\|_\infty \cdot \frac{1}{b-a} \int_a^b \left| f(t) - \frac{1}{b-a} \int_a^b f(s) ds \right| dt$$

for which the constant 1 cannot be replaced by a smaller constant.

If $m \leq g \leq M$ for a.e. $x \in [a, b]$, then $\|g - \frac{m+M}{2}\|_\infty \leq \frac{1}{2}(M-m)$ and by the first inequality in (1.6) we can deduce the following result obtained by Cheng and Sun [5]

$$(1.8) \quad |C(f, g)| \leq \frac{1}{2} (M-m) \cdot \frac{1}{b-a} \int_a^b \left| f(t) - \frac{1}{b-a} \int_a^b f(s) ds \right| dt.$$

The constant $\frac{1}{2}$ is best in (1.8) as shown by Cerone and Dragomir in [3].

The following result holds [11].

Theorem 1.1. Let $f : [a, b] \rightarrow \mathbb{C}$ be of bounded variation on $[a, b]$ and $g : [a, b] \rightarrow \mathbb{C}$ a Lebesgue integrable function on $[a, b]$. Then

$$(1.9) \quad |C(f, g)| \leq \frac{1}{2} \bigvee_a^b(f) \cdot \frac{1}{b-a} \int_a^b \left| g(t) - \frac{1}{b-a} \int_a^b g(s) ds \right| dt$$

where $\bigvee_a^b(f)$ denotes the total variation of f on the interval $[a, b]$.

The constant $\frac{1}{2}$ is best possible in (1.9).

We denote the variance of the function $f : [a, b] \rightarrow \mathbb{C}$ by $D(f)$ and defined as

$$(1.10) \quad D(f) = [C(f, \bar{f})]^{1/2} \left[\frac{1}{b-a} \int_a^b |f(t)|^2 dt - \left| \frac{1}{b-a} \int_a^b f(t) dt \right|^2 \right]^{1/2},$$

where \bar{f} denotes the complex conjugate function of f .

We have [11]:

Corollary 1.1. If the function $f : [a, b] \rightarrow \mathbb{C}$ is of bounded variation on $[a, b]$, then

$$(1.11) \quad D(f) \leq \frac{1}{2} \bigvee_a^b(f).$$

The constant $\frac{1}{2}$ is best possible in (1.11).

Now we can state the following result when both functions are of bounded variation [11]:

Corollary 1.2. If $f, g : [a, b] \rightarrow \mathbb{C}$ are of bounded variation on $[a, b]$, then

$$(1.12) \quad |C(f, g)| \leq \frac{1}{4} \bigvee_a^b(f) \bigvee_a^b(g).$$

The constant $\frac{1}{4}$ is best possible in (1.12).

Remark 1.1. We can consider the following quantity associated with a complex valued function $f : [a, b] \rightarrow \mathbb{C}$,

$$E(f) := |C(f, f)|^{1/2} = \left| \frac{1}{b-a} \int_a^b f^2(t) dt - \left(\frac{1}{b-a} \int_a^b f(t) dt \right)^2 \right|^{1/2}.$$

Utilising the above results we can state that

$$(1.13) \quad \begin{aligned} E^2(f) &\leq \frac{1}{2} \bigvee_a^b(f) \cdot \frac{1}{b-a} \int_a^b \left| f(t) - \frac{1}{b-a} \int_a^b f(s) ds \right| dt \\ &\leq \frac{1}{2} \bigvee_a^b(f) D(f) \leq \frac{1}{4} \left[\bigvee_a^b(f) \right]^2. \end{aligned}$$

If we consider

$$\begin{aligned} G(f) &:= \left| C(f, |f|) \right|^{1/2} \\ &= \left| \frac{1}{b-a} \int_a^b f(t) |f(t)| dt - \frac{1}{b-a} \int_a^b f(t) dt \cdot \frac{1}{b-a} \int_a^b |f(t)| dt \right|^{1/2}, \end{aligned}$$

then we also have

$$(1.14) \quad \begin{aligned} G^2(f) &\leq \frac{1}{2} \bigvee_a^b(f) \cdot \frac{1}{b-a} \int_a^b \left| |f(t)| - \frac{1}{b-a} \int_a^b |f(s)| ds \right| dt \\ &\leq \frac{1}{2} \bigvee_a^b(f) D(|f|) \leq \frac{1}{4} \bigvee_a^b(f) \bigvee_a^b(|f|) \leq \frac{1}{4} \left[\bigvee_a^b(f) \right]^2 \end{aligned}$$

and

$$(1.15) \quad \begin{aligned} G^2(f) &\leq \frac{1}{2} \bigvee_a^b(|f|) \cdot \frac{1}{b-a} \int_a^b \left| f(t) - \frac{1}{b-a} \int_a^b f(s) ds \right| dt \\ &\leq \frac{1}{2} \bigvee_a^b(|f|) D(f) \leq \frac{1}{4} \bigvee_a^b(f) \bigvee_a^b(|f|) \leq \frac{1}{4} \left[\bigvee_a^b(f) \right]^2. \end{aligned}$$

For recent related results see [1] and [9]-[13].

Motivated by the results presented above, we establish in this paper some new bounds for the magnitude of $C(f, g)$ in the case when one of the complex valued function, say f , is convex in absolute value while the other is Lebesgue integrable on $[a, b]$. Applications for functions of self-adjoint operators in Hilbert spaces via the spectral representation theorem are also given.

2. New Results for Čebyšev Functional

Recall that a function $g : [a, b] \rightarrow \mathbb{R}$ is *convex* (*strictly convex*) on the interval $[a, b]$, if

$$g((1-t)x + ty) \leq (<) (1-t)g(x) + tg(y)$$

for any $x, y \in [a, b]$ ($x \neq y$) and $t \in [0, 1]$ ($(0, 1)$).

We observe that the *constant function* $k(t) = k, t \in [a, b]$ and the *identity function* $e(t) = t, t \in [a, b]$ can then be interpreted as convex functions. However, they are not strictly convex functions on $[a, b]$.

We have the following result:

Theorem 2.1. Let $f : [a, b] \rightarrow \mathbb{C}$ be a measurable function such that $|f|$ is convex on $[a, b]$ and $g : [a, b] \rightarrow \mathbb{C}$ is a Lebesgue integrable function on $[a, b]$. Then

$$(2.1) \quad |C(f, g)| \leq \max\{|f(a)|, |f(b)|\} \frac{1}{b-a} \int_a^b \left| g(t) - \frac{1}{b-a} \int_a^b g(s) ds \right| dt.$$

The inequality (2.1) is sharp.

