Jensen's type trace inequalities for convex functions of selfadjoint operators in Hilbert spaces

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JENSEN'S TYPE TRACE INEQUALITIES FOR CONVEX FUNCTIONS OF SELFADJOINT OPERATORS IN HILBERT SPACES *

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Abstract. Some Jensen's type trace inequalities for convex functions of selfadjoint operators in Hilbert spaces are provided. Applications for some convex functions of interest are also given.

Keywords: Trace class operators; Hilbert-Schmidt operators; Trace; Convex functions; Jensen's inequality; Trace inequalities for matrices

1. Introduction

Let $(H, \langle \cdot, \cdot \rangle)$ be a complex Hilbert space and $\{e_i\}_{i \in I}$ an *orthonormal basis* of H. We say that $A \in \mathcal{B}(H)$ is a *Hilbert-Schmidt operator* if

$$\sum_{i \in I} \|Ae_i\|^2 < \infty. \tag{1.1}$$

It is well know that, if $\{e_i\}_{i \in I}$ and $\{f_j\}_{j \in J}$ are orthonormal bases for H and $A \in \mathcal{B}(H)$ then

$$\sum_{i \in I} \|Ae_i\|^2 = \sum_{j \in I} \|Af_j\|^2 = \sum_{j \in I} \|A^* f_j\|^2 \tag{1.2}$$

showing that the definition (1.1) is independent of the orthonormal basis and A is a Hilbert-Schmidt operator iff A^* is a Hilbert-Schmidt operator.

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Let $\mathcal{B}_2(H)$ the set of Hilbert-Schmidt operators in $\mathcal{B}(H)$. For $A \in \mathcal{B}_2(H)$ we define

$$\|A\|_2 := \left(\sum_{i\in I} \|Ae_i\|^2\right)^{1/2} \tag{1.3}$$

for $\{e_i\}_{i\in I}$ an orthonormal basis of H. This definition does not depend on the choice of the orthonormal basis.

Using the triangle inequality in $l^2(I)$, one checks that $\mathcal{B}_2(H)$ is a *vector space* and that $\|\cdot\|_2$ is a norm on $\mathcal{B}_2(H)$, which is usually called in the literature as the *Hilbert-Schmidt norm*.

Denote *the modulus* of an operator $A \in \mathcal{B}(H)$ by $|A| := (A^*A)^{1/2}$.

Because $\||A|x\| = \|Ax\|$ for all $x \in H$, A is Hilbert-Schmidt iff $|A|$ is Hilbert-Schmidt and $\|A\|_2 = \||A|\|_2$. From (1.2) we have that if $A \in \mathcal{B}_2(H)$, then $A^* \in \mathcal{B}_2(H)$ and $\|A\|_2 = \|A^*\|_2$.

The following theorem collects some of the most important properties of Hilbert-Schmidt operators:

Theorem 1.1. *We have*

(i) $(\mathcal{B}_2(H), \|\cdot\|_2)$ *is a Hilbert space with inner product*

$$\langle A, B\rangle_2 := \sum_{i\in I} \langle Ae_i, Be_i\rangle = \sum_{i\in I} \langle B^*Ae_i, e_i\rangle \tag{1.4}$$

and the definition does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$*;*

(ii) We have the inequalities

$$\|A\| \le \|A\|_2 \tag{1.5}$$

for any $A \in \mathcal{B}_2(H)$ *and*

$$\|AT\|_2, \|TA\|_2 \le \|T\| \|A\|_2 \tag{1.6}$$

for any $A \in \mathcal{B}_2(H)$ *and* $T \in \mathcal{B}(H)$*;*

(iii) $\mathcal{B}_2(H)$ *is an operator ideal in* $\mathcal{B}(H)$*, i.e.*

$$\mathcal{B}(H)\mathcal{B}_2(H)\mathcal{B}(H) \subseteq \mathcal{B}_2(H);$$

(iv) $\mathcal{B}_{fin}(H)$*, the space of operators of finite rank, is a dense subspace of* $\mathcal{B}_2(H)$*;*

(v) $\mathcal{B}_2(H) \subseteq \mathcal{K}(H)$*, where* $\mathcal{K}(H)$ *denotes the algebra of compact operators on* H*.*

If $\{e_i\}_{i\in I}$ an orthonormal basis of H, we say that $A \in \mathcal{B}(H)$ is *trace class* if

$$\|A\|_1 := \sum_{i\in I} \langle |A| e_i, e_i\rangle < \infty. \tag{1.7}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

The following proposition holds:

Proposition 1.1. *If $A \in \mathcal{B}(H)$, then the following are equivalent:*

(i) $A \in \mathcal{B}_1(H)$;

(ii) $|A|^{1/2} \in \mathcal{B}_2(H)$;

(ii) A (or $|A|$) is the product of two elements of $\mathcal{B}_2(H)$.

The following properties are also well known:

Theorem 1.2. *With the above notations:*

(i) We have

$$\|A\|_1 = \|A^*\|_1 \ \text{ and } \ \|A\|_2 \le \|A\|_1 \tag{1.8}$$

for any $A \in \mathcal{B}_1(H)$;

(ii) $\mathcal{B}_1(H)$ is an operator ideal in $\mathcal{B}(H)$, i.e.

$$\mathcal{B}(H)\,\mathcal{B}_1(H)\,\mathcal{B}(H) \subseteq \mathcal{B}_1(H);$$

(iii) We have

$$\mathcal{B}_2(H)\,\mathcal{B}_2(H) = \mathcal{B}_1(H);$$

(iv) We have

$$\|A\|_1 = \sup\{\langle A, B\rangle_2 \mid B \in \mathcal{B}_2(H),\ \ \|B\| \le 1\};$$

(v) $(\mathcal{B}_1(H), \|\cdot\|_1)$ is a Banach space.

(iv) We have the following isometric isomorphisms

$$\mathcal{B}_1(H) \cong K(H)^* \ \text{ and } \mathcal{B}_1(H)^* \cong \mathcal{B}(H),$$

where $K(H)^*$ is the dual space of $K(H)$ and $\mathcal{B}_1(H)^*$ is the dual space of $\mathcal{B}_1(H)$.

