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## INEQUALITIES OF HERMITE-HADAMARD TYPE FOR *HG*-CONVEX FUNCTIONS

**Abstract.** Some inequalities of Hermite-Hadamard type for *HG*-convex functions defined on positive intervals are given. Applications for special means are also provided.

**Key words:** *convex functions, integral inequalities, HG-convex functions*

**2010 Mathematical Subject Classification:** *26D15, 25D10*

**1. Introduction.** Following [4] (see also [26]) we say that the function  $f : I \subset \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$  is *HA-convex* if

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leq (1-t)f(x) + tf(y) \tag{1}$$

for all  $x, y \in I$  and  $t \in [0, 1]$ . If the inequality in (1) is reversed, then  $f$  is said to be *HA-concave*.

If  $I \subset (0, \infty)$  and  $f$  is convex and nondecreasing function then  $f$  is *HA-convex* and if  $f$  is *HA-convex* and nonincreasing function then  $f$  is convex.

If  $[a, b] \subset I \subset (0, \infty)$  and if we consider the function  $g : [\frac{1}{b}, \frac{1}{a}] \rightarrow \mathbb{R}$ , defined by  $g(t) = f(\frac{1}{t})$ , then we can state the following fact [4]:

**Lemma 1.** *The function  $f$  is HA-convex on  $[a, b]$  if and only if  $g$  is convex in the usual sense on  $[\frac{1}{b}, \frac{1}{a}]$ .*

Therefore, as examples of *HA-convex* functions we can take  $f(t) = g(\frac{1}{t})$ , where  $g$  is any convex function on  $[\frac{1}{b}, \frac{1}{a}]$ .

In the recent paper [16] we obtained the following characterization result as well:

**Lemma 2.** Let  $f, h : [a, b] \subset (0, \infty) \rightarrow \mathbb{R}$  be so that  $h(t) = tf(t)$  for  $t \in [a, b]$ . Then  $f$  is *HA-convex* on the interval  $[a, b]$  if and only if  $h$  is convex on  $[a, b]$ .

Following [4] (see also [26]) we say that the function  $f : I \subset \mathbb{R} \setminus \{0\} \rightarrow (0, \infty)$  is *HG-convex* if

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leq [f(x)]^{1-t} [f(y)]^t \quad (2)$$

for all  $x, y \in I$  and  $t \in [0, 1]$ . If the inequality in (2) is reversed, then  $f$  is said to be *HG-concave*.

By the geometric-mean - arithmetic mean inequality we have that any *HG-convex* function is *HA-convex*. The converse is obviously not true.

We observe that  $f : I \subset \mathbb{R} \setminus \{0\} \rightarrow (0, \infty)$  is *HG-convex* if and only if the function  $\ln f : I \subset \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$  is *HA-convex* on  $I$ .

Using Lemmas 1 and 2 we have:

**Theorem 1.** Let  $f : [a, b] \subset (0, \infty) \rightarrow (0, \infty)$  and define the associated functions  $G_f : [\frac{1}{b}, \frac{1}{a}] \rightarrow \mathbb{R}$  defined by  $G_f(t) = \ln f(\frac{1}{t})$  and  $H_f : [a, b] \subset (0, \infty) \rightarrow \mathbb{R}$  defined by  $H_f(t) = t \ln f(t)$ . Then the following statements are equivalent:

- (i) The function  $f$  is *HG-convex* on  $[a, b]$ ;
- (ii) The function  $G_f$  is convex on  $[\frac{1}{b}, \frac{1}{a}]$ ;
- (iii) The function  $H_f$  is convex on  $[a, b]$ .

For a convex function  $h : [c, d] \rightarrow \mathbb{R}$ , the following inequality is well known in the literature as the *Hermite-Hadamard inequality*

$$h\left(\frac{c+d}{2}\right) \leq \frac{1}{d-c} \int_c^d h(t) dt \leq \frac{h(c) + h(d)}{2}. \quad (3)$$

For related results, see [1]–[10], [12]–[28].

Motivated by the above results, we establish in this paper some inequalities of Hermite-Hadamard type for *HG-convex* functions defined on positive intervals. Applications for special means are also provided.

