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INEQUALITIES OF HERMITE-HADAMARD TYPE FOR HG-CONVEX FUNCTIONS

Abstract. Some inequalities of Hermite-Hadamard type for HG -convex functions defined on positive intervals are given. Applications for special means are also provided.

Key words: *convex functions, integral inequalities, HG-convex functions*

2010 Mathematical Subject Classification: *26D15, 25D10*

1. Introduction. Following [4] (see also [26]) we say that the function $f : I \subset \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$ is HA -convex if

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leq (1-t)f(x) + tf(y) \quad (1)$$

for all $x, y \in I$ and $t \in [0, 1]$. If the inequality in (1) is reversed, then f is said to be HA -concave.

If $I \subset (0, \infty)$ and f is convex and nondecreasing function then f is HA -convex and if f is HA -convex and nonincreasing function then f is convex.

If $[a, b] \subset I \subset (0, \infty)$ and if we consider the function $g : [\frac{1}{b}, \frac{1}{a}] \rightarrow \mathbb{R}$, defined by $g(t) = f(\frac{1}{t})$, then we can state the following fact [4]:

Lemma 1. *The function f is HA -convex on $[a, b]$ if and only if g is convex in the usual sense on $[\frac{1}{b}, \frac{1}{a}]$.*

Therefore, as examples of HA -convex functions we can take $f(t) = g(\frac{1}{t})$, where g is any convex function on $[\frac{1}{b}, \frac{1}{a}]$.

In the recent paper [16] we obtained the following characterization result as well:

Lemma 2. Let $f, h : [a, b] \subset (0, \infty) \rightarrow \mathbb{R}$ be so that $h(t) = tf(t)$ for $t \in [a, b]$. Then f is *HA-convex* on the interval $[a, b]$ if and only if h is convex on $[a, b]$.

Following [4] (see also [26]) we say that the function $f : I \subset \mathbb{R} \setminus \{0\} \rightarrow (0, \infty)$ is *HG-convex* if

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leq [f(x)]^{1-t} [f(y)]^t \quad (2)$$

for all $x, y \in I$ and $t \in [0, 1]$. If the inequality in (2) is reversed, then f is said to be *HG-concave*.

By the geometric-mean - arithmetic mean inequality we have that any *HG-convex* function is *HA-convex*. The converse is obviously not true.

We observe that $f : I \subset \mathbb{R} \setminus \{0\} \rightarrow (0, \infty)$ is *HG-convex* if and only if the function $\ln f : I \subset \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$ is *HA-convex* on I .

Using Lemmas 1 and 2 we have:

Theorem 1. Let $f : [a, b] \subset (0, \infty) \rightarrow (0, \infty)$ and define the associated functions $G_f : [\frac{1}{b}, \frac{1}{a}] \rightarrow \mathbb{R}$ defined by $G_f(t) = \ln f(\frac{1}{t})$ and $H_f : [a, b] \subset (0, \infty) \rightarrow \mathbb{R}$ defined by $H_f(t) = t \ln f(t)$. Then the following statements are equivalent:

- (i) The function f is *HG-convex* on $[a, b]$;
- (ii) The function G_f is convex on $[\frac{1}{b}, \frac{1}{a}]$;
- (iii) The function H_f is convex on $[a, b]$.

For a convex function $h : [c, d] \rightarrow \mathbb{R}$, the following inequality is well known in the literature as the *Hermite-Hadamard inequality*

$$h\left(\frac{c+d}{2}\right) \leq \frac{1}{d-c} \int_c^d h(t) dt \leq \frac{h(c) + h(d)}{2}. \quad (3)$$

For related results, see [1] – [10], [12] – [28].

Motivated by the above results, we establish in this paper some inequalities of Hermite-Hadamard type for *HG-convex* functions defined on positive intervals. Applications for special means are also provided.