We define the *trace* of a trace class operator $A \in \mathcal{B}_1(H)$ to be

$$\operatorname{tr}(A) := \sum_{i\in I} \langle Ae_i, e_i\rangle, \tag{1.9}$$

where $\{e_i\}_{i\in I}$ an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (1.9) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

Theorem 1.3. *We have*

(i) If $A \in \mathcal{B}_1(H)$ *then* $A^* \in \mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(A^*) = \overline{\operatorname{tr}(A)}; \tag{1.10}$$

(ii) If $A \in \mathcal{B}_1(H)$ *and* $T \in \mathcal{B}(H)$, *then* $AT, TA \in \mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(AT) = \operatorname{tr}(TA) \text{ and } |\operatorname{tr}(AT)| \le \|A\|_1 \|T\|; \tag{1.11}$$

(iii) $\operatorname{tr}(\cdot)$ *is a bounded linear functional on* $\mathcal{B}_1(H)$ *with* $\|\operatorname{tr}\| = 1$;

(iv) If $A, B \in \mathcal{B}_2(H)$ *then* $AB, BA \in \mathcal{B}_1(H)$ *and* $\operatorname{tr}(AB) = \operatorname{tr}(BA)$;

(v) $\mathcal{B}_{fin}(H)$ *is a dense subspace of* $\mathcal{B}_1(H)$.

Utilising the trace notation we obviously have that

$$\langle A, B\rangle_2 = \operatorname{tr}(B^*A) = \operatorname{tr}(AB^*) \text{ and } \|A\|_2^2 = \operatorname{tr}(A^*A) = \operatorname{tr}\left(|A|^2\right)$$

for any $A, B \in \mathcal{B}_2(H)$.

For the theory of trace functionals and their applications the reader is referred to [37].

For some classical trace inequalities see [5], [7], [34] and [48], which are continuations of the work of Bellman [2]. For related works the reader can refer to [1], [3], [5], [28], [31], [32], [33], [35] and [45].

Consider the orthonormal basis $\mathcal{E} := \{e_i\}_{i\in I}$ in the complex Hilbert space $(H, \langle\cdot,\cdot\rangle)$ and for a nonzero operator $B \in \mathcal{B}_2(H)$ let introduce the subset of indices from I defined by

$$I_{\mathcal{E},B} := \{i \in I : \ Be_i \neq 0\}.$$

We observe that $I_{\mathcal{E},B}$ is non-empty for any nonzero operator B and if $\ker(B) = 0$, i.e. B is injective, then $I_{\mathcal{E},B} = I$. We also have for $B \in \mathcal{B}_2(H)$ that

$$\operatorname{tr}\left(|B|^2\right) = \operatorname{tr}(B^*B) = \sum_{i\in I}\langle B^*Be_i, e_i\rangle = \sum_{i\in I}\|Be_i\|^2 = \sum_{i\in I_{\mathcal{E},B}}\|Be_i\|^2.$$

In the recent paper [26] we obtained, among others, the following result for convex functions:

Theorem 1.4. *Let* A *be a selfadjoint operator on the Hilbert space* H *and assume that* $Sp(A) \subseteq [m, M]$ *for some scalars* m, M *with* $m < M$. *If* f *is a continuous convex function on* $[m, M]$, $\mathcal{E} := \{e_i\}_{i\in I}$ *is an orthonormal basis in* H *and* $B \in \mathcal{B}_2(H)\setminus\{0\}$, *then* $\frac{\operatorname{tr}(|B|^2A)}{\operatorname{tr}(|B|^2)} \in [m, M]$ *and*

$$\begin{aligned} & f\left(\frac{\operatorname{tr}(|B|^2A)}{\operatorname{tr}(|B|^2)}\right)\operatorname{tr}\left(|B|^2\right) \\ & \le J_{\mathcal{E}}(f; A, B) \le \operatorname{tr}\left(|B|^2 f(A)\right) \\ & \le \tfrac{1}{M-m}\left(f(m)\operatorname{tr}\left[|B|^2(M1_H - A)\right] + f(M)\operatorname{tr}\left[|B|^2(A - m1_H)\right]\right), \end{aligned} \tag{1.12}$$

where

$$J_{\mathcal{E}}(f;A,B) := \sum_{i\in I_{\mathcal{E},B}} f\left(\frac{\langle B^*ABe_i,e_i\rangle}{\|Be_i\|^2}\right)\|Be_i\|^2. \tag{1.13}$$

For related functionals and their superadditivity and monotonicity properties see [26].

For some inequalities for convex functions see [8]-[12], [27] and [44]. For inequalities for functions of selfadjoint operators, see [14]-[23], [38], [40], [41], [42], [43] and the books [24], [25] and [29].

Motivated by the above results we establish in this paper other trace inequalities for convex functions of selfadjoint operators. Some examples for convex functions of interest are also given.

2. New Inequalities for Convex Functions

We recall the *gradient inequality* for the convex function $f:[m,M]\to\mathbb{R}$, namely

$$f(\varsigma)-f(\tau)\geq\delta_f(\tau)(\varsigma-\tau) \tag{2.1}$$

for any $\varsigma,\tau\in[m,M]$ where $\delta_f(\tau)\in\left[f'_-(\tau),f'_+(\tau)\right]$, (for $\tau=m$ we take $\delta_f(\tau)=f'_+(m)$ and for $\tau=M$ we take $\delta_f(\tau)=f'_-(M)$). Here $f'_+(m)$ and $f'_-(M)$ are the lateral derivatives of the convex function f.

The following result holds:

Theorem 2.1. *Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A)\subseteq[m,M]$ for some scalars m, M with $m<M$. If f is a continuous convex function on $[m,M]$ and $B\in\mathcal{B}_2(H)\setminus\{0\}$, then we have $\frac{\operatorname{tr}\left(|B|^2A\right)}{\operatorname{tr}\left(|B|^2\right)}\in[m,M]$,*

$$\begin{aligned}&\delta_f\left(\frac{\operatorname{tr}\left(|B|^2A\right)}{\operatorname{tr}\left(|B|^2\right)}\right)\frac{\operatorname{tr}\left(|B^*|^2A\right)-\operatorname{tr}\left(|B|^2A\right)}{\operatorname{tr}\left(|B|^2\right)}\\&\qquad\leq\frac{\operatorname{tr}\left(|B^*|^2f(A)\right)}{\operatorname{tr}\left(|B|^2\right)}-f\left(\frac{\operatorname{tr}\left(|B|^2A\right)}{\operatorname{tr}\left(|B|^2\right)}\right),\end{aligned} \tag{2.2}$$

where

$$\delta_f\left(\frac{\operatorname{tr}\left(|B|^2A\right)}{\operatorname{tr}\left(|B|^2\right)}\right)\in\left[f'_-\left(\frac{\operatorname{tr}\left(|B|^2A\right)}{\operatorname{tr}\left(|B|^2\right)}\right),f'_+\left(\frac{\operatorname{tr}\left(|B|^2A\right)}{\operatorname{tr}\left(|B|^2\right)}\right)\right]$$

and the Jensen's inequality

$$f\left(\frac{\operatorname{tr}\left(|B|^2A\right)}{\operatorname{tr}\left(|B|^2\right)}\right)\leq\frac{\operatorname{tr}\left(|B|^2f(A)\right)}{\operatorname{tr}\left(|B|^2\right)}. \tag{2.3}$$

Proof. Let $\mathcal{E} := \{e_i\}_{i \in I}$ be an orthonormal basis in H.

Utilising the gradient inequality (2.1) we get

$$f(\varsigma) - f\left(\frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}\right) \geq \delta_f\left(\frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}\right)\left(\varsigma - \frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}\right) \tag{2.4}$$

for any $\varsigma \in [m, M]$, since obviously, by $Sp(A) \subseteq [m, M]$ we have

$$m\|Be_i\|^2 \leq \langle ABe_i, Be_i\rangle \leq M\|Be_i\|^2,$$

for $i \in I$, which, by summation shows that

$$\frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)} \in [m, M].$$

The inequality (2.4) implies in the operator order of $\mathcal{B}(H)$ that

$$f(A) - f\left(\frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}\right)1_H \geq \delta_f\left(\frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}\right)\left(A - \frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}1_H\right), \tag{2.5}$$

which can be written as

$$\begin{aligned} &\langle f(A)y, y\rangle - f\left(\tfrac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}\right)\langle y, y\rangle \\ &\geq \delta_f\left(\tfrac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}\right)\left(\langle Ay, y\rangle - \tfrac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}\langle y, y\rangle\right), \end{aligned} \tag{2.6}$$

for any $y \in H$. This inequality is also of interest in itself.

Taking in (2.6) $y = Be_i$ we get

$$\begin{aligned} &\langle f(A)Be_i, Be_i\rangle - f\left(\tfrac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}\right)\langle Be_i, Be_i\rangle \\ &\geq \delta_f\left(\tfrac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}\right)\left(\langle ABe_i, Be_i\rangle - \tfrac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}\langle Be_i, Be_i\rangle\right), \end{aligned}$$

which is equivalent to

$$\begin{aligned} &\langle B^* f(A)Be_i, e_i\rangle - f\left(\tfrac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}\right)\left\langle |B|^2 e_i, e_i\right\rangle \\ &\geq \delta_f\left(\tfrac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}\right)\left(\langle B^* ABe_i, e_i\rangle - \tfrac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}\left\langle |B|^2 e_i, e_i\right\rangle\right), \end{aligned} \tag{2.7}$$

for any $i \in I$.

Summing in (2.7) we get

$$\sum_{i\in I}\langle B^{*}f(A)Be_{i},e_{i}\rangle-f\left(\frac{\operatorname{tr}\left(|B|^{2}A\right)}{\operatorname{tr}\left(|B|^{2}\right)}\right)\sum_{i\in I}\left\langle|B|^{2}e_{i},e_{i}\right\rangle \tag{2.8}$$
$$\geq\delta_{f}\left(\frac{\operatorname{tr}\left(|B|^{2}A\right)}{\operatorname{tr}\left(|B|^{2}\right)}\right)\left(\sum_{i\in I}\langle B^{*}ABe_{i},e_{i}\rangle-\frac{\operatorname{tr}\left(|B|^{2}A\right)}{\operatorname{tr}\left(|B|^{2}\right)}\sum_{i\in I}\left\langle|B|^{2}e_{i},e_{i}\right\rangle\right).$$

However

$$\sum_{i\in I}\langle B^{*}f(A)Be_{i},e_{i}\rangle=\sum_{i\in I}\langle BB^{*}f(A)e_{i},e_{i}\rangle$$
$$=\sum_{i\in I}\left\langle|B^{*}|^{2}f(A)e_{i},e_{i}\right\rangle=\operatorname{tr}\left(|B^{*}|^{2}f(A)\right)$$

and

$$\sum_{i\in I}\langle B^{*}ABe_{i},e_{i}\rangle=\sum_{i\in I}\langle BB^{*}Ae_{i},e_{i}\rangle=\operatorname{tr}\left(|B^{*}|^{2}A\right).$$

By (2.8) we get

$$\operatorname{tr}\left(|B^{*}|^{2}f(A)\right)-f\left(\frac{\operatorname{tr}\left(|B|^{2}A\right)}{\operatorname{tr}\left(|B|^{2}\right)}\right)\operatorname{tr}\left(|B|^{2}\right) \tag{2.9}$$
$$\geq\delta_{f}\left(\frac{\operatorname{tr}\left(|B|^{2}A\right)}{\operatorname{tr}\left(|B|^{2}\right)}\right)\left(\operatorname{tr}\left(|B^{*}|^{2}A\right)-\operatorname{tr}\left(|B|^{2}A\right)\right),$$

and the inequality (2.2) is thus proved.

Taking in (2.6) $y=B^{*}e_{i}$ we also get

$$\langle f(A)B^{*}e_{i},B^{*}e_{i}\rangle-f\left(\frac{\operatorname{tr}\left(|B|^{2}A\right)}{\operatorname{tr}\left(|B|^{2}\right)}\right)\langle B^{*}e_{i},B^{*}e_{i}\rangle$$
$$\geq\delta_{f}\left(\frac{\operatorname{tr}\left(|B|^{2}A\right)}{\operatorname{tr}\left(|B|^{2}\right)}\right)\left(\langle AB^{*}e_{i},B^{*}e_{i}\rangle-\frac{\operatorname{tr}\left(|B|^{2}A\right)}{\operatorname{tr}\left(|B|^{2}\right)}\langle B^{*}e_{i},B^{*}e_{i}\rangle\right),$$

which is equivalent to

$$\langle Bf(A)B^{*}e_{i},e_{i}\rangle-f\left(\frac{\operatorname{tr}\left(|B|^{2}A\right)}{\operatorname{tr}\left(|B|^{2}\right)}\right)\langle BB^{*}e_{i},e_{i}\rangle \tag{2.10}$$
$$\geq\delta_{f}\left(\frac{\operatorname{tr}\left(|B|^{2}A\right)}{\operatorname{tr}\left(|B|^{2}\right)}\right)\left(\langle BAB^{*}e_{i},e_{i}\rangle-\frac{\operatorname{tr}\left(|B|^{2}A\right)}{\operatorname{tr}\left(|B|^{2}\right)}\langle BB^{*}e_{i},e_{i}\rangle\right),$$

for any $i\in I$.