Proof. We use Sonin's identity

$$(2.2) \quad C(f, g) = \frac{1}{b-a} \int_a^b (f(t) - \lambda) \left[g(t) - \frac{1}{b-a} \int_a^b g(s) ds \right] dt,$$

for $\lambda = 0$ to get

$$(2.3) \quad C(f, g) = \frac{1}{b-a} \int_a^b f(t) \left[g(t) - \frac{1}{b-a} \int_a^b g(s) ds \right] dt.$$

Taking the modulus and utilizing the convexity of $|f|$ on $[a, b]$ we have

$$(2.4) \quad \begin{aligned} |C(f, g)| &\leq \frac{1}{b-a} \int_a^b |f(t)| \left| g(t) - \frac{1}{b-a} \int_a^b g(s) ds \right| dt \\ &\leq \frac{1}{b-a} \int_a^b \left[\frac{(b-t)|f(a)| + (t-a)|f(b)|}{b-a} \right] \\ &\quad \times \left| g(t) - \frac{1}{b-a} \int_a^b g(s) ds \right| dt \end{aligned}$$

If we denote the right side of (2.4) by I , then we have

$$\begin{aligned} I &\leq \sup_{t \in [a, b]} \left[\frac{(b-t)|f(a)| + (t-a)|f(b)|}{b-a} \right] \\ &\quad \times \frac{1}{b-a} \int_a^b \left| g(t) - \frac{1}{b-a} \int_a^b g(s) ds \right| dt \\ &= \max\{|f(a)|, |f(b)|\} \frac{1}{b-a} \int_a^b \left| g(t) - \frac{1}{b-a} \int_a^b g(s) ds \right| dt \end{aligned}$$

and by (2.4) we get (2.1).

Assume that the inequality (2.1) holds with a constant $K > 0$, namely

$$(2.5) \quad |C(f, g)| \leq K \max\{|f(a)|, |f(b)|\} \frac{1}{b-a} \int_a^b \left| g(t) - \frac{1}{b-a} \int_a^b g(s) ds \right| dt.$$

Consider the functions $f, g : [a, b] \rightarrow \mathbb{R}$ defined by

$$f(t) := \begin{cases} -1, & t \in \left[a, \frac{a+b}{2} \right] \\ 1, & t \in \left(\frac{a+b}{2}, b \right] \end{cases}$$

and $g : [a, b] \rightarrow \mathbb{R}, g(t) = t - \frac{a+b}{2}$.

We have $|f| = 1$, which satisfy the convexity condition with equality and

$$C(f, g) = \frac{1}{b-a} \int_a^b \left| t - \frac{a+b}{2} \right| dt = \frac{b-a}{4},$$

$$\max \{ |f(a)|, |f(b)| \} = 1$$

and

$$\frac{1}{b-a} \int_a^b \left| g(t) - \frac{1}{b-a} \int_a^b g(s) ds \right| dt = \frac{1}{b-a} \int_a^b \left| t - \frac{a+b}{2} \right| dt = \frac{b-a}{4}$$

and by (2.5) we have

$$\frac{b-a}{4} \leq K \frac{b-a}{4},$$

which shows that $K \geq 1$. \square

With the notations from the introduction we have:

Corollary 2.1. *Let $f : [a, b] \rightarrow \mathbb{C}$ be a measurable function such that $|f|$ is convex on $[a, b]$. Then*

$$(2.6) \quad \begin{aligned} & D^2(f), E^2(f) \\ & \leq \max \{ |f(a)|, |f(b)| \} \frac{1}{b-a} \int_a^b \left| f(t) - \frac{1}{b-a} \int_a^b f(s) ds \right| dt, \end{aligned}$$

and

$$(2.7) \quad G^2(f) \leq \max \{ |f(a)|, |f(b)| \} \frac{1}{b-a} \int_a^b \left| |f(t)| - \frac{1}{b-a} \int_a^b |f(s)| ds \right| dt.$$

We recall the p -logarithmic mean defined by

$$L_p^p(m, n) := \frac{m^{p+1} - n^{p+1}}{(p+1)(M-m)}, \quad m \neq n$$

where $p \neq -1, 0$ and $m, n > 0$.

The case of p -norm of the deviation

$$\left| f - \frac{1}{b-a} \int_a^b f(s) ds \right|$$

is as follows:

Theorem 2.2. Let $f : [a, b] \rightarrow \mathbb{C}$ be a measurable function such that $|f|$ is convex on $[a, b]$ and, for $p > 1$, $g : [a, b] \rightarrow \mathbb{C}$ is in the Lebesgue space $L_p[a, b]$. Then

$$(2.8) \quad |C(f, g)| \leq L_q(|f(a)|, |f(b)|) \left[\frac{1}{b-a} \int_a^b \left| g(t) - \frac{1}{b-a} \int_a^b g(s) ds \right|^p dt \right]^{1/p},$$

where $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

The inequality (2.8) is sharp.

Proof. Making use of Hölder's inequality, we have

$$(2.9) \quad \begin{aligned} (b-a)I &= \int_a^b \left[\frac{(b-t)|f(a)| + (t-a)|f(b)|}{b-a} \right] \left| g(t) - \frac{1}{b-a} \int_a^b g(s) ds \right| dt \\ &\leq \left(\int_a^b \left[\frac{(b-t)|f(a)| + (t-a)|f(b)|}{b-a} \right]^q dt \right)^{1/q} \\ &\quad \times \left(\int_a^b \left| g(t) - \frac{1}{b-a} \int_a^b g(s) ds \right|^p dt \right)^{1/p}. \end{aligned}$$

Observe that, by changing the variable $u = \frac{t-a}{b-a}$ we have

$$\begin{aligned} &\int_a^b \left[\frac{(b-t)|f(a)| + (t-a)|f(b)|}{b-a} \right]^q dt \\ &= (b-a) \int_0^1 [u|f(b)| + (1-u)|f(a)|]^q du. \end{aligned}$$

Changing the variable again

$$v = u|f(b)| + (1-u)|f(a)|$$

we have for $|f(a)| \neq |f(b)|$

$$\begin{aligned} (b-a) \int_0^1 [u|f(b)| + (1-u)|f(a)|]^q du &= \frac{b-a}{|f(b)| - |f(a)|} \int_0^1 v^q du \\ &= (b-a) L_q^q(|f(a)|, |f(b)|). \end{aligned}$$

For $|f(a)| = |f(b)|$ we also have

$$\begin{aligned} (b-a) \int_0^1 [u|f(b)| + (1-u)|f(a)|]^q du &= (b-a) |f(a)|^q \\ &= (b-a) L_q^q(|f(a)|, |f(b)|). \end{aligned}$$

Therefore

$$\begin{aligned} (b-a)I &\leq ((b-a) L_q^q(|f(a)|, |f(b)|))^{1/q} \left(\int_a^b \left| g(t) - \frac{1}{b-a} \int_a^b g(s) ds \right|^p dt \right)^{1/p} \\ &= (b-a)^{1/q} L_q(|f(a)|, |f(b)|) \left(\int_a^b \left| g(t) - \frac{1}{b-a} \int_a^b g(s) ds \right|^p dt \right)^{1/p}, \end{aligned}$$

which implies

$$I \leq L_q(|f(a)|, |f(b)|) \left(\frac{1}{b-a} \int_a^b \left| g(t) - \frac{1}{b-a} \int_a^b g(s) ds \right|^p dt \right)^{1/p}.$$

Making use of (2.9) we get the desired result (2.8).

Assume that

$$(2.10) \quad |C(f, g)| \leq KL_q(|f(a)|, |f(b)|) \times \left[\frac{1}{b-a} \int_a^b \left| g(t) - \frac{1}{b-a} \int_a^b g(s) ds \right|^p dt \right]^{1/p},$$

holds with a constant $K > 0$ for any $p > 1$ and f, g as above.