**2. Main Results.** The following result holds.

**Theorem 2.** Let  $f : [a, b] \subset (0, \infty) \rightarrow (0, \infty)$  be an *HG-convex* function on the interval  $[a, b]$ . Then for any  $\lambda \in [0, 1]$  we have the inequalities

$$f\left(\frac{2ab}{a+b}\right) \leq \left[ f\left(\frac{2ab}{(1-\lambda)a + (\lambda+1)b}\right) \right]^{1-\lambda} \left[ f\left(\frac{2ab}{(2-\lambda)a + \lambda b}\right) \right]^\lambda \leq$$

$$\begin{aligned} &\leq \exp \left( \frac{ab}{b-a} \int_a^b \frac{\ln f(t)}{t^2} dt \right) \leq \tag{4} \\ &\leq \sqrt{f \left( \frac{ab}{(1-\lambda)a + \lambda b} \right) [f(a)]^{1-\lambda} [f(b)]^\lambda} \leq \sqrt{f(a)f(b)}. \end{aligned}$$

If we take  $\lambda = \frac{1}{2}$  in (4), then we get

$$\begin{aligned} f \left( \frac{2ab}{a+b} \right) &\leq \sqrt{f \left( \frac{4ab}{a+3b} \right) f \left( \frac{4ab}{3a+b} \right)} \leq \\ &\leq \exp \left( \frac{ab}{b-a} \int_a^b \frac{\ln f(t)}{t^2} dt \right) \leq \tag{5} \\ &\leq \sqrt{f \left( \frac{2ab}{a+b} \right) \sqrt{f(a)f(b)}} \leq \sqrt{f(a)f(b)}. \end{aligned}$$

The *identric mean*  $I(a, b)$  for two distinct positive numbers  $a, b$  is defined by

$$I(a, b) := \frac{1}{e} \left( \frac{b^b}{a^a} \right)^{\frac{1}{b-a}}$$

while the *logarithmic mean* is defined by

$$L(a, b) := \frac{b-a}{\ln b - \ln a}.$$

**Theorem 3.** Let  $f : [a, b] \subset (0, \infty) \rightarrow (0, \infty)$  be an  $HG$ -convex function on the interval  $[a, b]$ . Then

$$f(L(a, b)) \leq \exp \left( \frac{1}{b-a} \int_a^b \ln f(t) dt \right) \leq [f(b)]^{\frac{(L(a,b)-a)b}{(b-a)L(a,b)}} [f(a)]^{\frac{(b-L(a,b))a}{(b-a)L(a,b)}}. \tag{6}$$

If we write the classical Hermite-Hadamard inequality for the function  $H_f$  that is convex on  $[a, b]$  when  $f : [a, b] \subset (0, \infty) \rightarrow (0, \infty)$  is an  $HG$ -convex function on  $[a, b]$  and perform the required calculations, we get:

**Theorem 4.** Let  $f : [a, b] \subset (0, \infty) \rightarrow (0, \infty)$  be an HG-convex function on the interval  $[a, b]$ . Then we have

$$\left[ f \left( \frac{a+b}{2} \right) \right]^{\frac{a+b}{2}} \leq \exp \left( \frac{1}{b-a} \int_a^b t \ln f(t) dt \right) \leq \sqrt{[f(b)]^b [f(a)]^a}. \quad (7)$$

We have the reverse inequalities as well:

**Theorem 5.** Let  $f : [a, b] \subset (0, \infty) \rightarrow (0, \infty)$  be an HG-convex function on the interval  $[a, b]$ . Then we have

$$\begin{aligned} 1 &\leq \frac{\exp \left( \frac{ab}{b-a} \int_a^b \frac{\ln f(t)}{t^2} dt \right)}{f \left( \frac{2ab}{a+b} \right)} \leq \\ &\leq \exp \left( \frac{1}{8} \left[ \frac{f'_-(b)}{f(b)} b^2 - \frac{f'_+(a)}{f(a)} a^2 \right] \left( \frac{b-a}{ab} \right) \right) \end{aligned} \quad (8)$$

and

$$\begin{aligned} 1 &\leq \frac{\sqrt{f(a) f(b)}}{\exp \left( \frac{ab}{b-a} \int_a^b \frac{\ln f(t)}{t^2} dt \right)} \leq \\ &\leq \exp \left( \frac{1}{8} \left[ \frac{f'_-(b)}{f(b)} b^2 - \frac{f'_+(a)}{f(a)} a^2 \right] \left( \frac{b-a}{ab} \right) \right). \end{aligned} \quad (9)$$

