Trace inequalities for positive operators via recent refinements and reverses of Young's inequality

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Trace inequalities for positive operators via recent refinements and reverses of Young's inequality

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Abstract: In this paper we obtain some trace inequalities for positive operators via recent refinements and reverses of Young's inequality due to Kittaneh-Manasrah, Liao-Wu-Zhao, Zuo-Shi-Fujii, Tominaga and Furuichi.

Keywords: Young's inequality, Hölder operator inequality, Operator means, Arithmetic mean-Geometric mean inequality

MSC: 47A63, 47A30, 26D15, 26D10, 15A60

1 Introduction

If $\{e_i\}_{i\in I}$ is an orthonormal basis of H, we say that $A \in \mathcal{B}(H)$ is *trace class* provided

$$\|A\|_1 := \sum_{i\in I} \langle |A| e_i, e_i\rangle < \infty.$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

The following properties are also well known:

(i) We have

$$\|A\|_1 = \left\|A^*\right\|_1$$

for any $A \in \mathcal{B}_1(H)$;

(ii) $\mathcal{B}_1(H)$ is an *operator ideal* in $\mathcal{B}(H)$, i.e.

$$\mathcal{B}(H)\,\mathcal{B}_1(H)\,\mathcal{B}(H) \subseteq \mathcal{B}_1(H);$$

(iii) $\left(\mathcal{B}_1(H), \|\cdot\|_1\right)$ is a *Banach space*.

We define the *trace* of a trace class operator $A \in \mathcal{B}_1(H)$ to be

$$\operatorname{tr}(A) := \sum_{i\in I} \langle Ae_i, e_i\rangle, \tag{1.1}$$

where $\{e_i\}_{i\in I}$ is an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (1.1) converges absolutely and it is independent from the choice of basis.

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The following results collect some properties of the trace:
(i) If $A \in \mathcal{B}_1(H)$ then $A^* \in \mathcal{B}_1(H)$ and

$$\operatorname{tr}\left(A^*\right) = \overline{\operatorname{tr}(A)};$$

(ii) If $A \in \mathcal{B}_1(H)$ and $T \in \mathcal{B}(H)$, then AT, $TA \in \mathcal{B}_1(H)$ and

$$\operatorname{tr}(AT) = \operatorname{tr}(TA) \text{ and } |\operatorname{tr}(AT)| \le \|A\|_1 \|T\|;$$

(iii) $\operatorname{tr}(\cdot)$ is a bounded linear functional on $\mathcal{B}_1(H)$ with $\|\operatorname{tr}\| = 1$;
(iv) $\mathcal{B}_{fin}(H)$, the space of operators of *finite rank*, is a dense subspace of $\mathcal{B}_1(H)$.

Now, for the finite dimensional case, it is well known that the trace functional is *submultiplicative*, that is, for *positive semidefinite matrices A* and *B* in $M_n(\mathbb{C})$,

$$0 \le \operatorname{tr}(AB) \le \operatorname{tr}(A)\operatorname{tr}(B).$$

Therefore

$$0 \le \operatorname{tr}(A^k) \le [\operatorname{tr}(A)]^k,$$

where k is any positive integer.

In 2000, Yang [31] proved a matrix trace inequality

$$\operatorname{tr}\left[(AB)^k\right] \le (\operatorname{tr} A)^k (\operatorname{tr} B)^k, \tag{1.2}$$

where A and B are positive semidefinite matrices over $\mathbb{C}$ of the same order n and k is any positive integer.

If $(H, \langle\cdot,\cdot\rangle)$ is a separable infinite-dimensional Hilbert space then the inequality (1.2) is also valid for any positive operators $A, B \in \mathcal{B}_1(H)$. This result was obtained by L. Liu in 2007, see [20].

In 2001, Yang et al. [32] improved (1.2) as follows:

$$\operatorname{tr}\left[(AB)^m\right] \le \left[\operatorname{tr}\left(A^{2m}\right)\operatorname{tr}\left(B^{2m}\right)\right]^{1/2}, \tag{1.3}$$

where A and B are positive semidefinite matrices over $\mathbb{C}$ of the same order and m is any positive integer.