Consider the functions $f, g : [a, b] \rightarrow \mathbb{R}$ defined by

$$f(t) := \begin{cases} -1, & t \in [a, \frac{a+b}{2}], \\ 1, & t \in (\frac{a+b}{2}, b] \end{cases}$$

and $g : [a, b] \rightarrow \mathbb{R}, g(t) = t - \frac{a+b}{2}$.

We have $|f| = 1$, which satisfies the convexity condition with equality and

$$C(f, g) = \frac{1}{b-a} \int_a^b \left| t - \frac{a+b}{2} \right| dt = \frac{b-a}{4}.$$

We also have $L_q(|f(a)|, |f(b)|) = 1$ and

$$\begin{aligned} & \left(\frac{1}{b-a} \int_a^b \left| g(t) - \frac{1}{b-a} \int_a^b g(s) ds \right|^p dt \right)^{1/p} \\ &= \left(\frac{1}{b-a} \int_a^b \left| t - \frac{a+b}{2} \right|^p dt \right)^{1/p} = \left(\frac{2}{b-a} \int_{\frac{a+b}{2}}^b \left(t - \frac{a+b}{2} \right)^p dt \right)^{1/p} \\ &= \left(\frac{2}{b-a} \frac{\left(\frac{b-a}{2} \right)^{p+1}}{p+1} \right)^{1/p} = \frac{b-a}{2(p+1)^{1/p}}. \end{aligned}$$

If we replace these values in (2.10) we get

$$(2.11) \quad \frac{b-a}{4} \leq \frac{K(b-a)}{2(p+1)^{1/p}}$$

for any $p > 1$.

Now, if we let $p \rightarrow 1+$ in (2.11) we get $K \geq 1$, which proves the desired sharpness. \square

The case $p = q = 2$ is of interest.

Corollary 2.2. Let $f : [a, b] \rightarrow \mathbb{C}$ be a measurable function such that $|f|$ is convex on $[a, b]$ and $g : [a, b] \rightarrow \mathbb{C}$ is in the Lebesgue space $L_2[a, b]$. Then

$$(2.12) \quad |C(f, g)| \leq \left(\frac{|f(a)|^2 + |f(a)f(b)| + |f(b)|^2}{3} \right)^{1/2} D(g).$$

The following particular cases are of interest as well:

Corollary 2.3. Let $f : [a, b] \rightarrow \mathbb{C}$ be a measurable function such that $|f|$ is convex on $[a, b]$. Then

$$(2.13) \quad \begin{aligned} D^2(f), E^2(f) &\leq L_q(|f(a)|, |f(b)|) \\ &\times \left[\frac{1}{b-a} \int_a^b |f(t) - \frac{1}{b-a} \int_a^b f(s) ds|^p dt \right]^{1/p}, \end{aligned}$$

and

$$(2.14) \quad \begin{aligned} G^2(f) &\leq L_q(|f(a)|, |f(b)|) \\ &\times \left[\frac{1}{b-a} \int_a^b \left| |f(t)| - \frac{1}{b-a} \int_a^b |f(s)| ds \right|^p dt \right]^{1/p}, \end{aligned}$$

where $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

In particular, we have

$$(2.15) \quad D^2(f), E^2(f) \leq \left(\frac{|f(a)|^2 + |f(a)f(b)| + |f(b)|^2}{3} \right)^{1/2} D(f),$$

and

$$(2.16) \quad G^2(f) \leq \left(\frac{|f(a)|^2 + |f(a)f(b)| + |f(b)|^2}{3} \right)^{1/2} D(|f|).$$

The first inequality in (2.15) is equivalent to

$$(2.17) \quad D(f) \leq \left(\frac{|f(a)|^2 + |f(a)f(b)| + |f(b)|^2}{3} \right)^{1/2}.$$

The following result also holds:

Theorem 2.3. Let $f : [a, b] \rightarrow \mathbb{C}$ be a measurable function such that $|f|$ is convex on $[a, b]$ and $g : [a, b] \rightarrow \mathbb{C}$ is essentially bounded on $[a, b]$. Then

$$(2.18) \quad |C(f, g)| \leq \frac{1}{2} [|f(a)| + |f(b)|] \sup_{t \in [a, b]} \left| g(t) - \frac{1}{b-a} \int_a^b g(s) ds \right|.$$

The inequality (2.18) is sharp.

Proof. We have

$$\begin{aligned} I &\leq \frac{1}{b-a} \int_a^b \left[\frac{(b-t)|f(a)| + (t-a)|f(b)|}{b-a} \right] \\ &\quad \times \left| g(t) - \frac{1}{b-a} \int_a^b g(s) ds \right| dt \\ &\leq \operatorname{ess\,sup}_{t \in [a, b]} \left| g(t) - \frac{1}{b-a} \int_a^b g(s) ds \right| \\ &\quad \times \frac{1}{b-a} \int_a^b \left[\frac{(b-t)|f(a)| + (t-a)|f(b)|}{b-a} \right] dt \\ &= \frac{|f(a)| + |f(b)|}{2} \operatorname{ess\,sup}_{t \in [a, b]} \left| g(t) - \frac{1}{b-a} \int_a^b g(s) ds \right| \end{aligned}$$

and by (2.4) and (2.19) we get the desired result (2.18).

Assume that the inequality (2.18) holds with a constant $D > 0$

$$(2.19) \quad |C(f, g)| \leq D [|f(a)| + |f(b)|] \sup_{t \in [a, b]} \left| g(t) - \frac{1}{b-a} \int_a^b g(s) ds \right|.$$

Consider the functions $f, g : [a, b] \rightarrow \mathbb{R}$ defined by

$$f(t) = g(t) := \begin{cases} -1, & t \in [a, \frac{a+b}{2}], \\ 1, & t \in (\frac{a+b}{2}, b]. \end{cases}$$

We have $|f| = 1$, which satisfies the convexity condition with equality and

$$C(f, g) := \frac{1}{b-a} \int_a^b dt = 1, \quad |f(a)| = |f(b)| = 1$$

while

$$\sup_{t \in [a, b]} \left| g(t) - \frac{1}{b-a} \int_a^b g(s) ds \right| = 1.$$

From (2.19) we have $1 \leq 2D$, i.e. $D \geq \frac{1}{2}$. \square

Corollary 2.4. *Let $f : [a, b] \rightarrow \mathbb{C}$ be a measurable function such that $|f|$ is convex on $[a, b]$. Then*

$$(2.20) \quad D^2(f), E^2(f) \leq \frac{1}{2} [|f(a)| + |f(b)|] \sup_{t \in [a, b]} \left| f(t) - \frac{1}{b-a} \int_a^b f(s) ds \right|$$

and

$$(2.21) \quad G^2(f) \leq \frac{1}{2} [|f(a)| + |f(b)|] \sup_{t \in [a, b]} \left| |f(t)| - \frac{1}{b-a} \int_a^b |f(s)| ds \right|.$$

3. Application for Riemann-Stieltjes Integral

The following representation is of interest in itself. The result was firstly obtained in [6] (see also [7]). For the sake a completeness we give here a short proof as well.