The following related result also holds:

**Theorem 6.** Let  $f : [a, b] \subset (0, \infty) \rightarrow (0, \infty)$  be an HG-convex function on the interval  $[a, b]$ . Then we have

$$\begin{aligned} 1 &\leq \frac{\sqrt{[f(a)]^a [f(b)]^b}}{\exp \left( \frac{1}{b-a} \int_a^b t \ln f(t) dt \right)} \leq \\ &\leq \left( \frac{f(b)}{f(a)} \right)^{\frac{1}{8}(b-a)} \exp \left( \frac{1}{8} (b-a) \left( \frac{bf'_-(b)}{f(b)} - \frac{af'_+(a)}{f(a)} \right) \right) \end{aligned} \quad (10)$$

and

$$\begin{aligned}
 1 &\leq \frac{\exp\left(\frac{1}{b-a} \int_a^b t \ln f(t) dt\right)}{\left[f\left(\frac{a+b}{2}\right)\right]^{\frac{a+b}{2}}} \leq \\
 &\leq \left(\frac{f(b)}{f(a)}\right)^{\frac{1}{8}(b-a)} \exp\left(\frac{1}{8}(b-a) \left(\frac{bf'_-(b)}{f(b)} - \frac{af'_+(a)}{f(a)}\right)\right). \tag{11}
 \end{aligned}$$

From a different perspective we have:

**Theorem 7.** Let  $f : [a, b] \subset (0, \infty) \rightarrow (0, \infty)$  be an HG-convex function on the interval  $[a, b]$ . Then

$$\exp\left(\frac{ab}{b-a} \int_a^b \frac{\ln f(t)}{t^2} dt\right) \leq \sqrt{f(x) [f(b)]^{\frac{a(b-x)}{x(b-a)}} [f(a)]^{\frac{b(x-a)}{x(b-a)}}} \tag{12}$$

for any  $x \in [a, b]$ .

If we take in (12),  $x = \frac{a+b}{2}$ , then we get from (12) that

$$\exp\left(\frac{ab}{b-a} \int_a^b \frac{\ln f(t)}{t^2} dt\right) \leq \sqrt{f\left(\frac{a+b}{2}\right) [f(b)]^{\frac{a}{a+b}} [f(a)]^{\frac{b}{a+b}}}. \tag{13}$$

**3. Proofs.** In [15], in order to improve İşcan’s inequality [26] for HA-convex functions  $g : [a, b] \subset (0, \infty) \rightarrow \mathbb{R}$ ,

$$g\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a} \int_a^b \frac{g(t)}{t^2} dt \leq \frac{g(a) + g(b)}{2}, \tag{14}$$

we obtained the following result:

$$\begin{aligned}
 g\left(\frac{2ab}{a+b}\right) &\leq (1-\lambda) g\left(\frac{2ab}{(1-\lambda)a + (\lambda+1)b}\right) + \\
 &+ \lambda g\left(\frac{2ab}{(2-\lambda)a + \lambda b}\right) \leq \frac{ab}{b-a} \int_a^b \frac{g(t)}{t^2} dt \leq
 \end{aligned}$$

$$\begin{aligned} &\leq \frac{1}{2} \left[ g \left( \frac{ab}{(1-\lambda)a + \lambda b} \right) + (1-\lambda)g(a) + \lambda g(b) \right] \leq \\ &\leq \frac{g(a) + g(b)}{2}, \end{aligned} \quad (15)$$

where  $\lambda \in [0, 1]$ .

Now, if  $f : [a, b] \subset (0, \infty) \rightarrow (0, \infty)$  is an *HG*-convex function on the interval  $[a, b]$ , then  $g := \ln f$  is *HA*-convex on  $[a, b]$ , and by (15) we get

$$\begin{aligned} \ln f \left( \frac{2ab}{a+b} \right) &\leq (1-\lambda) \ln f \left( \frac{2ab}{(1-\lambda)a + (\lambda+1)b} \right) + \\ &+ \lambda \ln f \left( \frac{2ab}{(2-\lambda)a + \lambda b} \right) \leq \frac{ab}{b-a} \int_a^b \frac{\ln f(t)}{t^2} dt \leq \\ &\leq \frac{1}{2} \left[ \ln f \left( \frac{ab}{(1-\lambda)a + \lambda b} \right) + (1-\lambda) \ln f(a) + \lambda \ln f(b) \right] \leq \\ &\leq \frac{\ln f(a) + \ln f(b)}{2}, \end{aligned} \quad (16)$$