Summing in (2.10) we get

$$\sum_{i\in I}\langle Bf(A)B^{*}e_{i},e_{i}\rangle-f\left(\frac{\operatorname{tr}\left(|B|^{2}A\right)}{\operatorname{tr}\left(|B|^{2}\right)}\right)\sum_{i\in I}\langle BB^{*}e_{i},e_{i}\rangle \tag{2.11}$$
$$\geq\delta_{f}\left(\frac{\operatorname{tr}\left(|B|^{2}A\right)}{\operatorname{tr}\left(|B|^{2}\right)}\right)\left(\sum_{i\in I}\langle BAB^{*}e_{i},e_{i}\rangle-\frac{\operatorname{tr}\left(|B|^{2}A\right)}{\operatorname{tr}\left(|B|^{2}\right)}\sum_{i\in I}\langle BB^{*}e_{i},e_{i}\rangle\right).$$

Since

$$\sum_{i\in I}\langle Bf(A)B^{*}e_i,e_i\rangle=\operatorname{tr}(Bf(A)B^{*})=\operatorname{tr}(B^{*}Bf(A))=\operatorname{tr}\left(|B|^2 f(A)\right),$$

$$\sum_{i\in I}\langle BB^{*}e_i,e_i\rangle=\operatorname{tr}(BB^{*})=\operatorname{tr}(B^{*}B)=\operatorname{tr}\left(|B|^2\right)$$

and

$$\sum_{i\in I}\langle BAB^{*}e_i,e_i\rangle=\operatorname{tr}(BAB^{*})=\operatorname{tr}(B^{*}BA)=\operatorname{tr}\left(|B|^2 A\right),$$

then by (2.11) we get

$$\operatorname{tr}\left(|B|^2 f(A)\right)-f\left(\frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}\right)\operatorname{tr}\left(|B|^2\right)\geq 0$$

and the inequality (2.3) is obtained. □

Remark 2.1. The inequality (2.3) is obviously not as good as the first part of (1.12). However it is the natural alternative of Jensen's inequality for trace and provides simple and nice examples for various convex functions of interest. The proof here is also simpler than the one from [26] and has some natural reverses as follows.

Corollary 2.1. *Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A)\subseteq[m,M]$ for some scalars m, M with $m<M$. If f is a continuous convex function on $[m,M]$ and $P\in\mathcal{B}_1(H)\setminus\{0\}$, $P\geq 0$ then $\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\in[m,M]$ and*

$$f\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)\leq\frac{\operatorname{tr}(Pf(A))}{\operatorname{tr}(P)}. \tag{2.12}$$

The proof follows by either (2.2) or (2.3) on choosing $B=P^{1/2}$, $P\in\mathcal{B}_1(H)\setminus\{0\}$, $P\geq 0$.

The following lemma is of interest in itself:

Lemma 2.1. *Let S be a selfadjoint operator such that $\gamma 1_H\leq S\leq\Gamma 1_H$ for some real constants $\Gamma\geq\gamma$. Then for any $B\in\mathcal{B}_2(H)\setminus\{0\}$ we have*

$$\begin{aligned}0&\leq\frac{\operatorname{tr}\left(|B|^2S^2\right)}{\operatorname{tr}\left(|B|^2\right)}-\left(\frac{\operatorname{tr}\left(|B|^2S\right)}{\operatorname{tr}\left(|B|^2\right)}\right)^2\\&\leq\tfrac{1}{2}(\Gamma-\gamma)\frac{1}{\operatorname{tr}\left(|B|^2\right)}\operatorname{tr}\left(|B|^2\left|S-\frac{\operatorname{tr}\left(|B|^2S\right)}{\operatorname{tr}\left(|B|^2\right)}1_H\right|\right)\\&\leq\tfrac{1}{2}(\Gamma-\gamma)\left[\frac{\operatorname{tr}\left(|B|^2S^2\right)}{\operatorname{tr}\left(|B|^2\right)}-\left(\frac{\operatorname{tr}\left(|B|^2S\right)}{\operatorname{tr}\left(|B|^2\right)}\right)^2\right]^{1/2}\leq\tfrac{1}{4}(\Gamma-\gamma)^2.\end{aligned} \tag{2.13}$$

Proof. The first inequality follows by Jensen's inequality (2.3) for the convex function $f(t) = t^2$.

Now, observe that

$$\frac{1}{\operatorname{tr}\left(|B|^2\right)} \operatorname{tr}\left(|B|^2\left(S-\frac{\Gamma+\gamma}{2}1_H\right)\left(S-\frac{\operatorname{tr}\left(|B|^2 S\right)}{\operatorname{tr}\left(|B|^2\right)}1_H\right)\right) \tag{2.14}$$

$$= \frac{1}{\operatorname{tr}\left(|B|^2\right)} \operatorname{tr}\left(|B|^2 S\left(S-\frac{\operatorname{tr}\left(|B|^2 S\right)}{\operatorname{tr}\left(|B|^2\right)}1_H\right)\right)$$

$$-\frac{\Gamma+\gamma}{2}\frac{1}{\operatorname{tr}\left(|B|^2\right)} \operatorname{tr}\left(|B|^2\left(S-\frac{\operatorname{tr}\left(|B|^2 S\right)}{\operatorname{tr}\left(|B|^2\right)}1_H\right)\right)$$

$$= \frac{\operatorname{tr}\left(|B|^2 S^2\right)}{\operatorname{tr}\left(|B|^2\right)} - \left(\frac{\operatorname{tr}\left(|B|^2 S\right)}{\operatorname{tr}\left(|B|^2\right)}\right)^2$$

since, obviously

$$\operatorname{tr}\left(|B|^2\left(S-\frac{\operatorname{tr}\left(|B|^2 S\right)}{\operatorname{tr}\left(|B|^2\right)}1_H\right)\right) = 0.$$