Lemma 3.1. *If $v : [a, b] \rightarrow \mathbb{C}$ is continuous (of bounded variation) on $[a, b]$ and $h : [a, b] \rightarrow \mathbb{C}$ is of bounded variation (continuous) on $[a, b]$, then we have the identity*

$$(3.1) \quad \begin{aligned} & \frac{v(b) \int_a^b (t-a) dh(t) + v(a) \int_a^b (b-t) dh(t)}{b-a} - \int_a^b v(t) dh(t) \\ &= \int_a^b h(t) dv(t) - \frac{v(b)-v(a)}{b-a} \int_a^b h(t) dt. \end{aligned}$$

Proof. Integrating by parts in the Riemann-Stieltjes integral we have

$$(3.2) \quad \begin{aligned} & \frac{v(b) \int_a^b (t-a) dh(t) + v(a) \int_a^b (b-t) dh(t)}{b-a} - \int_a^b v(t) dh(t) \\ &= \int_a^b \left[\frac{v(b)(t-a) + v(a)(b-t)}{b-a} - v(t) \right] dh(t) \\ &= \left[\frac{(t-a)v(b) + (b-t)v(a)}{b-a} - v(t) \right] h(t) \Big|_a^b \\ &\quad - \int_a^b h(t) d \left[\frac{(t-a)v(b) + (b-t)v(a)}{b-a} - v(t) \right] \\ &= [v(b) - v(a)] h(b) - [v(a) - v(a)] h(a) \\ &\quad - \int_a^b h(t) \left[\frac{v(b)-v(a)}{b-a} dt - dv(t) \right] \\ &= \int_a^b h(t) dv(t) - \frac{v(b)-v(a)}{b-a} \int_a^b h(t) dt \end{aligned}$$

and the identity is proven. \square

We can provide now the following application for Riemann-Stieltjes integral:

Proposition 3.1. *If $v : I \rightarrow \mathbb{C}$ is differentiable on the interior of the interval I denoted \mathring{I} and $[a, b] \subset \mathring{I}$, $|v'|$ is convex on $[a, b]$ and $h : [a, b] \rightarrow \mathbb{C}$ is integrable on $[a, b]$, then we have*

the inequalities

$$(3.3) \quad \left| \frac{v(b) \int_a^b (t-a) dh(t) + v(a) \int_a^b (b-t) dh(t)}{b-a} - \int_a^b v(t) dh(t) \right|$$

$$\leq \begin{cases} \max \{ |v'(a)|, |v'(b)| \} \int_a^b \left| h(t) - \frac{1}{b-a} \int_a^b h(s) ds \right| dt, \\ (b-a) L_q (|v'(a)|, |v'(b)|) \left[\frac{1}{b-a} \int_a^b \left| h(t) - \frac{1}{b-a} \int_a^b h(s) ds \right|^p dt \right]^{1/p} \\ \text{where } q > 1 \text{ and } \frac{1}{p} + \frac{1}{q} = 1, \\ \frac{1}{2} (b-a) [|v'(a)| + |v'(b)|] \sup_{t \in [a,b]} \left| h(t) - \frac{1}{b-a} \int_a^b h(s) ds \right|. \end{cases}$$

Proof. From (3.1) we have

$$(3.4) \quad \frac{v(b) \int_a^b (t-a) dh(t) + v(a) \int_a^b (b-t) dh(t)}{b-a} - \int_a^b v(t) dh(t)$$

$$= \int_a^b h(t) v'(t) dt - \frac{v(b)-v(a)}{b-a} \int_a^b h(t) dt = (b-a) C(v', h).$$

Since $|v'|$ is convex on $[a, b]$, then by applying Theorem 2.1-Theorem 2.3 for $f = v'$ and $g = h$ we deduce the desired result (3.3). \square

Remark 3.1. If $p = q = 2$, then by (3.3) we get

$$(3.5) \quad \left| \frac{v(b) \int_a^b (t-a) dh(t) + v(a) \int_a^b (b-t) dh(t)}{b-a} - \int_a^b v(t) dh(t) \right|$$

$$\leq (b-a) \left(\frac{|v'(a)|^2 + |v'(a)v'(b)| + |v'(b)|^2}{3} \right)^{1/2}$$

$$\times \left[\frac{1}{b-a} \int_a^b \left| h(t) - \frac{1}{b-a} \int_a^b h(s) ds \right|^2 dt \right]^{1/2},$$

provided that $|v'|$ is convex on $[a, b]$ and $h : [a, b] \rightarrow \mathbb{C}$ is integrable on $[a, b]$.

4. Applications for Self-adjoint Operators

We denote by $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on a complex Hilbert space $(H; \langle \cdot, \cdot \rangle)$. Let $A \in \mathcal{B}(H)$ be self-adjoint and let φ_λ be defined for all $\lambda \in \mathbb{R}$ as follows

$$\varphi_\lambda(s) := \begin{cases} 1, & \text{for } -\infty < s \leq \lambda, \\ 0, & \text{for } \lambda < s < +\infty. \end{cases}$$

Then for every $\lambda \in \mathbb{R}$ the operator

$$(4.1) \quad E_\lambda := \varphi_\lambda(A)$$

is a projection which reduces A .

The properties of these projections are collected in the following fundamental result concerning the spectral representation of bounded self-adjoint operators in Hilbert spaces, see for instance [15, p. 256]:

Theorem 4.1. Spectral Representation Theorem *Let A be a bounded self-adjoint operator on the Hilbert space H and let $m = \min \{ \lambda \mid \lambda \in Sp(A) \} =: \min Sp(A)$ and $M = \max \{ \lambda \mid \lambda \in Sp(A) \} =: \max Sp(A)$. Then there exists a family of projections $\{E_\lambda\}_{\lambda \in \mathbb{R}}$, called the spectral family of A , with the following properties*

- a) $E_\lambda \leq E_{\lambda'}$ for $\lambda \leq \lambda'$;
- b) $E_{m-0} = 0, E_M = I$ and $E_{\lambda+0} = E_\lambda$ for all $\lambda \in \mathbb{R}$;
- c) We have the representation

$$(4.2) \quad A = \int_{m-0}^M \lambda dE_\lambda.$$

More generally, for every continuous complex-valued function φ defined on \mathbb{R} and for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$(4.3) \quad \left\| \varphi(A) - \sum_{k=1}^n \varphi(\lambda'_k) [E_{\lambda_k} - E_{\lambda_{k-1}}] \right\| \leq \varepsilon$$

whenever

$$(4.4) \quad \begin{cases} \lambda_0 < m = \lambda_1 < \dots < \lambda_{n-1} < \lambda_n = M, \\ \lambda_k - \lambda_{k-1} \leq \delta \text{ for } 1 \leq k \leq n, \\ \lambda'_k \in [\lambda_{k-1}, \lambda_k] \text{ for } 1 \leq k \leq n \end{cases}$$

this means that

$$(4.5) \quad \varphi(A) = \int_{m-0}^M \varphi(\lambda) dE_\lambda,$$

where the integral is of Riemann-Stieltjes type.