Stronger results than inequalities (1.2) and (1.3) had been obtained in the last 70s by Lieb and Thirring in [19].

In [25] the authors have proved many trace inequalities for sums and products of matrices. For instance, if A and B are positive semidefinite matrices in $M_n(\mathbb{C})$, then

$$\operatorname{tr}\left[(AB)^k\right] \le \min\left\{\|A\|^k \operatorname{tr}\left(B^k\right), \|B\|^k \operatorname{tr}\left(A^k\right)\right\}$$

for any positive integer k. Also, if $A, B \in M_n(\mathbb{C})$ then for $r \ge 1$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ we have the following *Young type inequality*

$$\operatorname{tr}\left(\left|AB^*\right|^r\right) \le \operatorname{tr}\left[\left(\frac{|A|^p}{p} + \frac{|B|^q}{q}\right)^r\right]. \tag{1.4}$$

Ando [1] proved a strong form of Young's inequality - it was shown that if A and B are in $M_n(\mathbb{C})$, then there is a *unitary matrix U* such that

$$\left|AB^*\right| \le U\left(\frac{1}{p}|A|^p + \frac{1}{q}|B|^q\right)U^*,$$

where $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, which immediately gives the trace inequality

$$\operatorname{tr}\left(\left|AB^*\right|\right) \le \frac{1}{p}\operatorname{tr}\left(|A|^p\right) + \frac{1}{q}\operatorname{tr}\left(|B|^q\right).$$

This inequality can also be obtained from (1.4) by taking $r = 1$.

The following Hölder's type inequality has been obtained by Ruskai in [23]

$$|\operatorname{tr}(AB)| \le \operatorname{tr}(|AB|) \le \left[\operatorname{tr}\left(|A|^p\right)\right]^{1/p}\left[\operatorname{tr}\left(|B|^q\right)\right]^{1/q}$$

where $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $A, B \in \mathcal{B}(H)$ with $|A|^p, |B|^q \in \mathcal{B}_1(H)$.

In particular, for $p = 2$ we get the Schwarz inequality

$$|\operatorname{tr}(AB)| \le \operatorname{tr}(|AB|) \le \left[\operatorname{tr}\left(|A|^2\right)\right]^{1/2}\left[\operatorname{tr}\left(|B|^2\right)\right]^{1/2}$$

with $|A|^2, |B|^2 \in \mathcal{B}_1(H)$.

Assume that A, B are positive invertible operators on a complex Hilbert space $(H, \langle\cdot,\cdot\rangle)$. We use the following notation

$$A\sharp_\nu B := A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^\nu A^{1/2}, \tag{1.5}$$

for the *weighted geometric mean*. When $\nu = \frac{1}{2}$, we write $A\sharp B$ for brevity.

We have the following Hölder type trace inequality for the weighted geometric mean [9]: If A, B are positive invertible operators, $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $A^p, B^q \in \mathcal{B}_1(H)$, then $B^q\sharp_{1/p}A^p \in \mathcal{B}_1(H)$ and

$$\operatorname{tr}\left(B^q\sharp_{1/p}A^p\right) \le \left[\operatorname{tr}\left(A^p\right)\right]^{1/p}\left[\operatorname{tr}\left(B^q\right)\right]^{1/q}.$$

In particular, if $A^2, B^2 \in \mathcal{B}_1(H)$, then $B^2\sharp A^2 \in \mathcal{B}_1(H)$ and

$$\left[\operatorname{tr}\left(B^2\sharp A^2\right)\right]^2 \le \operatorname{tr}\left(A^2\right)\operatorname{tr}\left(B^2\right).$$

Also, if A, B are positive invertible operators, $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $C \in \mathcal{B}_1(H)$, $C \ge 0$ then CA^p, CB^q, $C\left(B^q\sharp_{1/p}A^p\right) \in \mathcal{B}_1(H)$ and

$$\operatorname{tr}\left(C\left(B^q\sharp_{1/p}A^p\right)\right) \le \left[\operatorname{tr}\left(CA^p\right)\right]^{1/p}\left[\operatorname{tr}\left(CB^q\right)\right]^{1/q}.$$

In particular, if $C \in \mathcal{B}_1(H)$, then $CA^2, CB^2, C\left(B^2\sharp A^2\right) \in \mathcal{B}_1(H)$ and

$$\left[\operatorname{tr}\left(C\left(B^2\sharp A^2\right)\right)\right]^2 \le \operatorname{tr}\left(CA^2\right)\operatorname{tr}\left(CB^2\right).$$

Related inequalities may be found in [9] as well.