Now, since $\gamma 1_H \le S \le \Gamma 1_H$ then

$$\left|S-\frac{\Gamma+\gamma}{2}1_H\right| \le \frac{1}{2}(\Gamma-\gamma).$$

Taking the modulus in (2.14) and using the properties of trace, we have

$$\frac{\operatorname{tr}\left(|B|^2 S^2\right)}{\operatorname{tr}\left(|B|^2\right)} - \left(\frac{\operatorname{tr}\left(|B|^2 S\right)}{\operatorname{tr}\left(|B|^2\right)}\right)^2 \tag{2.15}$$

$$= \frac{1}{\operatorname{tr}\left(|B|^2\right)} \left|\operatorname{tr}\left(|B|^2\left(S-\frac{\Gamma+\gamma}{2}1_H\right)\left(S-\frac{\operatorname{tr}\left(|B|^2 S\right)}{\operatorname{tr}\left(|B|^2\right)}1_H\right)\right)\right|$$

$$\le \frac{1}{\operatorname{tr}\left(|B|^2\right)} \operatorname{tr}\left(|B|^2\left|\left(S-\frac{\Gamma+\gamma}{2}1_H\right)\left(S-\frac{\operatorname{tr}\left(|B|^2 S\right)}{\operatorname{tr}\left(|B|^2\right)}1_H\right)\right|\right)$$

$$\le \frac{1}{2}(\Gamma-\gamma)\frac{1}{\operatorname{tr}\left(|B|^2\right)} \operatorname{tr}\left(|B|^2\left|S-\frac{\operatorname{tr}\left(|B|^2 S\right)}{\operatorname{tr}\left(|B|^2\right)}1_H\right|\right),$$

which proves the first part of (2.13).

By Schwarz inequality for trace we also have

$$\frac{1}{\operatorname{tr}\left(|B|^2\right)} \operatorname{tr}\left(|B|^2\left|S-\frac{\operatorname{tr}\left(|B|^2 S\right)}{\operatorname{tr}\left(|B|^2\right)}1_H\right|\right) \tag{2.16}$$

$$\le \left[\frac{1}{\operatorname{tr}\left(|B|^2\right)} \operatorname{tr}\left(|B|^2\left(S-\frac{\operatorname{tr}\left(|B|^2 S\right)}{\operatorname{tr}\left(|B|^2\right)}1_H\right)^2\right)\right]^{1/2}$$

$$= \left[\frac{\operatorname{tr}\left(|B|^2 S^2\right)}{\operatorname{tr}\left(|B|^2\right)} - \left(\frac{\operatorname{tr}\left(|B|^2 S\right)}{\operatorname{tr}\left(|B|^2\right)}\right)^2\right]^{1/2}.$$

From (2.15) and (2.16) we get

$$\frac{\operatorname{tr}\left(|B|^2 S^2\right)}{\operatorname{tr}\left(|B|^2\right)} - \left(\frac{\operatorname{tr}\left(|B|^2 S\right)}{\operatorname{tr}\left(|B|^2\right)}\right)^2 \le \frac{1}{2}(\Gamma-\gamma)\left[\frac{\operatorname{tr}\left(|B|^2 S^2\right)}{\operatorname{tr}\left(|B|^2\right)} - \left(\frac{\operatorname{tr}\left(|B|^2 S\right)}{\operatorname{tr}\left(|B|^2\right)}\right)^2\right]^{1/2},$$

which implies that

$$\left[\frac{\operatorname{tr}\left(|B|^2 S^2\right)}{\operatorname{tr}\left(|B|^2\right)} - \left(\frac{\operatorname{tr}\left(|B|^2 S\right)}{\operatorname{tr}\left(|B|^2\right)}\right)^2\right]^{1/2} \le \frac{1}{2}(\Gamma-\gamma).$$

By (2.16) we then obtain

$$\frac{1}{\operatorname{tr}\left(|B|^2\right)} \operatorname{tr}\left(|B|^2 \left|S - \frac{\operatorname{tr}\left(|B|^2 S\right)}{\operatorname{tr}\left(|B|^2\right)} 1_H\right|\right) \le \left[\frac{\operatorname{tr}\left(|B|^2 S^2\right)}{\operatorname{tr}\left(|B|^2\right)} - \left(\frac{\operatorname{tr}\left(|B|^2 S\right)}{\operatorname{tr}\left(|B|^2\right)}\right)^2\right]^{1/2} \le \frac{1}{2}(\Gamma-\gamma)$$

that proves the last part of (2.13). □

Remark 2.2. Let S be a selfadjoint operator such that $\gamma 1_H \le S \le \Gamma 1_H$ for some real constants $\Gamma \ge \gamma$. Then for any $P \in \mathcal{B}_1(H) \setminus \{0\}$, $P \ge 0$ we have

$$\begin{aligned}0 &\le \frac{\operatorname{tr}\left(PS^2\right)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PS)}{\operatorname{tr}(P)}\right)^2 \\ &\le \frac{1}{2}(\Gamma-\gamma)\frac{1}{\operatorname{tr}(P)}\operatorname{tr}\left(P\left|S - \frac{\operatorname{tr}(PS)}{\operatorname{tr}(P)}1_H\right|\right) \\ &\le \frac{1}{2}(\Gamma-\gamma)\left[\frac{\operatorname{tr}\left(PS^2\right)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PS)}{\operatorname{tr}(P)}\right)^2\right]^{1/2} \le \frac{1}{4}(\Gamma-\gamma)^2.\end{aligned} \tag{2.17}$$

The following result provides reverses for the inequalities (2.2) and (2.3) above:

Theorem 2.2. *Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $m < M$. If f is a continuously differentiable convex function on $[m, M]$ and $B \in \mathcal{B}_2(H) \setminus \{0\}$, then we have*

$$\begin{aligned}0 &\le \frac{\operatorname{tr}\left(|B^*|^2 f(A)\right)}{\operatorname{tr}\left(|B|^2\right)} - f\left(\frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}\right) \\ &\le \frac{\operatorname{tr}\left(|B^*|^2 f'(A)A\right)}{\operatorname{tr}\left(|B|^2\right)} - \frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)} \cdot \frac{\operatorname{tr}\left(|B^*|^2 f'(A)\right)}{\operatorname{tr}\left(|B|^2\right)}\end{aligned} \tag{2.18}$$

and

$$\begin{aligned}0 &\le \frac{\operatorname{tr}\left(|B|^2 f(A)\right)}{\operatorname{tr}\left(|B|^2\right)} - f\left(\frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}\right) \\ &\le \frac{\operatorname{tr}\left(|B|^2 f'(A)A\right)}{\operatorname{tr}\left(|B|^2\right)} - \frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)} \cdot \frac{\operatorname{tr}\left(|B|^2 f'(A)\right)}{\operatorname{tr}\left(|B|^2\right)} =: \mathcal{K}(f', B, A).\end{aligned} \tag{2.19}$$

Moreover, we have

$$\mathcal{K}(f', B, A) \tag{2.20}$$
$$\leq \begin{cases} \frac{1}{2}\left[f'(M) - f'(m)\right] \dfrac{\operatorname{tr}\left(|B|^2 \left| A - \frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)} 1_H \right|\right)}{\operatorname{tr}\left(|B|^2\right)} \\ \\ \frac{1}{2}(M-m) \dfrac{\operatorname{tr}\left(|B|^2 \left| f'(A) - \frac{\operatorname{tr}\left(|B|^2 f'(A)\right)}{\operatorname{tr}\left(|B|^2\right)} 1_H \right|\right)}{\operatorname{tr}\left(|B|^2\right)} \end{cases}$$
$$\leq \begin{cases} \frac{1}{2}\left[f'(M) - f'(m)\right] \left[\dfrac{\operatorname{tr}\left(|B|^2 A^2\right)}{\operatorname{tr}\left(|B|^2\right)} - \left(\dfrac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}\right)^2\right]^{1/2} \\ \\ \frac{1}{2}(M-m) \left[\dfrac{\operatorname{tr}\left(|B|^2 \left[f'(A)\right]^2\right)}{\operatorname{tr}\left(|B|^2\right)} - \left(\dfrac{\operatorname{tr}\left(|B|^2 f'(A)\right)}{\operatorname{tr}\left(|B|^2\right)}\right)^2\right]^{1/2} \end{cases}$$
$$\leq \frac{1}{4}\left[f'(M) - f'(m)\right](M-m).$$

Proof. By the gradient inequality we have

$$f(\tau) - f(\varsigma) \leq f'(\tau)(\tau - \varsigma) \tag{2.21}$$

for any $\tau, \varsigma \in [m, M]$.

This inequality implies in the operator order

$$f(A) - f\left(\frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}\right) 1_H \leq f'(A)\left(A - \frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)} 1_H\right)$$

that is equivalent to

$$\langle f(A) y, y\rangle - f\left(\frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}\right)\langle y, y\rangle \tag{2.22}$$
$$\leq \langle f'(A) A y, y\rangle - \frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}\langle f'(A) y, y\rangle$$

for any $y \in H$, which is of interest in itself as well.

Let $\mathcal{E} := \{e_i\}_{i \in I}$ be an orthonormal basis in H.

If we take in (2.22) $y = Be_i$ and sum, then we get

$$\sum_{i \in I}\langle f(A) Be_i, Be_i\rangle - f\left(\frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}\right)\sum_{i \in I}\langle Be_i, Be_i\rangle$$
$$\leq \sum_{i \in I}\langle f'(A) ABe_i, Be_i\rangle - \frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}\sum_{i \in I}\langle f'(A) Be_i, Be_i\rangle,$$

which is equivalent to

$$\begin{aligned}&\sum_{i\in I}\langle B^{*}f(A)Be_i,e_i\rangle - f\left(\frac{\operatorname{tr}\left(|B|^2A\right)}{\operatorname{tr}\left(|B|^2\right)}\right)\sum_{i\in I}\langle B^{*}Be_i,e_i\rangle\\&\le \sum_{i\in I}\langle B^{*}f'(A)ABe_i,e_i\rangle - \frac{\operatorname{tr}\left(|B|^2A\right)}{\operatorname{tr}\left(|B|^2\right)}\sum_{i\in I}\langle B^{*}f'(A)Be_i,e_i\rangle\end{aligned}$$

and the inequality (2.18) is obtained.

If we take in (2.22) $y=B^{*}e_i$ and sum, then we get

$$\begin{aligned}&\sum_{i\in I}\langle f(A)B^{*}e_i,B^{*}e_i\rangle - f\left(\frac{\operatorname{tr}\left(|B|^2A\right)}{\operatorname{tr}\left(|B|^2\right)}\right)\sum_{i\in I}\langle B^{*}e_i,B^{*}e_i\rangle\\&\le \sum_{i\in I}\langle f'(A)AB^{*}e_i,B^{*}e_i\rangle - \frac{\operatorname{tr}\left(|B|^2A\right)}{\operatorname{tr}\left(|B|^2\right)}\sum_{i\in I}\langle f'(A)B^{*}e_i,B^{*}e_i\rangle\end{aligned}$$

that is equivalent to

$$\begin{aligned}&\sum_{i\in I}\langle Bf(A)B^{*}e_i,e_i\rangle - f\left(\frac{\operatorname{tr}\left(|B|^2A\right)}{\operatorname{tr}\left(|B|^2\right)}\right)\sum_{i\in I}\langle BB^{*}e_i,e_i\rangle\\&\le \sum_{i\in I}\langle Bf'(A)AB^{*}e_i,e_i\rangle - \frac{\operatorname{tr}\left(|B|^2A\right)}{\operatorname{tr}\left(|B|^2\right)}\sum_{i\in I}\langle Bf'(A)B^{*}e_i,e_i\rangle\end{aligned}\tag{2.23}$$

and the inequality (2.19) is obtained.