Corollary 4.1. *With the assumptions of Theorem 4.1 for A, E_λ and φ we have the representations*

$$(4.6) \quad \varphi(A)x = \int_{m-0}^M \varphi(\lambda) dE_\lambda x \text{ for all } x \in H$$

and

$$(4.7) \quad \langle \varphi(A)x, y \rangle = \int_{m-0}^M \varphi(\lambda) d \langle E_\lambda x, y \rangle \text{ for all } x, y \in H.$$

In particular,

$$(4.8) \quad \langle \varphi(A)x, x \rangle = \int_{m-0}^M \varphi(\lambda) d \langle E_\lambda x, x \rangle \text{ for all } x \in H.$$

Moreover, we have the equality

$$(4.9) \quad \|\varphi(A)x\|^2 = \int_{m-0}^M |\varphi(\lambda)|^2 d \|E_\lambda x\|^2 \text{ for all } x \in H.$$

The next result shows that it is legitimate to talk about "the" spectral family of the bounded self-adjoint operator A since it is uniquely determined by the requirements a), b) and c) in Theorem 4.1, see for instance [15, p. 258]:

Theorem 4.2. *Let A be a bounded self-adjoint operator on the Hilbert space H and let $m = \min Sp(A)$ and $M = \max Sp(A)$. If $\{E_\lambda\}_{\lambda \in \mathbb{R}}$ is a family of projections satisfying the requirements a), b) and c) in Theorem 4.1, then $F_\lambda = E_\lambda$ for all $\lambda \in \mathbb{R}$ where E_λ is defined by (4.1).*

By the above two theorems, the spectral family $\{E_\lambda\}_{\lambda \in \mathbb{R}}$ uniquely determines and in turn is uniquely determined by the bounded self-adjoint operator A .

We can state now the following generalized trapezoid inequality for functions of self-adjoint operators:

Theorem 4.3. *Let A be a bounded self-adjoint operator on the Hilbert space H and let $m = \min \{\lambda \mid \lambda \in Sp(A)\} =: \min Sp(A)$ and $M = \max \{\lambda \mid \lambda \in Sp(A)\} =: \max Sp(A)$. Consider also the spectral family $\{E_\lambda\}_{\lambda \in \mathbb{R}}$ of A .*

If $f : I \rightarrow \mathbb{C}$ is differentiable on the interior of the interval I , denoted \mathring{I} and $[m, M] \subset \mathring{I}$, $|f'|$ is convex on $[m, M]$, then we have the inequalities

$$(4.10) \quad \begin{aligned} & \left| \left\langle \left[\frac{f(m)(M1_H - A) + f(M)(A - m1_H)}{M - m} \right] x, y \right\rangle - \langle f(A)x, y \rangle \right| \\ & \leq \frac{1}{2} \left[|f'(m)| + |f'(M)| \right] (M - m) \\ & \times \sup_{t \in [m, M]} \left[\frac{t-m}{M-m} \bigvee_{m-0}^t (\langle E_{(\cdot)} x, y \rangle) + \frac{M-t}{M-m} \bigvee_t^M (\langle E_{(\cdot)} x, y \rangle) \right] \\ & \leq \frac{1}{2} \left[|f'(m)| + |f'(M)| \right] (M - m) \bigvee_{m-0}^M (\langle E_{(\cdot)} x, y \rangle) \\ & \leq \frac{1}{2} \left[|f'(m)| + |f'(M)| \right] (M - m) \|x\| \|y\| \end{aligned}$$

for any $x, y \in H$.

Proof. Let $x, y \in H$ and consider $h : \mathbb{R} \rightarrow \mathbb{C}$, $h(t) := \langle E_t x, y \rangle$. If we use the third inequality in (3.3) for the interval $[m - \varepsilon, M]$ with small $\varepsilon > 0$, we have

$$(4.11) \quad \left| \frac{f(M) \int_{m-\varepsilon}^M (t-m+\varepsilon) d\langle E_t x, y \rangle + f(m-\varepsilon) \int_{m-\varepsilon}^M (M-t) d\langle E_t x, y \rangle}{M-m+\varepsilon} - \int_{m-\varepsilon}^M f(t) d\langle E_t x, y \rangle \right| \\ \leq \frac{1}{2} \left[|f'(M)| + |f'(m-\varepsilon)| \right] (M-m+\varepsilon) \\ \times \sup_{t \in [m-\varepsilon, M]} \left| \langle E_t x, y \rangle - \frac{1}{M-m+\varepsilon} \int_{m-\varepsilon}^M \langle E_s x, y \rangle ds \right|.$$

Taking the limit over $\varepsilon \rightarrow 0+$ and using the Spectral representation theorem, we have

$$(4.12) \quad \left| \left\langle \left[\frac{f(m)(M1_H - A) + f(M)(A - m1_H)}{M-m} \right] x, y \right\rangle - \langle f(A)x, y \rangle \right| \\ \leq \frac{1}{2} \left[|f'(m)| + |f'(M)| \right] (M-m) \\ \times \sup_{t \in [m, M]} \left| \langle E_t x, y \rangle - \frac{1}{M-m} \int_{m-0}^M \langle E_s x, y \rangle ds \right|$$

for any $x, y \in H$.

It is well known that if $p : [a, b] \rightarrow \mathbb{C}$ is a bounded function, $v : [a, b] \rightarrow \mathbb{C}$ is of bounded variation and the Riemann-Stieltjes integral $\int_a^b p(t) dv(t)$ exists, then the following inequality holds

$$(4.13) \quad \left| \int_a^b p(t) dv(t) \right| \leq \sup_{t \in [a, b]} |p(t)| \bigvee_a^b(v),$$

where $\bigvee_a^b(v)$ denotes the total variation of v on $[a, b]$.

Now, a simple integration by parts in the Riemann-Stieltjes integral reveals the following equality of interest

$$(4.14) \quad \langle E_t x, y \rangle - \frac{1}{M-m} \int_{m-0}^M \langle E_s x, y \rangle ds \\ = \frac{1}{M-m} \left[\int_{m-0}^t (s-m) d\langle E_s x, y \rangle + \int_t^M (s-M) d\langle E_s x, y \rangle \right]$$

that holds for any $t \in [m, M]$ and for any $x, y \in H$.

Since the function $v(s) := \langle E_s x, y \rangle$ is of bounded variation on $[m, M]$ for any

$x, y \in H$, then on applying the inequality (4.13), we get

$$\begin{aligned}
(4.15) \quad & \left| \langle E_t x, y \rangle - \frac{1}{M-m} \int_{m-0}^M \langle E_s x, y \rangle ds \right| \\
& \leq \frac{1}{M-m} \left[\left| \int_{m-0}^t (s-m) d \langle E_s x, y \rangle \right| + \left| \int_t^M (s-M) d \langle E_s x, y \rangle \right| \right] \\
& \leq \frac{t-m}{M-m} \bigvee_{m-0}^t (\langle E_{(\cdot)} x, y \rangle) + \frac{M-t}{M-m} \bigvee_t^M (\langle E_{(\cdot)} x, y \rangle) \\
& \leq \max \left\{ \frac{t-m}{M-m}, \frac{M-t}{M-m} \right\} \bigvee_{m-0}^M (\langle E_{(\cdot)} x, y \rangle) \\
& = \left[\frac{1}{2} + \left| t - \frac{m+M}{2} \right| \right] \bigvee_{m-0}^M (\langle E_{(\cdot)} x, y \rangle)
\end{aligned}$$

that holds for any $t \in [m, M]$ and for any $x, y \in H$.