For the theory of trace functionals and their applications the reader is referred to [27].

For some classical trace inequalities see [4], [6], [22] and [33], which are continuations of the work of Bellman [2]. For related works the reader can refer to [1], [3], [4], [12], [17], [20], [21], [24] and [30].

Motivated by the above results, we establish in this paper some new trace inequalities via recent scalar Young type inequalities.

2 Trace Inequalities Via Kittaneh-Manasrah Results

Kittaneh and Manasrah [15], [16] provided a refinement and a reverse for *Young's inequality* as follows:

$$r\left(\sqrt{a} - \sqrt{b}\right)^2 \le (1-\nu)a + \nu b - a^{1-\nu}b^\nu \le R\left(\sqrt{a} - \sqrt{b}\right)^2, \tag{2.1}$$

where $a, b > 0$, $\nu \in [0,1]$, $r = \min\{1-\nu, \nu\}$ and $R = \max\{1-\nu, \nu\}$. The case $\nu = \frac{1}{2}$ reduces (2.1) to an identity.

We can give a simple direct proof for (2.1) as follows. Recall the following result obtained by the author in 2006 [7] that provides a refinement and a reverse for the weighted Jensen's discrete inequality:

$$n \min_{j\in\{1,2,\dots,n\}}\{p_j\}\left[\frac{1}{n}\sum_{j=1}^n \Phi(x_j) - \Phi\left(\frac{1}{n}\sum_{j=1}^n x_j\right)\right] \tag{2.2}$$

$$\leq \frac{1}{P_n}\sum_{j=1}^{n} p_j\Phi\left(x_j\right) - \Phi\left(\frac{1}{P_n}\sum_{j=1}^{n} p_j x_j\right)$$
$$\leq n \max_{j\in\{1,2,\ldots,n\}}\left\{p_j\right\}\left[\frac{1}{n}\sum_{j=1}^{n}\Phi\left(x_j\right) - \Phi\left(\frac{1}{n}\sum_{j=1}^{n} x_j\right)\right],$$

where $\Phi : C \to \mathbb{R}$ is a convex function defined on convex subset C of the linear space X, $\left\{x_j\right\}_{j\in\{1,2,\ldots,n\}}$ are vectors in C and $\left\{p_j\right\}_{j\in\{1,2,\ldots,n\}}$ are nonnegative numbers with $P_n = \sum_{j=1}^{n} p_j > 0$. For $n = 2$, we deduce from (2.2) that

$$2\min\{\nu, 1-\nu\}\left[\frac{\Phi(x)+\Phi(y)}{2} - \Phi\left(\frac{x+y}{2}\right)\right] \leq \nu\Phi(x) + (1-\nu)\Phi(y) - \Phi[\nu x + (1-\nu)y] \tag{2.3}$$
$$\leq 2\max\{\nu, 1-\nu\}\left[\frac{\Phi(x)+\Phi(y)}{2} - \Phi\left(\frac{x+y}{2}\right)\right]$$

for any $x, y \in \mathbb{R}$ and $\nu \in [0, 1]$. If we take $\Phi(x) = \exp(x)$, then we get from (2.3)

$$2\min\{\nu, 1-\nu\}\left[\frac{\exp(x)+\exp(y)}{2} - \exp\left(\frac{x+y}{2}\right)\right] \leq \nu\exp(x) + (1-\nu)\exp(y) - \exp[\nu x + (1-\nu)y] \tag{2.4}$$
$$\leq 2\max\{\nu, 1-\nu\}\left[\frac{\exp(x)+\exp(y)}{2} - \exp\left(\frac{x+y}{2}\right)\right]$$

for any $x, y \in \mathbb{R}$ and $\nu \in [0, 1]$. Further, denote $\exp(x) = a$, $\exp(y) = b$ with $a, b > 0$, then from (2.4) we obtain the inequality (2.1).