2. Main Results. The following result holds.

Theorem 2. Let $f : [a, b] \subset (0, \infty) \rightarrow (0, \infty)$ be an *HG-convex* function on the interval $[a, b]$. Then for any $\lambda \in [0, 1]$ we have the inequalities

$$f\left(\frac{2ab}{a+b}\right) \leq \left[f\left(\frac{2ab}{(1-\lambda)a + (\lambda+1)b}\right)\right]^{1-\lambda} \left[f\left(\frac{2ab}{(2-\lambda)a + \lambda b}\right)\right]^{\lambda} \leq$$

$$\begin{aligned} &\leq \exp \left(\frac{ab}{b-a} \int_a^b \frac{\ln f(t)}{t^2} dt \right) \leq \\ &\leq \sqrt{f \left(\frac{ab}{(1-\lambda)a + \lambda b} \right) [f(a)]^{1-\lambda} [f(b)]^\lambda} \leq \sqrt{f(a)f(b)}. \end{aligned} \quad (4)$$

If we take $\lambda = \frac{1}{2}$ in (4), then we get

$$\begin{aligned} f \left(\frac{2ab}{a+b} \right) &\leq \sqrt{f \left(\frac{4ab}{a+3b} \right) f \left(\frac{4ab}{3a+b} \right)} \leq \\ &\leq \exp \left(\frac{ab}{b-a} \int_a^b \frac{\ln f(t)}{t^2} dt \right) \leq \\ &\leq \sqrt{f \left(\frac{2ab}{a+b} \right) \sqrt{f(a)f(b)}} \leq \sqrt{f(a)f(b)}. \end{aligned} \quad (5)$$

The *identric mean* $I(a, b)$ for two distinct positive numbers a, b is defined by

$$I(a, b) := \frac{1}{e} \left(\frac{b^b}{a^a} \right)^{\frac{1}{b-a}}$$

while the *logarithmic mean* is defined by

$$L(a, b) := \frac{b-a}{\ln b - \ln a}.$$

Theorem 3. Let $f : [a, b] \subset (0, \infty) \rightarrow (0, \infty)$ be an HG -convex function on the interval $[a, b]$. Then

$$f(L(a, b)) \leq \exp \left(\frac{1}{b-a} \int_a^b \ln f(t) dt \right) \leq [f(b)]^{\frac{(L(a,b)-a)b}{(b-a)L(a,b)}} [f(a)]^{\frac{(b-L(a,b))a}{(b-a)L(a,b)}}. \quad (6)$$

If we write the classical Hermite-Hadamard inequality for the function H_f that is convex on $[a, b]$ when $f : [a, b] \subset (0, \infty) \rightarrow (0, \infty)$ is an HG -convex function on $[a, b]$ and perform the required calculations, we get:

Theorem 4. Let $f : [a, b] \subset (0, \infty) \rightarrow (0, \infty)$ be an HG-convex function on the interval $[a, b]$. Then we have

$$\left[f \left(\frac{a+b}{2} \right) \right]^{\frac{a+b}{2}} \leq \exp \left(\frac{1}{b-a} \int_a^b t \ln f(t) dt \right) \leq \sqrt{[f(b)]^b [f(a)]^a}. \quad (7)$$

We have the reverse inequalities as well:

Theorem 5. Let $f : [a, b] \subset (0, \infty) \rightarrow (0, \infty)$ be an HG-convex function on the interval $[a, b]$. Then we have

$$\begin{aligned} 1 &\leq \frac{\exp \left(\frac{ab}{b-a} \int_a^b \frac{\ln f(t)}{t^2} dt \right)}{f \left(\frac{2ab}{a+b} \right)} \leq \\ &\leq \exp \left(\frac{1}{8} \left[\frac{f'_-(b)}{f(b)} b^2 - \frac{f'_+(a)}{f(a)} a^2 \right] \left(\frac{b-a}{ab} \right) \right) \end{aligned} \quad (8)$$

and

$$\begin{aligned} 1 &\leq \frac{\sqrt{f(a) f(b)}}{\exp \left(\frac{ab}{b-a} \int_a^b \frac{\ln f(t)}{t^2} dt \right)} \leq \\ &\leq \exp \left(\frac{1}{8} \left[\frac{f'_-(b)}{f(b)} b^2 - \frac{f'_+(a)}{f(a)} a^2 \right] \left(\frac{b-a}{ab} \right) \right). \end{aligned} \quad (9)$$

The following related result also holds:

Theorem 6. Let $f : [a, b] \subset (0, \infty) \rightarrow (0, \infty)$ be an HG-convex function on the interval $[a, b]$. Then we have

$$\begin{aligned} 1 &\leq \frac{\sqrt{[f(a)]^a [f(b)]^b}}{\exp \left(\frac{1}{b-a} \int_a^b t \ln f(t) dt \right)} \leq \\ &\leq \left(\frac{f(b)}{f(a)} \right)^{\frac{1}{8}(b-a)} \exp \left(\frac{1}{8} (b-a) \left(\frac{b f'_-(b)}{f(b)} - \frac{a f'_+(a)}{f(a)} \right) \right) \end{aligned} \quad (10)$$

and

$$1 \leq \frac{\exp \left(\frac{1}{b-a} \int_a^b t \ln f(t) dt \right)}{\left[f \left(\frac{a+b}{2} \right) \right]^{\frac{a+b}{2}}} \leq \left(\frac{f(b)}{f(a)} \right)^{\frac{1}{8}(b-a)} \exp \left(\frac{1}{8} (b-a) \left(\frac{bf'_-(b)}{f(b)} - \frac{af'_+(a)}{f(a)} \right) \right). \quad (11)$$

From a different perspective we have:

Theorem 7. Let $f : [a, b] \subset (0, \infty) \rightarrow (0, \infty)$ be an HG-convex function on the interval $[a, b]$. Then

$$\exp \left(\frac{ab}{b-a} \int_a^b \frac{\ln f(t)}{t^2} dt \right) \leq \sqrt{f(x) [f(b)]^{\frac{a(b-x)}{x(b-a)}} [f(a)]^{\frac{b(x-a)}{x(b-a)}}} \quad (12)$$

for any $x \in [a, b]$.

If we take in (12), $x = \frac{a+b}{2}$, then we get from (12) that

$$\exp \left(\frac{ab}{b-a} \int_a^b \frac{\ln f(t)}{t^2} dt \right) \leq \sqrt{f \left(\frac{a+b}{2} \right) [f(b)]^{\frac{a}{a+b}} [f(a)]^{\frac{b}{a+b}}}. \quad (13)$$

3. Proofs. In [15], in order to improve İşcan's inequality [26] for HA-convex functions $g : [a, b] \subset (0, \infty) \rightarrow \mathbb{R}$,

$$g \left(\frac{2ab}{a+b} \right) \leq \frac{ab}{b-a} \int_a^b \frac{g(t)}{t^2} dt \leq \frac{g(a) + g(b)}{2}, \quad (14)$$

we obtained the following result:

$$g \left(\frac{2ab}{a+b} \right) \leq (1-\lambda) g \left(\frac{2ab}{(1-\lambda)a + (\lambda+1)b} \right) + \lambda g \left(\frac{2ab}{(2-\lambda)a + \lambda b} \right) \leq \frac{ab}{b-a} \int_a^b \frac{g(t)}{t^2} dt \leq$$

$$\begin{aligned}
&\leq \frac{1}{2} \left[g \left(\frac{ab}{(1-\lambda)a + \lambda b} \right) + (1-\lambda)g(a) + \lambda g(b) \right] \leq \\
&\leq \frac{g(a) + g(b)}{2},
\end{aligned} \tag{15}$$

where $\lambda \in [0, 1]$.

Now, if $f : [a, b] \subset (0, \infty) \rightarrow (0, \infty)$ is an *HG*-convex function on the interval $[a, b]$, then $g := \ln f$ is *HA*-convex on $[a, b]$, and by (15) we get

$$\begin{aligned}
&\ln f \left(\frac{2ab}{a+b} \right) \leq (1-\lambda) \ln f \left(\frac{2ab}{(1-\lambda)a + (\lambda+1)b} \right) + \\
&+ \lambda \ln f \left(\frac{2ab}{(2-\lambda)a + \lambda b} \right) \leq \frac{ab}{b-a} \int_a^b \frac{\ln f(t)}{t^2} dt \leq \\
&\leq \frac{1}{2} \left[\ln f \left(\frac{ab}{(1-\lambda)a + \lambda b} \right) + (1-\lambda) \ln f(a) + \lambda \ln f(b) \right] \leq \\
&\leq \frac{\ln f(a) + \ln f(b)}{2},
\end{aligned} \tag{16}$$