This implies that

$$\begin{aligned}
(4.16) \quad & \sup_{t \in [m, M]} \left| \langle E_t x, y \rangle - \frac{1}{M-m} \int_{m-0}^M \langle E_s x, y \rangle ds \right| \\
& \leq \sup_{t \in [m, M]} \left[\frac{t-m}{M-m} \bigvee_{m-0}^t (\langle E_{(\cdot)} x, y \rangle) + \frac{M-t}{M-m} \bigvee_t^M (\langle E_{(\cdot)} x, y \rangle) \right] \\
& \leq \sup_{t \in [m, M]} \left[\frac{1}{2} + \left| t - \frac{m+M}{2} \right| \right] \bigvee_{m-0}^M (\langle E_{(\cdot)} x, y \rangle) = \bigvee_{m-0}^M (\langle E_{(\cdot)} x, y \rangle)
\end{aligned}$$

for any $x, y \in H$.

The proof of the inequality

$$\bigvee_{m-0}^M (\langle E_{(\cdot)} x, y \rangle) \leq \|x\| \|y\|$$

for any $x, y \in H$, can be found in [13, p. 9]. \square

We also have:

Theorem 4.4. *Let A be a bonded self-adjoint operator on the Hilbert space H and let $m = \min \{ \lambda \mid \lambda \in Sp(A) \} =: \min Sp(A)$ and $M = \max \{ \lambda \mid \lambda \in Sp(A) \} =: \max Sp(A)$. Consider also the spectral family $\{E_\lambda\}_{\lambda \in \mathbb{R}}$ of A .*

If $f : I \rightarrow \mathbb{C}$ is differentiable on \mathring{I} , $[m, M] \subset \mathring{I}$ and $|f'|$ is convex on $[m, M]$, then we have the inequalities

$$\begin{aligned}
(4.17) \quad & \left| \left\langle \left[\frac{f(m)(M1_H - A) + f(M)(A - m1_H)}{M-m} \right] x, y \right\rangle - \langle f(A) x, y \rangle \right| \\
& \leq \max \{ |f'(m)|, |f'(M)| \} \int_{m-0}^M \left| \langle E_t x, y \rangle - \frac{1}{M-m} \int_{m-0}^M \langle E_s x, y \rangle ds \right| dt \\
& \leq \frac{1}{2} \max \{ |f'(m)|, |f'(M)| \} (M-m) \|x\| \|y\|
\end{aligned}$$

for any $x, y \in H$.

Proof. Let $x, y \in H$ and consider $h : \mathbb{R} \rightarrow \mathbb{C}$, $h(t) := \langle E_t x, y \rangle$. If we use the first inequality in (3.3) for the interval $[m - \varepsilon, M]$ with small $\varepsilon > 0$, we have

$$(4.18) \quad \left| \frac{f(M) \int_{m-\varepsilon}^M (t-m+\varepsilon) d\langle E_t x, y \rangle + f(m-\varepsilon) \int_{m-\varepsilon}^M (M-t) d\langle E_t x, y \rangle}{M-m+\varepsilon} - \int_{m-\varepsilon}^M f(t) d\langle E_t x, y \rangle \right| \leq \max \left\{ |f'(M)|, |f'(m-\varepsilon)| \right\} (M-m+\varepsilon) \times \frac{1}{M-m+\varepsilon} \int_{m-\varepsilon}^M \left| \langle E_t x, y \rangle - \frac{1}{M-m+\varepsilon} \int_{m-\varepsilon}^M \langle E_s x, y \rangle ds \right| dt.$$

Taking the limit over $\varepsilon \rightarrow 0+$ and using the Spectral representation theorem, we have

$$(4.19) \quad \left| \left\langle \left[\frac{f(m)(M1_H - A) + f(M)(A - m1_H)}{M-m} \right] x, y \right\rangle - \langle f(A)x, y \rangle \right| \leq \max \left\{ |f'(m)|, |f'(M)| \right\} \int_{m-0}^M \left| \langle E_t x, y \rangle - \frac{1}{M-m} \int_{m-0}^M \langle E_s x, y \rangle ds \right| dt$$

for any $x, y \in H$.

By the Schwarz inequality in H we have that

$$(4.20) \quad \begin{aligned} & \int_{m-0}^M \left| \langle E_t x, y \rangle - \frac{1}{M-m} \int_{m-0}^M \langle E_s x, y \rangle ds \right| dt \\ &= \int_{m-0}^M \left| \left\langle \left[E_t x - \frac{1}{M-m} \int_{m-0}^M E_s x ds \right], y \right\rangle \right| dt \\ &\leq \|y\| \int_{m-0}^M \left\| E_t x - \frac{1}{M-m} \int_{m-0}^M E_s x ds \right\| dt \end{aligned}$$

for any $x, y \in H$.

On utilizing the Cauchy-Buniakovski-Schwarz integral inequality we may state that

$$(4.21) \quad \begin{aligned} & \int_{m-0}^M \left\| E_t x - \frac{1}{M-m} \int_{m-0}^M E_s x ds \right\| dt \\ &\leq (M-m)^{1/2} \left(\int_{m-0}^M \left\| E_t x - \frac{1}{M-m} \int_{m-0}^M E_s x ds \right\|^2 dt \right)^{1/2} \end{aligned}$$

for any $x \in H$.

Observe that the following equalities of interest hold and they can be easily proved by direct calculations

$$(4.22) \quad \begin{aligned} & \frac{1}{M-m} \int_{m-0}^M \left\| E_t x - \frac{1}{M-m} \int_{m-0}^M E_s x ds \right\|^2 dt \\ &= \frac{1}{M-m} \int_{m-0}^M \|E_t x\|^2 dt - \left\| \frac{1}{M-m} \int_{m-0}^M E_s x ds \right\|^2 \end{aligned}$$

and

$$(4.23) \quad \begin{aligned} & \frac{1}{M-m} \int_{m-0}^M \|E_t x\|^2 dt - \left\| \frac{1}{M-m} \int_{m-0}^M E_s x ds \right\|^2 \\ &= \frac{1}{M-m} \int_{m-0}^M \left\langle E_t x - \frac{1}{M-m} \int_{m-0}^M E_s x ds, E_t x - \frac{1}{2} x \right\rangle dt \end{aligned}$$

for any $x \in H$.

By (4.21), (4.22) and (4.23) we get

$$(4.24) \quad \begin{aligned} & \int_{m-0}^M \left\| E_t x - \frac{1}{M-m} \int_{m-0}^M E_s x ds \right\| dt \\ & \leq (M-m)^{1/2} \left(\int_{m-0}^M \left\langle E_t x - \frac{1}{M-m} \int_{m-0}^M E_s x ds, E_t x - \frac{1}{2} x \right\rangle dt \right)^{1/2} \end{aligned}$$

for any $x \in H$.