We have:

Theorem 1. *Let A, B be two positive operators and $P, Q \in \mathcal{B}_1(H)$ with $P, Q > 0$. Then for any $\nu \in [0, 1]$ we have*

$$r\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} - 2\frac{\operatorname{tr}\left(PA^{1/2}\right)}{\operatorname{tr}(P)}\frac{\operatorname{tr}\left(QB^{1/2}\right)}{\operatorname{tr}(Q)} + \frac{\operatorname{tr}(QB)}{\operatorname{tr}(Q)}\right) \leq (1-\nu)\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} + \nu\frac{\operatorname{tr}(QB)}{\operatorname{tr}(Q)} - \frac{\operatorname{tr}\left(PA^{1-\nu}\right)}{\operatorname{tr}(P)}\frac{\operatorname{tr}\left(QB^{\nu}\right)}{\operatorname{tr}(Q)} \tag{2.5}$$
$$\leq R\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} - 2\frac{\operatorname{tr}\left(PA^{1/2}\right)}{\operatorname{tr}(P)}\frac{\operatorname{tr}\left(QB^{1/2}\right)}{\operatorname{tr}(Q)} + \frac{\operatorname{tr}(QB)}{\operatorname{tr}(Q)}\right),$$

where $r = \min\{1-\nu, \nu\}$ and $R = \max\{1-\nu, \nu\}$.

Proof. Fix $b > 0$, and by using the functional calculus for the operator A, we have from (2.1) that

$$r\left(\langle Ax, x\rangle - 2\sqrt{b}\left\langle A^{1/2}x, x\right\rangle + b\langle x, x\rangle\right) \leq (1-\nu)\langle Ax, x\rangle + \nu b\langle x, x\rangle - b^{\nu}\left\langle A^{1-\nu}x, x\right\rangle \tag{2.6}$$
$$\leq R\left(\langle Ax, x\rangle - 2\sqrt{b}\left\langle A^{1/2}x, x\right\rangle + b\langle x, x\rangle\right)$$

for any $x \in H$.

Now, fix $x \in H \setminus \{0\}$. Then by using the functional calculus for the operator B, we have by (2.6) that

$$r\left(\langle Ax, x\rangle\|y\|^2 - 2\left\langle A^{1/2}x, x\right\rangle\left\langle B^{1/2}y, y\right\rangle + \|x\|^2\langle By, y\rangle\right) \tag{2.7}$$
$$\leq (1-\nu)\langle Ax, x\rangle\|y\|^2 + \nu\|x\|^2\langle By, y\rangle - \left\langle B^{\nu}y, y\right\rangle\left\langle A^{1-\nu}x, x\right\rangle$$
$$\leq R\left(\langle Ax, x\rangle\|y\|^2 - 2\left\langle A^{1/2}x, x\right\rangle\left\langle B^{1/2}y, y\right\rangle + \|x\|^2\langle By, y\rangle\right)$$

for any $x, y \in H$ and $\nu \in [0, 1]$.

This inequality is of interest in itself as well.

Now, let $x = P^{1/2}e$, $y = Q^{1/2}f$ where $e, f \in H$. Then by (2.7) we get

$$r\left(\left\langle P^{1/2}AP^{1/2}e,e\right\rangle \langle Qf,f\rangle - 2\left\langle P^{1/2}A^{1/2}P^{1/2}e,e\right\rangle \left\langle Q^{1/2}B^{1/2}Q^{1/2}f,f\right\rangle \right.$$
$$\left. + \langle Pe,e\rangle \left\langle Q^{1/2}BQ^{1/2}f,f\right\rangle\right) \le (1-\nu)\left\langle P^{1/2}AP^{1/2}e,e\right\rangle \langle Qf,f\rangle + \nu\langle Pe,e\rangle\left\langle Q^{1/2}BQ^{1/2}f,f\right\rangle$$
$$-\left\langle P^{1/2}A^{1-\nu}P^{1/2}e,e\right\rangle\left\langle Q^{1/2}B^{\nu}Q^{1/2}f,f\right\rangle \le R\left(\left\langle P^{1/2}AP^{1/2}e,e\right\rangle\langle Qf,f\rangle\right.$$
$$\left. -2\left\langle P^{1/2}A^{1/2}P^{1/2}e,e\right\rangle\left\langle Q^{1/2}B^{1/2}Q^{1/2}f,f\right\rangle + \langle Pe,e\rangle\left\langle Q^{1/2}BQ^{1/2}f,f\right\rangle\right) \tag{2.8}$$

for any $e, f \in H$.