that is equivalent to

$$\begin{aligned}
&\ln f \left(\frac{2ab}{a+b} \right) \leq \\
&\leq \ln \left(\left[f \left(\frac{2ab}{(1-\lambda)a + (\lambda+1)b} \right) \right]^{1-\lambda} \left[f \left(\frac{2ab}{(2-\lambda)a + \lambda b} \right) \right]^\lambda \right) \leq \\
&\leq \frac{ab}{b-a} \int_a^b \frac{\ln f(t)}{t^2} dt \leq \ln \sqrt{f \left(\frac{ab}{(1-\lambda)a + \lambda b} \right) [f(a)]^{1-\lambda} [f(b)]^\lambda} \leq \\
&\leq \ln \sqrt{f(a)f(b)},
\end{aligned}$$

and by taking the exponential we get the desired result (4).

We have the following result for *HA*-convex functions [15]:

Lemma 3. *Let $g : [a, b] \subset (0, \infty) \rightarrow \mathbb{R}$ be an *HA*-convex function on the interval $[a, b]$. Then*

$$g(L(a, b)) \leq \frac{1}{b-a} \int_a^b g(x) dx \leq \frac{(L(a, b) - a)bg(b) + (b - L(a, b))ag(a)}{(b-a)L(a, b)}. \tag{17}$$

If $f : [a, b] \subset (0, \infty) \rightarrow (0, \infty)$ is an HG -convex function on the interval $[a, b]$, then $g := \ln f$ is HA -convex on $[a, b]$, and by (17) we have

$$\begin{aligned} \ln f(L(a, b)) &\leq \frac{1}{b-a} \int_a^b \ln f(x) dx \leq \\ &\leq \frac{(L(a, b) - a)b \ln f(b) + (b - L(a, b))a \ln f(a)}{(b-a)L(a, b)} = \\ &= \ln \left([f(b)]^{\frac{(L(a, b) - a)b}{(b-a)L(a, b)}} [f(a)]^{\frac{(b - L(a, b))a}{(b-a)L(a, b)}} \right). \end{aligned} \quad (18)$$

By taking the exponential in (18) we get the desired result (6).

We use the following results obtained by the author in [10] and [11]:

Lemma 4. *Let $h : [\alpha, \beta] \rightarrow \mathbb{R}$ be a convex function on $[\alpha, \beta]$. Then we have the inequalities*

$$0 \leq \frac{h(\alpha) + h(\beta)}{2} - \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} h(t) dt \leq \frac{1}{8} [h'_-(\beta) - h'_+(\alpha)] (\beta - \alpha) \quad (19)$$

and

$$0 \leq \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} h(t) dt - h\left(\frac{\alpha + \beta}{2}\right) \leq \frac{1}{8} [h'_-(\beta) - h'_+(\alpha)] (\beta - \alpha). \quad (20)$$

The constant $\frac{1}{8}$ is best possible in (19) and (20).

If $\ell : [a, b] \subset (0, \infty) \rightarrow \mathbb{R}$ is an HA -convex function on the interval $[a, b]$, then the function $g : [\frac{1}{b}, \frac{1}{a}] \rightarrow \mathbb{R}$, $g(s) = \ell(\frac{1}{s})$, is convex on $[\frac{1}{b}, \frac{1}{a}]$.

Now, by (19) and (20) we have

$$\begin{aligned} 0 &\leq \frac{g(\frac{1}{a}) + g(\frac{1}{b})}{2} - \frac{1}{\frac{1}{a} - \frac{1}{b}} \int_{\frac{1}{b}}^{\frac{1}{a}} g(t) dt \leq \\ &\leq \frac{1}{8} \left[g'_-\left(\frac{1}{a}\right) - g'_+\left(\frac{1}{b}\right) \right] \left(\frac{1}{a} - \frac{1}{b} \right) \end{aligned} \quad (21)$$

and

$$\begin{aligned}
 0 &\leq \frac{1}{\frac{1}{a} - \frac{1}{b}} \int_{\frac{1}{b}}^{\frac{1}{a}} g(t) dt - g\left(\frac{\frac{1}{a} + \frac{1}{b}}{2}\right) \leq \\
 &\leq \frac{1}{8} \left[g'_-\left(\frac{1}{a}\right) - g'_+\left(\frac{1}{b}\right) \right] \left(\frac{1}{a} - \frac{1}{b}\right). \tag{22}
 \end{aligned}$$