that is equivalent to

$$\begin{aligned} &\ln f \left( \frac{2ab}{a+b} \right) \leq \\ &\leq \ln \left( \left[ f \left( \frac{2ab}{(1-\lambda)a + (\lambda+1)b} \right) \right]^{1-\lambda} \left[ f \left( \frac{2ab}{(2-\lambda)a + \lambda b} \right) \right]^\lambda \right) \leq \\ &\leq \frac{ab}{b-a} \int_a^b \frac{\ln f(t)}{t^2} dt \leq \ln \sqrt{f \left( \frac{ab}{(1-\lambda)a + \lambda b} \right) [f(a)]^{1-\lambda} [f(b)]^\lambda} \leq \\ &\leq \ln \sqrt{f(a)f(b)}, \end{aligned}$$

and by taking the exponential we get the desired result (4).

We have the following result for *HA*-convex functions [15]:

**Lemma 3.** *Let  $g : [a, b] \subset (0, \infty) \rightarrow \mathbb{R}$  be an *HA*-convex function on the interval  $[a, b]$ . Then*

$$g(L(a, b)) \leq \frac{1}{b-a} \int_a^b g(x) dx \leq \frac{(L(a, b) - a)bg(b) + (b - L(a, b))ag(a)}{(b-a)L(a, b)}. \quad (17)$$

If  $f : [a, b] \subset (0, \infty) \rightarrow (0, \infty)$  is an  $HG$ -convex function on the interval  $[a, b]$ , then  $g := \ln f$  is  $HA$ -convex on  $[a, b]$ , and by (17) we have

$$\begin{aligned} \ln f(L(a, b)) &\leq \frac{1}{b-a} \int_a^b \ln f(x) dx \leq \\ &\leq \frac{(L(a, b) - a)b \ln f(b) + (b - L(a, b))a \ln f(a)}{(b-a)L(a, b)} = \\ &= \ln \left( [f(b)]^{\frac{(L(a, b) - a)b}{(b-a)L(a, b)}} [f(a)]^{\frac{(b - L(a, b))a}{(b-a)L(a, b)}} \right). \end{aligned} \tag{18}$$

By taking the exponential in (18) we get the desired result (6).

We use the following results obtained by the author in [10] and [11]:

**Lemma 4.** Let  $h : [\alpha, \beta] \rightarrow \mathbb{R}$  be a convex function on  $[\alpha, \beta]$ . Then we have the inequalities

$$0 \leq \frac{h(\alpha) + h(\beta)}{2} - \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} h(t) dt \leq \frac{1}{8} [h'_-(\beta) - h'_+(\alpha)] (\beta - \alpha) \tag{19}$$

and

$$0 \leq \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} h(t) dt - h\left(\frac{\alpha + \beta}{2}\right) \leq \frac{1}{8} [h'_-(\beta) - h'_+(\alpha)] (\beta - \alpha). \tag{20}$$

The constant  $\frac{1}{8}$  is best possible in (19) and (20).

If  $\ell : [a, b] \subset (0, \infty) \rightarrow \mathbb{R}$  is an  $HA$ -convex function on the interval  $[a, b]$ , then the function  $g : [\frac{1}{b}, \frac{1}{a}] \rightarrow \mathbb{R}$ ,  $g(s) = \ell(\frac{1}{s})$ , is convex on  $[\frac{1}{b}, \frac{1}{a}]$ .

Now, by (19) and (20) we have

$$\begin{aligned} 0 &\leq \frac{g\left(\frac{1}{a}\right) + g\left(\frac{1}{b}\right)}{2} - \frac{1}{\frac{1}{a} - \frac{1}{b}} \int_{\frac{1}{b}}^{\frac{1}{a}} g(t) dt \leq \\ &\leq \frac{1}{8} \left[ g'_-\left(\frac{1}{a}\right) - g'_+\left(\frac{1}{b}\right) \right] \left( \frac{1}{a} - \frac{1}{b} \right) \end{aligned} \tag{21}$$



and

$$\begin{aligned}
 0 &\leq \frac{1}{\frac{1}{a} - \frac{1}{b}} \int_{\frac{1}{b}}^{\frac{1}{a}} g(t) dt - g\left(\frac{\frac{1}{a} + \frac{1}{b}}{2}\right) \leq \\
 &\leq \frac{1}{8} \left[ g'_-\left(\frac{1}{a}\right) - g'_+\left(\frac{1}{b}\right) \right] \left(\frac{1}{a} - \frac{1}{b}\right). \tag{22}
 \end{aligned}$$