Let $\{e_i\}_{i\in I}$ and $\{f_j\}_{j\in J}$ be two orthonormal bases of H. If we take in (2.8) $e = e_i$, $i \in I$ and $f = f_j$, $j \in J$ and summing over $i \in I$ and $j \in J$, then we get

$$r\left(\sum_{i\in I}\left\langle P^{1/2}AP^{1/2}e_i,e_i\right\rangle\sum_{j\in J}\langle Qf_j,f_j\rangle - 2\sum_{i\in I}\left\langle P^{1/2}A^{1/2}P^{1/2}e_i,e_i\right\rangle\sum_{j\in J}\left\langle Q^{1/2}B^{1/2}Q^{1/2}f_j,f_j\right\rangle\right.$$
$$\left.+\sum_{i\in I}\langle Pe_i,e_i\rangle\sum_{j\in J}\left\langle Q^{1/2}BQ^{1/2}f_j,f_j\right\rangle\right)$$
$$\le (1-\nu)\sum_{i\in I}\left\langle P^{1/2}AP^{1/2}e_i,e_i\right\rangle\sum_{j\in J}\langle Qf_j,f_j\rangle + \nu\sum_{i\in I}\langle Pe_i,e_i\rangle\sum_{j\in J}\left\langle Q^{1/2}BQ^{1/2}f_j,f_j\right\rangle$$
$$-\sum_{i\in I}\left\langle P^{1/2}A^{1-\nu}P^{1/2}e_i,e_i\right\rangle\sum_{j\in J}\left\langle Q^{1/2}B^{\nu}Q^{1/2}f_j,f_j\right\rangle$$
$$\le R\left(\sum_{i\in I}\left\langle P^{1/2}AP^{1/2}e_i,e_i\right\rangle\sum_{j\in J}\langle Qf_j,f_j\rangle - 2\sum_{i\in I}\left\langle P^{1/2}A^{1/2}P^{1/2}e_i,e_i\right\rangle\sum_{j\in J}\left\langle Q^{1/2}B^{1/2}Q^{1/2}f_j,f_j\right\rangle\right.$$
$$\left.+\sum_{i\in I}\langle Pe_i,e_i\rangle\sum_{j\in J}\left\langle Q^{1/2}BQ^{1/2}f_j,f_j\right\rangle\right).$$

Using the properties of the trace we get

$$r\left(\operatorname{tr}(PA)\operatorname{tr}(Q) - 2\operatorname{tr}\left(PA^{1/2}\right)\operatorname{tr}\left(QB^{1/2}\right) + \operatorname{tr}(P)\operatorname{tr}(QB)\right)$$
$$\le (1-\nu)\operatorname{tr}(PA)\operatorname{tr}(Q) + \nu\operatorname{tr}(P)\operatorname{tr}(QB) - \operatorname{tr}\left(PA^{1-\nu}\right)\operatorname{tr}\left(QB^{\nu}\right)$$
$$\le R\left(\operatorname{tr}(PA)\operatorname{tr}(Q) - 2\operatorname{tr}\left(PA^{1/2}\right)\operatorname{tr}\left(QB^{1/2}\right) + \operatorname{tr}(P)\operatorname{tr}(QB)\right)$$

and the inequality (2.5) is proved. □

Corollary 1. *Let A be a positive operator and $P \in \mathcal{B}_1(H)$ with $P > 0$. Then for any $\nu \in [0,1]$ we have*

$$2r\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}\left(PA^{1/2}\right)}{\operatorname{tr}(P)}\right)^2\right) \le \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}\left(PA^{1-\nu}\right)}{\operatorname{tr}(P)}\frac{\operatorname{tr}\left(PA^{\nu}\right)}{\operatorname{tr}(P)}$$
$$\le 2R\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}\left(PA^{1/2}\right)}{\operatorname{tr}(P)}\right)^2\right), \tag{2.9}$$

where $r = \min\{1-\nu,\nu\}$ and $R = \max\{1-\nu,\nu\}$.