We also have

$$g'_\pm(s) = \ell'_{\mp}\left(\frac{1}{s}\right) \left(-\frac{1}{s^2}\right)$$

and then

$$g'_-\left(\frac{1}{a}\right) = -\ell'_+(a) a^2 \quad \text{and} \quad g'_+\left(\frac{1}{b}\right) = -\ell'_-(b) b^2.$$

From (21) and (22) we have

$$\begin{aligned}
 0 &\leq \frac{\ell(a) + \ell(b)}{2} - \frac{ab}{b-a} \int_{\frac{1}{b}}^{\frac{1}{a}} \ell\left(\frac{1}{s}\right) ds \leq \\
 &\leq \frac{1}{8} [\ell'_-(b) b^2 - \ell'_+(a) a^2] \left(\frac{b-a}{ab}\right) \tag{23}
 \end{aligned}$$

and

$$\begin{aligned}
 0 &\leq \frac{ab}{b-a} \int_{\frac{1}{b}}^{\frac{1}{a}} \ell\left(\frac{1}{s}\right) ds - \ell\left(\frac{2ab}{a+b}\right) \leq \\
 &\leq \frac{1}{8} [\ell'_-(b) b^2 - \ell'_+(a) a^2] \left(\frac{b-a}{ab}\right). \tag{24}
 \end{aligned}$$

If we change the variable $\frac{1}{s} = u$, then $ds = -\frac{du}{u^2}$ and (23) and (24) can be written as

$$\begin{aligned}
 0 &\leq \frac{\ell(a) + \ell(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{\ell(t)}{t^2} dt \leq \\
 &\leq \frac{1}{8} [\ell'_-(b) b^2 - \ell'_+(a) a^2] \left(\frac{b-a}{ab}\right) \tag{25}
 \end{aligned}$$

and

$$\begin{aligned}
 0 &\leq \frac{ab}{b-a} \int_a^b \frac{\ell(t)}{t^2} dt - \ell\left(\frac{2ab}{a+b}\right) \leq \\
 &\leq \frac{1}{8} [\ell'_-(b)b^2 - \ell'_+(a)a^2] \left(\frac{b-a}{ab}\right). \quad (26)
 \end{aligned}$$

If $f : [a, b] \subset (0, \infty) \rightarrow (0, \infty)$ is an HG -convex function on the interval $[a, b]$, then $\ell := \ln f$ is HA -convex on $[a, b]$, and by (25) and (26) we have

$$\begin{aligned}
 0 &\leq \ln \sqrt{f(a)f(b)} - \frac{ab}{b-a} \int_a^b \frac{\ln f(t)}{t^2} dt \leq \\
 &\leq \frac{1}{8} \left[\frac{f'_-(b)}{f(b)} b^2 - \frac{f'_+(a)}{f(a)} a^2 \right] \left(\frac{b-a}{ab}\right) \quad (27)
 \end{aligned}$$

and

$$\begin{aligned}
 0 &\leq \frac{ab}{b-a} \int_a^b \frac{\ln f(t)}{t^2} dt - \ln f\left(\frac{2ab}{a+b}\right) \leq \\
 &\leq \frac{1}{8} \left[\frac{f'_-(b)}{f(b)} b^2 - \frac{f'_+(a)}{f(a)} a^2 \right] \left(\frac{b-a}{ab}\right), \quad (28)
 \end{aligned}$$

and the Theorem 5 is proved.