We also have

$$g'_\pm(s) = \ell'_{\mp}\left(\frac{1}{s}\right) \left(-\frac{1}{s^2}\right)$$

and then

$$g'_-\left(\frac{1}{a}\right) = -\ell'_+(a) a^2 \quad \text{and} \quad g'_+\left(\frac{1}{b}\right) = -\ell'_-(b) b^2.$$

From (21) and (22) we have

$$\begin{aligned}
 0 &\leq \frac{\ell(a) + \ell(b)}{2} - \frac{ab}{b-a} \int_{\frac{1}{b}}^{\frac{1}{a}} \ell\left(\frac{1}{s}\right) ds \leq \\
 &\leq \frac{1}{8} [\ell'_-(b) b^2 - \ell'_+(a) a^2] \left(\frac{b-a}{ab}\right) \tag{23}
 \end{aligned}$$

and

$$\begin{aligned}
 0 &\leq \frac{ab}{b-a} \int_{\frac{1}{b}}^{\frac{1}{a}} \ell\left(\frac{1}{s}\right) ds - \ell\left(\frac{2ab}{a+b}\right) \leq \\
 &\leq \frac{1}{8} [\ell'_-(b) b^2 - \ell'_+(a) a^2] \left(\frac{b-a}{ab}\right). \tag{24}
 \end{aligned}$$

If we change the variable  $\frac{1}{s} = u$ , then  $ds = -\frac{du}{u^2}$  and (23) and (24) can be written as

$$\begin{aligned}
 0 &\leq \frac{\ell(a) + \ell(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{\ell(t)}{t^2} dt \leq \\
 &\leq \frac{1}{8} [\ell'_-(b) b^2 - \ell'_+(a) a^2] \left(\frac{b-a}{ab}\right) \tag{25}
 \end{aligned}$$

and

$$\begin{aligned}
 0 &\leq \frac{ab}{b-a} \int_a^b \frac{\ell(t)}{t^2} dt - \ell\left(\frac{2ab}{a+b}\right) \leq \\
 &\leq \frac{1}{8} [\ell'_-(b)b^2 - \ell'_+(a)a^2] \left(\frac{b-a}{ab}\right). \tag{26}
 \end{aligned}$$

If  $f : [a, b] \subset (0, \infty) \rightarrow (0, \infty)$  is an  $HG$ -convex function on the interval  $[a, b]$ , then  $\ell := \ln f$  is  $HA$ -convex on  $[a, b]$ , and by (25) and (26) we have

$$\begin{aligned}
 0 &\leq \ln \sqrt{f(a)f(b)} - \frac{ab}{b-a} \int_a^b \frac{\ln f(t)}{t^2} dt \leq \\
 &\leq \frac{1}{8} \left[ \frac{f'_-(b)}{f(b)} b^2 - \frac{f'_+(a)}{f(a)} a^2 \right] \left(\frac{b-a}{ab}\right) \tag{27}
 \end{aligned}$$

and

$$\begin{aligned}
 0 &\leq \frac{ab}{b-a} \int_a^b \frac{\ln f(t)}{t^2} dt - \ln f\left(\frac{2ab}{a+b}\right) \leq \\
 &\leq \frac{1}{8} \left[ \frac{f'_-(b)}{f(b)} b^2 - \frac{f'_+(a)}{f(a)} a^2 \right] \left(\frac{b-a}{ab}\right), \tag{28}
 \end{aligned}$$

and the Theorem 5 is proved.