Now, since f is continuously convex on $[m,M]$, then f' is monotonic nondecreasing on $[m,M]$ and $f'(m)\le f'(t)\le f'(M)$ for any $t\in[m,M]$. We also observe that

$$\begin{aligned}&\frac{1}{\operatorname{tr}\left(|B|^2\right)}\operatorname{tr}\left(|B|^2\left[f'(A)-\frac{f'(m)+f'(M)}{2}1_H\right]\left[A-\frac{\operatorname{tr}\left(|B|^2A\right)}{\operatorname{tr}\left(|B|^2\right)}1_H\right]\right)\\&=\frac{1}{\operatorname{tr}\left(|B|^2\right)}\operatorname{tr}\left(|B|^2f'(A)\left[A-\frac{\operatorname{tr}\left(|B|^2A\right)}{\operatorname{tr}\left(|B|^2\right)}1_H\right]\right)\\&\quad-\frac{f'(m)+f'(M)}{2}\frac{1}{\operatorname{tr}\left(|B|^2\right)}\operatorname{tr}\left(|B|^2\left[A-\frac{\operatorname{tr}\left(|B|^2A\right)}{\operatorname{tr}\left(|B|^2\right)}1_H\right]\right)\\&=\mathcal{K}(f',B,A).\end{aligned}\tag{2.24}$$

Since

$$\left|f'(A)-\frac{f'(m)+f'(M)}{2}1_H\right|\le\frac{1}{2}\left[f'(M)-f'(m)\right]1_H,$$

then by taking the modulus in (2.24) and utilizing the properties of trace we have

$$\begin{aligned}0&\le\mathcal{K}(f',B,A)\\&\le\frac{1}{\operatorname{tr}\left(|B|^2\right)}\\&\quad\times\operatorname{tr}\left(|B|^2\left|\left[f'(A)-\frac{f'(m)+f'(M)}{2}1_H\right]\left[A-\frac{\operatorname{tr}\left(|B|^2A\right)}{\operatorname{tr}\left(|B|^2\right)}1_H\right]\right|\right)\\&\le\frac{1}{2}\left[f'(M)-f'(m)\right]\frac{1}{\operatorname{tr}\left(|B|^2\right)}\operatorname{tr}\left(|B|^2\left|A-\frac{\operatorname{tr}\left(|B|^2A\right)}{\operatorname{tr}\left(|B|^2\right)}1_H\right|\right),\end{aligned}\tag{2.25}$$

and the first inequality in the first branch of (2.20) is proved.

We have $m1_H \leq A \leq M1_H$ and by applying Lemma 2.1 we can state that

$$\frac{1}{\operatorname{tr}(|B|^2)} \operatorname{tr}\left(|B|^2 \left|A - \frac{\operatorname{tr}(|B|^2 A)}{\operatorname{tr}(|B|^2)} 1_H\right|\right) \leq \left[\frac{\operatorname{tr}(|B|^2 A^2)}{\operatorname{tr}(|B|^2)} - \left(\frac{\operatorname{tr}(|B|^2 A)}{\operatorname{tr}(|B|^2)}\right)^2\right]^{1/2} \leq \frac{1}{2}(M - m). \tag{2.26}$$

Making use of (2.25) and (2.26) we deduce the second and the third inequalities in the first branch of (2.20).

We observe that $\mathcal{K}(f', B, A)$ can be also represented as

$$\mathcal{K}(f', B, A) = \frac{1}{\operatorname{tr}(|B|^2)} \operatorname{tr}\left(|B|^2 \left[f'(A) - \frac{\operatorname{tr}(|B|^2 f'(A))}{\operatorname{tr}(|B|^2)} 1_H\right] \left(A - \frac{m+M}{2} 1_H\right)\right).$$

Applying a similar argument as above for this representation, we get the second branch of the inequality (2.20).

The proof is complete. □

Corollary 2.2. *Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $m < M$. If f is a continuously differentiable convex function on $[m, M]$ and $P \in \mathcal{B}_1(H) \setminus \{0\}$, $P \geq 0$, then we have*

$$\begin{aligned}
0 &\leq \frac{\operatorname{tr}(Pf(A))}{\operatorname{tr}(P)} - f\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right) \\
&\leq \frac{\operatorname{tr}(Pf'(A)A)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} \cdot \frac{\operatorname{tr}(Pf'(A))}{\operatorname{tr}(P)} \\
&\leq \begin{cases} \frac{1}{2}[f'(M) - f'(m)] \frac{\operatorname{tr}\left(P\left|A - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} 1_H\right|\right)}{\operatorname{tr}(P)} \\[2ex] \frac{1}{2}(M - m) \frac{\operatorname{tr}\left(P\left|f'(A) - \frac{\operatorname{tr}(Pf'(A))}{\operatorname{tr}(P)} 1_H\right|\right)}{\operatorname{tr}(P)} \end{cases} \\
&\leq \begin{cases} \frac{1}{2}[f'(M) - f'(m)] \left[\frac{\operatorname{tr}(PA^2)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^2\right]^{1/2} \\[2ex] \frac{1}{2}(M - m) \left[\frac{\operatorname{tr}\left(P[f'(A)]^2\right)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(Pf'(A))}{\operatorname{tr}(P)}\right)^2\right]^{1/2} \end{cases} \\
&\leq \frac{1}{4}[f'(M) - f'(m)](M - m).
\end{aligned} \tag{2.27}$$

Remark 2.3. Let $\mathcal{M}_n(\mathbb{C})$ be the space of all square matrices of order n with complex elements and $A \in \mathcal{M}_n(\mathbb{C})$ be a Hermitian matrix such that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $m<M$. If f is a continuously differentiable convex function on $[m,M]$, then by taking $P=I_n$, the identity matrix, in (2.27) we get

$$
\begin{aligned}
0 &\le \frac{\operatorname{tr}(f(A))}{n} - f\left(\frac{\operatorname{tr}(A)}{n}\right) \\
&\le \frac{\operatorname{tr}\left(f'(A)A\right)}{n} - \frac{\operatorname{tr}(A)}{n}\cdot\frac{\operatorname{tr}\left(f'(A)\right)}{n} \\
&\le \begin{cases} \frac{1}{2}\left[f'(M)-f'(m)\right]\frac{\operatorname{tr}\left(\left|A-\frac{\operatorname{tr}(A)}{n}I_n\right|\right)}{n} \\[2ex] \frac{1}{2}(M-m)\frac{\operatorname{tr}\left(\left|f'(A)-\frac{\operatorname{tr}\left(f'(A)\right)}{n}I_n\right|\right)}{n} \end{cases} \\
&\le \begin{cases} \frac{1}{2}\left[f'(M)-f'(m)\right]\left[\frac{\operatorname{tr}\left(A^2\right)}{n}-\left(\frac{\operatorname{tr}(A)}{n}\right)^2\right]^{1/2} \\[2ex] \frac{1}{2}(M-m)\left[\frac{\operatorname{tr}\left(\left[f'(A)\right]^2\right)}{n}-\left(\frac{\operatorname{tr}\left(f'(A)\right)}{n}\right)^2\right]^{1/2} \end{cases} \\
&\le \frac{1}{4}\left[f'(M)-f'(m)\right](M-m). \qquad (2.28)
\end{aligned}
$$