If $f : [a, b] \subset (0, \infty) \rightarrow (0, \infty)$ is an HG -convex function on the interval $[a, b]$, then H_f is convex on $[a, b]$ and by (19) and (20) we have after appropriate calculations

$$\begin{aligned}
 0 &\leq \ln \sqrt{[f(a)]^a [f(b)]^b} - \frac{1}{b-a} \int_a^b t \ln f(t) dt \leq \\
 &\leq \frac{1}{8} \left[\ln f(b) + \frac{bf'_-(b)}{f(b)} - \ln f(a) - \frac{af'_+(a)}{f(a)} \right] (b-a) = \\
 &= \ln \left(\frac{f(b)}{f(a)} \right)^{\frac{1}{8}(b-a)} + \frac{1}{8} (b-a) \left(\frac{bf'_-(b)}{f(b)} - \frac{af'_+(a)}{f(a)} \right)
 \end{aligned}$$

and

$$\begin{aligned}
 0 &\leq \frac{1}{b-a} \int_a^b t \ln f(t) dt - \ln \left(\left[f \left(\frac{a+b}{2} \right) \right]^{\frac{a+b}{2}} \right) \leq \\
 &\leq \ln \left(\frac{f(b)}{f(a)} \right)^{\frac{1}{8}(b-a)} + \frac{1}{8} (b-a) \left(\frac{bf'_-(b)}{f(b)} - \frac{af'_+(a)}{f(a)} \right).
 \end{aligned}$$

These inequalities are equivalent to

$$\begin{aligned}
 0 &\leq \ln \left(\frac{\sqrt{[f(a)]^a [f(b)]^b}}{\exp \left(\frac{1}{b-a} \int_a^b t \ln f(t) dt \right)} \right) \leq \\
 &\leq \ln \left[\left(\frac{f(b)}{f(a)} \right)^{\frac{1}{8}(b-a)} \exp \left(\frac{1}{8} (b-a) \left(\frac{bf'_-(b)}{f(b)} - \frac{af'_+(a)}{f(a)} \right) \right) \right]
 \end{aligned}$$

and

$$\begin{aligned}
 0 &\leq \ln \left(\frac{\exp \left(\frac{1}{b-a} \int_a^b t \ln f(t) dt \right)}{\left[f \left(\frac{a+b}{2} \right) \right]^{\frac{a+b}{2}}} \right) \leq \\
 &\leq \ln \left[\left(\frac{f(b)}{f(a)} \right)^{\frac{1}{8}(b-a)} \exp \left(\frac{1}{8} (b-a) \left(\frac{bf'_-(b)}{f(b)} - \frac{af'_+(a)}{f(a)} \right) \right) \right]
 \end{aligned}$$

and by taking the exponential we get the desired results (10) and (11).

The following lemma is of interest in itself:

Lemma 5. *Let $g : [a, b] \subset (0, \infty) \rightarrow \mathbb{R}$ be a HA-convex function on the interval $[a, b]$. Then*

$$\frac{1}{2x} \left(\frac{g(b)a(b-x) + g(a)b(x-a)}{b-a} + xg(x) \right) \geq \frac{ab}{b-a} \int_a^b \frac{g(y)}{y^2} dy \quad (29)$$

for any $x \in [a, b]$.

Proof. Since $h(t) = tg(t)$ for $t \in [a, b]$ is convex, then by the gradient inequality for convex functions we have

$$xg(x) - yg(y) \geq (g(y) + yg'_-(y))(x - y)$$

for any $x, y \in (a, b)$.

This is equivalent to

$$xg(x) - xg(y) \geq yg'_-(y)(x - y) \quad (30)$$

for any $x, y \in (a, b)$.

From (30) we have, by division with $xy^2 > 0$, that

$$\frac{1}{y^2}g(x) - \frac{1}{y^2}g(y) \geq \frac{g'_-(y)}{y} \left(1 - \frac{y}{x}\right)$$

for any $x, y \in (a, b)$.

Taking the integral mean over y we have

$$\begin{aligned} & g(x) \frac{1}{b-a} \int_a^b \frac{1}{y^2} dy - \frac{1}{b-a} \int_a^b \frac{g(y)}{y^2} dy \geq \\ & \geq \frac{1}{b-a} \int_a^b \frac{g'_-(y)}{y} dy - \frac{1}{x} \frac{1}{b-a} \int_a^b g'_-(y) dy \end{aligned}$$

that is equivalent to

$$\begin{aligned} & \frac{g(x)}{ab} - \frac{1}{b-a} \int_a^b \frac{g(y)}{y^2} dy \geq \\ & \geq \frac{1}{b-a} \left[\frac{g(b)}{b} - \frac{g(a)}{a} + \int_a^b \frac{g(y)}{y^2} dy \right] - \frac{1}{x} \frac{g(b) - g(a)}{b-a} = \\ & = \frac{1}{b-a} \left(\frac{g(b)}{b} - \frac{g(a)}{a} \right) + \frac{1}{b-a} \int_a^b \frac{g(y)}{y^2} dy - \frac{1}{x} \frac{g(b) - g(a)}{b-a}, \end{aligned}$$