If  $f : [a, b] \subset (0, \infty) \rightarrow (0, \infty)$  is an  $HG$ -convex function on the interval  $[a, b]$ , then  $H_f$  is convex on  $[a, b]$  and by (19) and (20) we have after appropriate calculations

$$\begin{aligned}
 0 &\leq \ln \sqrt{[f(a)]^a [f(b)]^b} - \frac{1}{b-a} \int_a^b t \ln f(t) dt \leq \\
 &\leq \frac{1}{8} \left[ \ln f(b) + \frac{bf'_-(b)}{f(b)} - \ln f(a) - \frac{af'_+(a)}{f(a)} \right] (b-a) = \\
 &= \ln \left(\frac{f(b)}{f(a)}\right)^{\frac{1}{8}(b-a)} + \frac{1}{8} (b-a) \left(\frac{bf'_-(b)}{f(b)} - \frac{af'_+(a)}{f(a)}\right)
 \end{aligned}$$

and

$$0 \leq \frac{1}{b-a} \int_a^b t \ln f(t) dt - \ln \left( \left[ f \left( \frac{a+b}{2} \right) \right]^{\frac{a+b}{2}} \right) \leq \\ \leq \ln \left( \frac{f(b)}{f(a)} \right)^{\frac{1}{8}(b-a)} + \frac{1}{8} (b-a) \left( \frac{bf'_-(b)}{f(b)} - \frac{af'_+(a)}{f(a)} \right).$$

These inequalities are equivalent to

$$0 \leq \ln \left( \frac{\sqrt{[f(a)]^a [f(b)]^b}}{\exp \left( \frac{1}{b-a} \int_a^b t \ln f(t) dt \right)} \right) \leq \\ \leq \ln \left[ \left( \frac{f(b)}{f(a)} \right)^{\frac{1}{8}(b-a)} \exp \left( \frac{1}{8} (b-a) \left( \frac{bf'_-(b)}{f(b)} - \frac{af'_+(a)}{f(a)} \right) \right) \right]$$

and

$$0 \leq \ln \left( \frac{\exp \left( \frac{1}{b-a} \int_a^b t \ln f(t) dt \right)}{\left[ f \left( \frac{a+b}{2} \right) \right]^{\frac{a+b}{2}}} \right) \leq \\ \leq \ln \left[ \left( \frac{f(b)}{f(a)} \right)^{\frac{1}{8}(b-a)} \exp \left( \frac{1}{8} (b-a) \left( \frac{bf'_-(b)}{f(b)} - \frac{af'_+(a)}{f(a)} \right) \right) \right]$$

and by taking the exponential we get the desired results (10) and (11).

The following lemma is of interest in itself:

**Lemma 5.** *Let  $g : [a, b] \subset (0, \infty) \rightarrow \mathbb{R}$  be a HA-convex function on the interval  $[a, b]$ . Then*

$$\frac{1}{2x} \left( \frac{g(b) a (b-x) + g(a) b (x-a)}{b-a} + xg(x) \right) \geq \frac{ab}{b-a} \int_a^b \frac{g(y)}{y^2} dy \quad (29)$$

for any  $x \in [a, b]$ .

**Proof.** Since  $h(t) = tg(t)$  for  $t \in [a, b]$  is convex, then by the gradient inequality for convex functions we have

$$xg(x) - yg(y) \geq (g(y) + yg'_-(y))(x - y)$$

for any  $x, y \in (a, b)$ .

This is equivalent to

$$xg(x) - xg(y) \geq yg'_-(y)(x - y) \tag{30}$$

for any  $x, y \in (a, b)$ .

From (30) we have, by division with  $xy^2 > 0$ , that

$$\frac{1}{y^2}g(x) - \frac{1}{y^2}g(y) \geq \frac{g'_-(y)}{y} \left(1 - \frac{y}{x}\right)$$

for any  $x, y \in (a, b)$ .

Taking the integral mean over  $y$  we have

$$\begin{aligned} &g(x) \frac{1}{b-a} \int_a^b \frac{1}{y^2} dy - \frac{1}{b-a} \int_a^b \frac{g(y)}{y^2} dy \geq \\ &\geq \frac{1}{b-a} \int_a^b \frac{g'_-(y)}{y} dy - \frac{1}{x} \frac{1}{b-a} \int_a^b g'_-(y) dy \end{aligned}$$