On making use of the Schwarz inequality in H we also have

$$(4.25) \quad \begin{aligned} & \int_{m-0}^M \left\langle E_t x - \frac{1}{M-m} \int_{m-0}^M E_s x ds, E_t x - \frac{1}{2} x \right\rangle dt \\ & \leq \int_{m-0}^M \left\| E_t x - \frac{1}{M-m} \int_{m-0}^M E_s x ds \right\| \left\| E_t x - \frac{1}{2} x \right\| dt \\ & = \frac{1}{2} \|x\| \int_{m-0}^M \left\| E_t x - \frac{1}{M-m} \int_{m-0}^M E_s x ds \right\| dt, \end{aligned}$$

where we used the fact that E_t are projectors, and in this case we have

$$\left\| E_t x - \frac{1}{2} x \right\|^2 = \|E_t x\|^2 - \langle E_t x, x \rangle + \frac{1}{4} \|x\|^2 = \frac{1}{4} \|x\|^2$$

for any $t \in [m, M]$ for any $x \in H$.

From (4.24) and (4.25) we get

$$(4.26) \quad \begin{aligned} & \int_{m-0}^M \left\| E_t x - \frac{1}{M-m} \int_{m-0}^M E_s x ds \right\| dt \\ & \leq (M-m)^{1/2} \left(\frac{1}{2} \|x\| \int_{m-0}^M \left\| E_t x - \frac{1}{M-m} \int_{m-0}^M E_s x ds \right\| dt \right)^{1/2}, \end{aligned}$$

which is clearly equivalent with the following inequality of interest in itself

$$(4.27) \quad \int_{m-0}^M \left\| E_t x - \frac{1}{M-m} \int_{m-0}^M E_s x ds \right\| dt \leq \frac{1}{2} \|x\| (M-m)$$

for any $x \in H$.

From (4.20) we then get

$$\frac{1}{M-m} \int_{m-0}^M \left| \langle E_t x, y \rangle - \frac{1}{M-m} \int_{m-0}^M \langle E_s x, y \rangle ds \right| dt \leq \frac{1}{2} \|x\| \|y\|$$

for any $x, y \in H$. \square

Finally, we also have:

Theorem 4.5. *Let A be a bounded self-adjoint operator on the Hilbert space H and let $m = \min \{ \lambda \mid \lambda \in \text{Sp}(A) \} =: \min \text{Sp}(A)$ and $M = \max \{ \lambda \mid \lambda \in \text{Sp}(A) \} =: \max \text{Sp}(A)$. Consider also the spectral family $\{E_\lambda\}_{\lambda \in \mathbb{R}}$ of A .*

If $f : I \rightarrow \mathbb{C}$ is differentiable on \mathring{I} , $[m, M] \subset \mathring{I}$ and $|f'|$ is convex on $[m, M]$, then we have the inequalities

$$\begin{aligned}
 (4.28) \quad & \left| \left\langle \left[\frac{f(m)(M1_H - A) + f(M)(A - m1_H)}{M - m} \right] x, y \right\rangle - \langle f(A)x, y \rangle \right| \\
 & \leq \left(\frac{|f'(M)|^2 + |f'(M)f'(m)| + |f'(m)|^2}{3} \right)^{1/2} (M - m) \\
 & \times \left(\frac{1}{M - m} \int_{m-0}^M \left| \langle E_t x, y \rangle - \frac{1}{M - m} \int_{m-0}^M \langle E_s x, y \rangle ds \right|^2 dt \right)^{1/2} \\
 & \leq \frac{1}{2} \left(\frac{|f'(M)|^2 + |f'(M)f'(m)| + |f'(m)|^2}{3} \right)^{1/2} (M - m) \|x\| \|y\|
 \end{aligned}$$

for any $x, y \in H$.

Proof. Utilising the inequality (3.5) we can prove in a similar manner as above the first inequality in (4.28).

By the Schwarz inequality in H we have that

$$\begin{aligned}
 (4.29) \quad & \frac{1}{M - m} \int_{m-0}^M \left| \langle E_t x, y \rangle - \frac{1}{M - m} \int_{m-0}^M \langle E_s x, y \rangle ds \right|^2 dt \\
 & = \frac{1}{M - m} \int_{m-0}^M \left| \left\langle \left[E_t x - \frac{1}{M - m} \int_{m-0}^M E_s x ds \right], y \right\rangle \right|^2 dt \\
 & \leq \|y\|^2 \frac{1}{M - m} \int_{m-0}^M \left\| E_t x - \frac{1}{M - m} \int_{m-0}^M E_s x ds \right\|^2 dt
 \end{aligned}$$

for any $x, y \in H$.

As in the proof of Theorem 4.4 we also have

$$\begin{aligned}
 (4.30) \quad & \frac{1}{M - m} \int_{m-0}^M \left\| E_t x - \frac{1}{M - m} \int_{m-0}^M E_s x ds \right\|^2 dt \\
 & \leq \frac{1}{2} \|x\| \frac{1}{M - m} \int_{m-0}^M \left\| E_t x - \frac{1}{M - m} \int_{m-0}^M E_s x ds \right\| dt \leq \frac{1}{4} \|x\|^2.
 \end{aligned}$$

By (4.29) and (4.30) we then get

$$\frac{1}{M - m} \int_{m-0}^M \left| \langle E_t x, y \rangle - \frac{1}{M - m} \int_{m-0}^M \langle E_s x, y \rangle ds \right|^2 dt \leq \frac{1}{4} \|x\|^2 \|y\|^2,$$

namely

$$\left[\frac{1}{M - m} \int_{m-0}^M \left| \langle E_t x, y \rangle - \frac{1}{M - m} \int_{m-0}^M \langle E_s x, y \rangle ds \right|^2 dt \right]^{1/2} \leq \frac{1}{2} \|x\| \|y\|,$$

for any $x, y \in H$.

This proves the last part of (4.28). \square

Example 4.1. a) Let A be a bonded self-adjoint operator on the Hilbert space H and let $m = \min\{\lambda \mid \lambda \in Sp(A)\} =: \min Sp(A) \geq 0$ and $M = \max\{\lambda \mid \lambda \in Sp(A)\} =: \max Sp(A)$. Consider also the spectral family $\{E_\lambda\}_{\lambda \in \mathbb{R}}$ of A . Then by Theorem 4.3-4.5 we have for $f(t) = t^p$, $p \geq 2$ that

$$\begin{aligned}
 (4.31) \quad & \left| \left\langle \left[\frac{m^p(M1_H - A) + M^p(A - m1_H)}{M - m} \right] x, y \right\rangle - \langle A^p x, y \rangle \right| \\
 & \leq \frac{1}{2} p \left(m^{p-1} + M^{p-1} \right) (M - m) \\
 & \times \sup_{t \in [m, M]} \left[\frac{t-m}{M-m} \bigvee_{m-0}^t (\langle E_{(\cdot)} x, y \rangle) + \frac{M-t}{M-m} \bigvee_t^M (\langle E_{(\cdot)} x, y \rangle) \right] \\
 & \leq \frac{1}{2} p \left(m^{p-1} + M^{p-1} \right) (M - m) \bigvee_{m-0}^M (\langle E_{(\cdot)} x, y \rangle) \\
 & \leq \frac{1}{2} p \left(m^{p-1} + M^{p-1} \right) (M - m) \|x\| \|y\|,
 \end{aligned}$$