for any $x \in (a, b)$. This can be written as

$$\frac{1}{x} \frac{g(b) - g(a)}{b - a} - \frac{1}{b - a} \left(\frac{g(b)}{b} - \frac{g(a)}{a} \right) \geq \frac{2}{b - a} \int_a^b \frac{g(y)}{y^2} dy - \frac{g(x)}{ab}$$

or as

$$\frac{1}{2} \left(\frac{1}{b - a} \left[g(b) \frac{b - x}{xb} + g(a) \frac{x - a}{ax} \right] + \frac{g(x)}{ab} \right) \geq \frac{1}{b - a} \int_a^b \frac{g(y)}{y^2} dy.$$

This is equivalent to the desired result (29). \square

If $f : [a, b] \subset (0, \infty) \rightarrow (0, \infty)$ is an HG -convex function on the interval $[a, b]$, then $g := \ln f$ is HA -convex on $[a, b]$, and by (29) we have

$$\begin{aligned} \frac{1}{2x} \left(\frac{a(b - x) \ln f(b) + b(x - a) \ln f(a)}{b - a} + x \ln f(x) \right) &\geq \\ &\geq \frac{ab}{b - a} \int_a^b \frac{\ln f(y)}{y^2} dy \end{aligned}$$

for any $x \in [a, b]$.

This is clearly equivalent to

$$\ln \left(\sqrt{[f(b)]^{\frac{a(b-x)}{x(b-a)}} [f(a)]^{\frac{b(x-a)}{x(b-a)}}} \sqrt{f(x)} \right) \geq \frac{ab}{b - a} \int_a^b \frac{\ln f(y)}{y^2} dy \quad (31)$$

for any $x \in [a, b]$.

If we take the exponential in (31), then we get the desired result (12).

4. Applications. Consider the function $f : [a, b] \subset (0, \infty) \rightarrow (0, \infty)$, $f(t) = t$. Using the geometric mean - harmonic mean inequality, we have

$$f \left(\frac{xy}{tx + (1 - t)y} \right) = \frac{xy}{tx + (1 - t)y} \leq x^{1-t} y^t = [f(x)]^{1-t} [f(y)]^t,$$

which shows that f is HG -convex on $[a, b]$.

We need the following integrals

$$\frac{1}{b - a} \int_a^b \ln f(t) dt = \frac{1}{b - a} \int_a^b \ln t dt = \ln I(a, b),$$

$$\begin{aligned} \frac{1}{b-a} \int_a^b t \ln f(t) dt &= \frac{1}{b-a} \int_a^b t \ln t dt = \\ &= \frac{1}{2} A(a, b) \ln I(a^2, b^2) = \ln \left[I(a^2, b^2)^{\frac{1}{2} A(a, b)} \right] \end{aligned}$$

and

$$\int_a^b \frac{\ln f(t)}{t^2} dt = \int_a^b \frac{\ln t}{t^2} dt = \frac{b-a}{ab} \ln [I(a^{-1}, b^{-1})]^{-1}$$

giving that

$$\frac{ab}{b-a} \int_a^b \frac{\ln f(t)}{t^2} dt = \ln [I(a^{-1}, b^{-1})]^{-1}.$$

Now, if we write the inequality (4) for the function $f : [a, b] \subset (0, \infty) \rightarrow (0, \infty)$, $f(t) = t$, we get

$$\begin{aligned} H(a, b) &\leq \left(\frac{2ab}{(1-\lambda)a + (\lambda+1)b} \right)^{1-\lambda} \left(\frac{2ab}{(2-\lambda)a + \lambda b} \right)^{\lambda} \leq \\ &\leq [I(a^{-1}, b^{-1})]^{-1} \leq \sqrt{\left(\frac{ab}{(1-\lambda)a + \lambda b} \right)^{a^{1-\lambda} b^{\lambda}}} \leq G(a, b), \end{aligned} \quad (32)$$

where $H(a, b) := \frac{2ab}{a+b}$ is the *harmonic mean*.