that is equivalent to

$$\begin{aligned} &\frac{g(x)}{ab} - \frac{1}{b-a} \int_a^b \frac{g(y)}{y^2} dy \geq \\ &\geq \frac{1}{b-a} \left[ \frac{g(b)}{b} - \frac{g(a)}{a} + \int_a^b \frac{g(y)}{y^2} dy \right] - \frac{1}{x} \frac{g(b) - g(a)}{b-a} = \\ &= \frac{1}{b-a} \left( \frac{g(b)}{b} - \frac{g(a)}{a} \right) + \frac{1}{b-a} \int_a^b \frac{g(y)}{y^2} dy - \frac{1}{x} \frac{g(b) - g(a)}{b-a}, \end{aligned}$$

for any  $x \in (a, b)$ . This can be written as

$$\frac{1}{x} \frac{g(b) - g(a)}{b - a} - \frac{1}{b - a} \left( \frac{g(b)}{b} - \frac{g(a)}{a} \right) \geq \frac{2}{b - a} \int_a^b \frac{g(y)}{y^2} dy - \frac{g(x)}{ab}$$

or as

$$\frac{1}{2} \left( \frac{1}{b - a} \left[ g(b) \frac{b - x}{xb} + g(a) \frac{x - a}{ax} \right] + \frac{g(x)}{ab} \right) \geq \frac{1}{b - a} \int_a^b \frac{g(y)}{y^2} dy.$$

This is equivalent to the desired result (29).  $\square$

If  $f : [a, b] \subset (0, \infty) \rightarrow (0, \infty)$  is an  $HG$ -convex function on the interval  $[a, b]$ , then  $g := \ln f$  is  $HA$ -convex on  $[a, b]$ , and by (29) we have

$$\begin{aligned} \frac{1}{2x} \left( \frac{a(b-x) \ln f(b) + b(x-a) \ln f(a)}{b-a} + x \ln f(x) \right) &\geq \\ &\geq \frac{ab}{b-a} \int_a^b \frac{\ln f(y)}{y^2} dy \end{aligned}$$

for any  $x \in [a, b]$ .

This is clearly equivalent to

$$\ln \left( \sqrt{[f(b)]^{\frac{a(b-x)}{x(b-a)}} [f(a)]^{\frac{b(x-a)}{x(b-a)}}} \sqrt{f(x)} \right) \geq \frac{ab}{b-a} \int_a^b \frac{\ln f(y)}{y^2} dy \quad (31)$$

for any  $x \in [a, b]$ .

If we take the exponential in (31), then we get the desired result (12).

**4. Applications.** Consider the function  $f : [a, b] \subset (0, \infty) \rightarrow (0, \infty)$ ,  $f(t) = t$ . Using the geometric mean - harmonic mean inequality, we have

$$f \left( \frac{xy}{tx + (1-t)y} \right) = \frac{xy}{tx + (1-t)y} \leq x^{1-t} y^t = [f(x)]^{1-t} [f(y)]^t,$$

which shows that  $f$  is  $HG$ -convex on  $[a, b]$ .

We need the following integrals

$$\frac{1}{b-a} \int_a^b \ln f(t) dt = \frac{1}{b-a} \int_a^b \ln t dt = \ln I(a, b),$$

$$\begin{aligned} \frac{1}{b-a} \int_a^b t \ln f(t) dt &= \frac{1}{b-a} \int_a^b t \ln t dt = \\ &= \frac{1}{2} A(a, b) \ln I(a^2, b^2) = \ln \left[ I(a^2, b^2)^{\frac{1}{2}A(a,b)} \right] \end{aligned}$$

and

$$\int_a^b \frac{\ln f(t)}{t^2} dt = \int_a^b \frac{\ln t}{t^2} dt = \frac{b-a}{ab} \ln [I(a^{-1}, b^{-1})]^{-1}$$

giving that

$$\frac{ab}{b-a} \int_a^b \frac{\ln f(t)}{t^2} dt = \ln [I(a^{-1}, b^{-1})]^{-1}.$$

Now, if we write the inequality (4) for the function  $f : [a, b] \subset (0, \infty) \rightarrow (0, \infty)$ ,  $f(t) = t$ , we get

$$\begin{aligned} H(a, b) &\leq \left( \frac{2ab}{(1-\lambda)a + (\lambda+1)b} \right)^{1-\lambda} \left( \frac{2ab}{(2-\lambda)a + \lambda b} \right)^\lambda \leq \\ &\leq [I(a^{-1}, b^{-1})]^{-1} \leq \sqrt{\left( \frac{ab}{(1-\lambda)a + \lambda b} \right)^{a^{1-\lambda} b^\lambda}} \leq G(a, b), \end{aligned} \tag{32}$$

where  $H(a, b) := \frac{2ab}{a+b}$  is the *harmonic mean*.