$$\begin{aligned}
 (4.32) \quad & \left| \left\langle \left[\frac{m^p(M1_H - A) + M^p(A - m1_H)}{M - m} \right] x, y \right\rangle - \langle A^p x, y \rangle \right| \\
 & \leq p M^{p-1} \int_{m-0}^M \left| \langle E_t x, y \rangle - \frac{1}{M-m} \int_{m-0}^M \langle E_s x, y \rangle ds \right| dt \\
 & \leq \frac{1}{2} p M^{p-1} (M - m) \|x\| \|y\|
 \end{aligned}$$

and

$$\begin{aligned}
 (4.33) \quad & \left| \left\langle \left[\frac{m^p(M1_H - A) + M^p(A - m1_H)}{M - m} \right] x, y \right\rangle - \langle A^p x, y \rangle \right| \\
 & \leq p \left(\frac{M^{2(p-1)} + (Mm)^{p-1} + m^{2(p-1)}}{3} \right)^{1/2} (M - m) \\
 & \times \left(\frac{1}{M-m} \int_{m-0}^M \left| \langle E_t x, y \rangle - \frac{1}{M-m} \int_{m-0}^M \langle E_s x, y \rangle ds \right|^2 dt \right)^{1/2} \\
 & \leq \frac{1}{2} \left(\frac{M^{2(p-1)} + (Mm)^{p-1} + m^{2(p-1)}}{3} \right)^{1/2} (M - m) \|x\| \|y\|
 \end{aligned}$$

for any $x, y \in H$.

b) With the assumptions of a) and if $m > 0$, then by Theorem 4.3-4.5 we have for $f(t) = \ln t$, that

$$\begin{aligned}
 (4.34) \quad & \left| \left\langle \left[\frac{\ln m(M1_H - A) + \ln M(A - m1_H)}{M - m} \right] x, y \right\rangle - \langle \ln A x, y \rangle \right| \\
 & \leq \frac{m+M}{2mM} (M - m) \\
 & \times \sup_{t \in [m, M]} \left[\frac{t-m}{M-m} \bigvee_{m-0}^t (\langle E_{(\cdot)} x, y \rangle) + \frac{M-t}{M-m} \bigvee_t^M (\langle E_{(\cdot)} x, y \rangle) \right] \\
 & \leq \frac{m+M}{2mM} (M - m) \bigvee_{m-0}^M (\langle E_{(\cdot)} x, y \rangle) \leq \frac{m+M}{2mM} (M - m) \|x\| \|y\|,
 \end{aligned}$$

$$\begin{aligned}
 (4.35) \quad & \left| \left\langle \left[\frac{\ln m(M1_H - A) + \ln M(A - m1_H)}{M - m} \right] x, y \right\rangle - \langle \ln Ax, y \rangle \right| \\
 & \leq \frac{1}{m} \int_{m-0}^M \left| \langle E_t x, y \rangle - \frac{1}{M-m} \int_{m-0}^M \langle E_s x, y \rangle ds \right| dt \\
 & \leq \frac{1}{2m} \|x\| \|y\| (M - m)
 \end{aligned}$$

and

$$\begin{aligned}
 (4.36) \quad & \left| \left\langle \left[\frac{\ln m(M1_H - A) + \ln M(A - m1_H)}{M - m} \right] x, y \right\rangle - \langle \ln Ax, y \rangle \right| \\
 & \leq \left(\frac{M^2 + mM + m^2}{3m^2M^2} \right)^{1/2} (M - m) \\
 & \times \left(\frac{1}{M-m} \int_{m-0}^M \left| \langle E_t x, y \rangle - \frac{1}{M-m} \int_{m-0}^M \langle E_s x, y \rangle ds \right|^2 dt \right)^{1/2} \\
 & \leq \frac{1}{2mM} \left(\frac{M^2 + mM + m^2}{3} \right)^{1/2} (M - m) \|x\| \|y\|
 \end{aligned}$$

for any $x, y \in H$.

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REFERENCES

1. W. ALOMARI: *New Čebyšev type inequalities and applications for functions of self-adjoint operators on complex Hilbert spaces*, Chinese Journal of Mathematics, **2014** (2014), Article ID 363050, 10 pages, <http://dx.doi.org/10.1155/2014/363050>
2. P. CERONE AND S. S. DRAGOMIR: *New bounds for the Čebyšev functional*, Appl. Math. Lett., **18** (2005), 603–611.
3. P. CERONE AND S. S. DRAGOMIR: *A refinement of the Grüss inequality and applications*, Tamkang J. Math. **38**(2007), No. 1, 37–49. Preprint available at RGMIA Res. Rep. Coll., **5**(2) (2002), Art. 14.
4. P. L. CHEBYSHEV: *Sur les expressions approximatives des intégrals définis par les autres prises entre les même limites*, Proc. Math. Soc. Charkov, **2** (1882), 93–98.
5. X.-L. CHENG AND J. SUN: *Note on the perturbed trapezoid inequality*, J. Inequal. Pure Appl. Math., **3**(2) (2002), Art. 29.
6. S. S. DRAGOMIR: *Inequalities of Grüss type for the Stieltjes integral and applications*, Kragujevac J. Math., **26** (2004), 89–112.
7. S. S. DRAGOMIR: *Inequalities for Stieltjes integrals with convex integrators and applications*, Appl. Math. Lett., **20** (2007), 123–130.
8. S. S. DRAGOMIR: *Čebyšev's type inequalities for functions of selfadjoint operators in Hilbert spaces*, Preprint RGMIA Res. Rep. Coll., **11**(e) (2008), Art. 9.
9. S. S. DRAGOMIR: *Grüss' type inequalities for functions of selfadjoint operators in Hilbert spaces*, Preprint RGMIA Res. Rep. Coll., **11**(e) (2008), Art. 11.
10. S. S. DRAGOMIR: *Inequalities for the Čebyšev functional of two functions of selfadjoint operators in Hilbert spaces*, Preprint RGMIA Res. Rep. Coll., **11**(e) (2008), Art. 17.

11. S. S. DRAGOMIR: *New Grüss' type inequalities for functions of bounded variation and applications*. Appl. Math. Lett. **25** (2012), no. 10, 1475–1479.
12. S. S. DRAGOMIR: *Operator Inequalities of the Jensen, Čebyšev and Grüss Type*. Springer Briefs in Mathematics. Springer, New York, 2012. xii+121 pp. ISBN: 978-1-4614-1520-6
13. S. S. DRAGOMIR: *Operator Inequalities of Ostrowski and Trapezoidal Type*. Springer Briefs in Mathematics. Springer, New York, 2012. x+112 pp. ISBN: 978-1-4614-1778-1
14. G. GRÜSS: *Über das maximum des absoluten Betrages von $\frac{1}{b-a} \int_a^b f(x)g(x)dx - \frac{1}{(b-a)^2} \int_a^b f(x)dx \cdot \int_a^b g(x)dx$* , Math. Z., **39** (1934), 215–226.
15. G. HELMBERG: *Introduction to Spectral Theory in Hilbert Space*, John Wiley & Sons, Inc. -New York, 1969.
16. A. LUPAŞ; *The best constant in an integral inequality* Mathematica (Cluj), **15** (38) (1973), No. 2, 219–222.
17. D. S. MITRINOVIĆ, J. E. PEČARIĆ AND A. M. FINK: *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, Dordrecht, 1993.
18. A. M. OSTROWSKI: *On an integral inequality*, Aequationes Math., **4** (1970), 358–373.

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