If we use the inequality (6) for $f(t) = t$, then we have

$$(L(a, b) \leq) I(a, b) \leq b^{\frac{(L(a, b)-a)b}{(b-a)L(a, b)}} a^{\frac{(b-L(a, b))a}{(b-a)L(a, b)}}. \quad (33)$$

If we use the inequality (7) for $f(t) = t$, $t \in [a, b]$, then we also get

$$[A(a, b)]^{A(a, b)} \leq I(a^2, b^2)^{\frac{1}{2} A(a, b)} \leq G(b^b, a^a). \quad (34)$$

From (8) and (9) for $f(t) = t$ we have

$$1 \leq \frac{[I(a^{-1}, b^{-1})]^{-1}}{H(a, b)} \leq \exp \left(\frac{(b-a)^2}{8ab} \right) \quad (35)$$

and

$$1 \leq \frac{G(a, b)}{[I(a^{-1}, b^{-1})]^{-1}} \leq \exp \left(\frac{(b-a)^2}{8ab} \right). \quad (36)$$

From (10) and (11) we also have

$$1 \leq \frac{G(a^a, b^b)}{[I(a^2, b^2)]^{\frac{1}{2}A(a,b)}} \leq \left(\frac{b}{a}\right)^{\frac{1}{8}(b-a)} \quad (37)$$

and

$$1 \leq \frac{[I(a^2, b^2)]^{\frac{1}{2}A(a,b)}}{[A(a, b)]^{A(a,b)}} \leq \left(\frac{b}{a}\right)^{\frac{1}{8}(b-a)}. \quad (38)$$

Finally, from (13) we obtain

$$[I(a^{-1}, b^{-1})]^{-1} \leq \sqrt{A(a, b) b^{\frac{a}{a+b}} a^{\frac{b}{a+b}}}. \quad (39)$$

Now consider the function $f : [a, b] \subset (0, \infty) \rightarrow (0, \infty)$, $f(t) = \exp(t)$. Using the harmonic mean-arithmetic mean inequality we have

$$\begin{aligned} f\left(\frac{xy}{tx + (1-t)y}\right) &= \exp\left(\frac{xy}{tx + (1-t)y}\right) \leq \exp((1-t)x + ty) = \\ &= [\exp(x)]^{1-t} [\exp(y)]^t = [f(x)]^{1-t} [f(y)]^t \end{aligned}$$

for any $x, y \in [a, b]$ and $t \in [0, 1]$.

Now, if we use the inequality (4) for the HG -convex function $f : [a, b] \subset (0, \infty) \rightarrow (0, \infty)$, $f(t) = \exp(t)$, then we get, after suitable calculations, that

$$\begin{aligned} H(a, b) &\leq \frac{2(1-\lambda)ab}{(1-\lambda)a + (\lambda+1)b} + \frac{2\lambda ab}{(2-\lambda)a + \lambda b} \leq \\ &\leq \frac{G^2(a, b)}{L(a, b)} \leq \frac{1}{2} \left(\frac{ab}{(1-\lambda)a + \lambda b} + (1-\lambda)a + \lambda b \right) \leq A(a, b), \end{aligned}$$

for any $\lambda \in [0, 1]$.

If we use the inequalities (8) and (9) for the HG -convex function $f : [a, b] \subset (0, \infty) \rightarrow (0, \infty)$, $f(t) = \exp(t)$, then, by performing the required calculations, we get

$$0 \leq \frac{G^2(a, b)}{L(a, b)} - H(a, b) \leq \frac{1}{4} \frac{A(a, b)}{G^2(a, b)} (b-a)^2 \quad (40)$$

and

$$0 \leq A(a, b) - \frac{G^2(a, b)}{L(a, b)} \leq \frac{1}{4} \frac{A(a, b)}{G^2(a, b)} (b-a)^2. \quad (41)$$

From the inequality (13) we also have

$$\frac{G^2(a, b)}{L(a, b)} \leq \frac{1}{2} \left(A(a, b) + b^{\frac{a}{a+b}} a^{\frac{b}{a+b}} \right). \quad (42)$$

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