If we use the inequality (6) for  $f(t) = t$ , then we have

$$(L(a, b) \leq) I(a, b) \leq b^{\frac{(L(a,b)-a)b}{(b-a)L(a,b)}} a^{\frac{(b-L(a,b))a}{(b-a)L(a,b)}}. \tag{33}$$

If we use the inequality (7) for  $f(t) = t$ ,  $t \in [a, b]$ , then we also get

$$[A(a, b)]^{A(a,b)} \leq I(a^2, b^2)^{\frac{1}{2}A(a,b)} \leq G(b^b, a^a). \tag{34}$$

From (8) and (9) for  $f(t) = t$  we have

$$1 \leq \frac{[I(a^{-1}, b^{-1})]^{-1}}{H(a, b)} \leq \exp \left( \frac{(b-a)^2}{8ab} \right) \tag{35}$$

and

$$1 \leq \frac{G(a, b)}{[I(a^{-1}, b^{-1})]^{-1}} \leq \exp \left( \frac{(b-a)^2}{8ab} \right). \tag{36}$$

From (10) and (11) we also have

$$1 \leq \frac{G(a^a, b^b)}{[I(a^2, b^2)]^{\frac{1}{2}A(a,b)}} \leq \left(\frac{b}{a}\right)^{\frac{1}{8}(b-a)} \quad (37)$$

and

$$1 \leq \frac{[I(a^2, b^2)]^{\frac{1}{2}A(a,b)}}{[A(a, b)]^{A(a,b)}} \leq \left(\frac{b}{a}\right)^{\frac{1}{8}(b-a)}. \quad (38)$$

Finally, from (13) we obtain

$$[I(a^{-1}, b^{-1})]^{-1} \leq \sqrt{A(a, b) b^{\frac{a}{a+b}} a^{\frac{b}{a+b}}}. \quad (39)$$

Now consider the function  $f : [a, b] \subset (0, \infty) \rightarrow (0, \infty)$ ,  $f(t) = \exp(t)$ . Using the harmonic mean-arithmetical mean inequality we have

$$\begin{aligned} f\left(\frac{xy}{tx + (1-t)y}\right) &= \exp\left(\frac{xy}{tx + (1-t)y}\right) \leq \exp((1-t)x + ty) = \\ &= [\exp(x)]^{1-t} [\exp(y)]^t = [f(x)]^{1-t} [f(y)]^t \end{aligned}$$

for any  $x, y \in [a, b]$  and  $t \in [0, 1]$ .

Now, if we use the inequality (4) for the  $HG$ -convex function  $f : [a, b] \subset (0, \infty) \rightarrow (0, \infty)$ ,  $f(t) = \exp(t)$ , then we get, after suitable calculations, that

$$\begin{aligned} H(a, b) &\leq \frac{2(1-\lambda)ab}{(1-\lambda)a + (\lambda+1)b} + \frac{2\lambda ab}{(2-\lambda)a + \lambda b} \leq \\ &\leq \frac{G^2(a, b)}{L(a, b)} \leq \frac{1}{2} \left( \frac{ab}{(1-\lambda)a + \lambda b} + (1-\lambda)a + \lambda b \right) \leq A(a, b), \end{aligned}$$

for any  $\lambda \in [0, 1]$ .

If we use the inequalities (8) and (9) for the  $HG$ -convex function  $f : [a, b] \subset (0, \infty) \rightarrow (0, \infty)$ ,  $f(t) = \exp(t)$ , then, by performing the required calculations, we get

$$0 \leq \frac{G^2(a, b)}{L(a, b)} - H(a, b) \leq \frac{1}{4} \frac{A(a, b)}{G^2(a, b)} (b-a)^2 \quad (40)$$

and

$$0 \leq A(a, b) - \frac{G^2(a, b)}{L(a, b)} \leq \frac{1}{4} \frac{A(a, b)}{G^2(a, b)} (b-a)^2. \quad (41)$$

From the inequality (13) we also have

$$\frac{G^2(a, b)}{L(a, b)} \leq \frac{1}{2} \left( A(a, b) + b^{\frac{a}{a+b}} a^{\frac{b}{a+b}} \right). \quad (42)$$

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