3. Some Examples

We consider the power function $f:(0,\infty)\to(0,\infty)$, $f(t)=t^r$ with $t\in\mathbb{R}\setminus\{0\}$. For $r\in(-\infty,0)\cup[1,\infty)$, f is convex while for $r\in(0,1)$, f is concave. Denote $\mathcal{B}_1^+(H):=\{P \text{ with } P\in\mathcal{B}_1(H) \text{ and } P\ge 0\}$.

Let $r\ge 1$ and A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A)\subseteq[m,M]$ for some scalars m, M with $0\le m<M$. If $P\in\mathcal{B}_1^+(H)\setminus\{0\}$, then

$$
\begin{aligned}
0 &\le \frac{\operatorname{tr}(PA^r)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^r \\
&\le r\left[\frac{\operatorname{tr}(PA^r)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\cdot\frac{\operatorname{tr}\left(PA^{r-1}\right)}{\operatorname{tr}(P)}\right] \\
&\le \begin{cases} \frac{1}{2}r\left(M^{r-1}-m^{r-1}\right)\frac{\operatorname{tr}\left(P\left|A-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}1_H\right|\right)}{\operatorname{tr}(P)} \\[2ex] \frac{1}{2}r(M-m)\frac{\operatorname{tr}\left(P\left|A^{r-1}-\frac{\operatorname{tr}\left(PA^{r-1}\right)}{\operatorname{tr}(P)}1_H\right|\right)}{\operatorname{tr}(P)} \end{cases} \\
&\le \begin{cases} \frac{1}{2}r\left(M^{r-1}-m^{r-1}\right)\left[\frac{\operatorname{tr}\left(PA^2\right)}{\operatorname{tr}(P)}-\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^2\right]^{1/2} \\[2ex] \frac{1}{2}r(M-m)\left[\frac{\operatorname{tr}\left(PA^{2(r-1)}\right)}{\operatorname{tr}(P)}-\left(\frac{\operatorname{tr}\left(PA^{r-1}\right)}{\operatorname{tr}(P)}\right)^2\right]^{1/2} \end{cases} \\
&\le \frac{1}{4}r\left(M^{r-1}-m^{r-1}\right)(M-m). \qquad (3.1)
\end{aligned}
$$

Consider the convex function $f:(0,\infty)\to(0,\infty)$, $f(t)=-\ln t$ and let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A)\subseteq[m,M]$ for some scalars m, M with $0<m<M$. If $P\in\mathcal{B}_1^+(H)\setminus\{0\}$, then

$$
\begin{aligned}
(3.2)\qquad 0 &\le \ln\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)-\frac{\operatorname{tr}(P\ln A)}{\operatorname{tr}(P)}\\
&\le \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\cdot\frac{\operatorname{tr}\left(PA^{-1}\right)}{\operatorname{tr}(P)}-1\\
&\le \begin{cases}\frac{M-m}{2mM}\frac{\operatorname{tr}\left(P\left|A-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}1_H\right|\right)}{\operatorname{tr}(P)}\\[2ex] \frac{1}{2}(M-m)\frac{\operatorname{tr}\left(P\left|A^{-1}-\frac{\operatorname{tr}\left(PA^{-1}\right)}{\operatorname{tr}(P)}1_H\right|\right)}{\operatorname{tr}(P)}\end{cases}\\
&\le \begin{cases}\frac{M-m}{2mM}\left[\frac{\operatorname{tr}\left(PA^{2}\right)}{\operatorname{tr}(P)}-\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^2\right]^{1/2}\\[2ex] \frac{1}{2}(M-m)\left[\frac{\operatorname{tr}\left(PA^{-2}\right)}{\operatorname{tr}(P)}-\left(\frac{\operatorname{tr}\left(PA^{-1}\right)}{\operatorname{tr}(P)}\right)^2\right]^{1/2}\end{cases}\\
&\le \frac{(M-m)^2}{4mM}.
\end{aligned}
$$

Consider the convex function $f(t)=t\ln t$ and let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A)\subseteq[m,M]$ for some scalars m, M with $0<m<M$. If $P\in\mathcal{B}_1^+(H)\setminus\{0\}$, then

$$
\begin{aligned}
(3.3)\qquad 0 &\le \frac{\operatorname{tr}(PA\ln A)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\ln\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)\\
&\le \frac{\operatorname{tr}(PA\ln(eA))}{\operatorname{tr}(P)}-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\cdot\frac{\operatorname{tr}(P\ln(eA))}{\operatorname{tr}(P)}\\
&\le \begin{cases}\frac{1}{2}\ln\left(\frac{M}{m}\right)\frac{\operatorname{tr}\left(P\left|A-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}1_H\right|\right)}{\operatorname{tr}(P)}\\[2ex] \frac{1}{2}(M-m)\frac{\operatorname{tr}\left(P\left|\ln(eA)-\frac{\operatorname{tr}(P\ln(eA))}{\operatorname{tr}(P)}1_H\right|\right)}{\operatorname{tr}(P)}\end{cases}\\
&\le \begin{cases}\frac{1}{2}\ln\left(\frac{M}{m}\right)\left[\frac{\operatorname{tr}\left(PA^{2}\right)}{\operatorname{tr}(P)}-\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^2\right]^{1/2}\\[2ex] \frac{1}{2}(M-m)\left[\frac{\operatorname{tr}\left(P[\ln(eA)]^{2}\right)}{\operatorname{tr}(P)}-\left(\frac{\operatorname{tr}(P\ln(eA))}{\operatorname{tr}(P)}\right)^2\right]^{1/2}\end{cases}\\
&\le \frac{1}{4}(M-m)\ln\left(\frac{M}{m}\right).
\end{aligned}
$$

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