Remark 1. *If P, Q are positive invertible operators with $P, Q \in \mathcal{B}_1(H)$, then by (2.9) for $A = P^{-1/2}QP^{-1/2}$ we get*

$$2r\left(\frac{\operatorname{tr}(Q)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(P\sharp Q)}{\operatorname{tr}(P)}\right)^2\right) \le \frac{\operatorname{tr}(Q)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(P\sharp_{1-\nu}Q)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(P\sharp_{\nu}Q)}{\operatorname{tr}(P)} \le 2R\left(\frac{\operatorname{tr}(Q)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(P\sharp Q)}{\operatorname{tr}(P)}\right)^2\right),$$

where the operator weighted geometric mean is defined in (1.5).

Corollary 2. *Let A, B two positive operators and $P, Q \in \mathcal{B}_1(H)$ with $P, Q > 0$. If $p, q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$, then we have*

$$t\left(\frac{\operatorname{tr}(PA^p)}{\operatorname{tr}(P)}-2\frac{\operatorname{tr}\left(PA^{p/2}\right)}{\operatorname{tr}(P)}\frac{\operatorname{tr}\left(QB^{q/2}\right)}{\operatorname{tr}(Q)}+\frac{\operatorname{tr}(QB^q)}{\operatorname{tr}(Q)}\right)\le\frac{1}{p}\frac{\operatorname{tr}(PA^p)}{\operatorname{tr}(P)}+\frac{1}{q}\frac{\operatorname{tr}(QB^q)}{\operatorname{tr}(Q)}-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(QB)}{\operatorname{tr}(Q)} \tag{2.10}$$

$$\le T\left(\frac{\operatorname{tr}(PA^p)}{\operatorname{tr}(P)}-2\frac{\operatorname{tr}\left(PA^{p/2}\right)}{\operatorname{tr}(P)}\frac{\operatorname{tr}\left(QB^{q/2}\right)}{\operatorname{tr}(Q)}+\frac{\operatorname{tr}(QB^q)}{\operatorname{tr}(Q)}\right),$$

where $t=\min\left\{\frac{1}{p},\frac{1}{q}\right\}$ and $T=\max\left\{\frac{1}{p},\frac{1}{q}\right\}$.

The proof follows by (2.5) on replacing A with A^p, B with B^q and $\nu=\frac{1}{q}$.

Remark 2. *If P, Q, S, V are positive invertible operators with $P, Q, S, V \in \mathcal{B}_1(H)$, then by (2.10) we get for $A=P^{-1/2}SP^{-1/2}$ and $B=Q^{-1/2}VQ^{-1/2}$ that*

$$t\left(\frac{\operatorname{tr}(P\sharp_p S)}{\operatorname{tr}(P)}-2\frac{\operatorname{tr}(P\sharp_{p/2}S)}{\operatorname{tr}(P)}\frac{\operatorname{tr}\left(Q\sharp_{q/2}V\right)}{\operatorname{tr}(Q)}+\frac{\operatorname{tr}(Q\sharp_q V)}{\operatorname{tr}(Q)}\right)\le\frac{1}{p}\frac{\operatorname{tr}(P\sharp_p S)}{\operatorname{tr}(P)}+\frac{1}{q}\frac{\operatorname{tr}(Q\sharp_q V)}{\operatorname{tr}(Q)}-\frac{\operatorname{tr}(S)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(V)}{\operatorname{tr}(Q)} \tag{2.11}$$

$$\le T\left(\frac{\operatorname{tr}(P\sharp_p S)}{\operatorname{tr}(P)}-2\frac{\operatorname{tr}(P\sharp_{p/2}S)}{\operatorname{tr}(P)}\frac{\operatorname{tr}\left(Q\sharp_{q/2}V\right)}{\operatorname{tr}(Q)}+\frac{\operatorname{tr}(Q\sharp_q V)}{\operatorname{tr}(Q)}\right),$$

where $t=\min\left\{\frac{1}{p},\frac{1}{q}\right\}$ and $T=\max\left\{\frac{1}{p},\frac{1}{q}\right\}$.

In particular, if we take in (2.11) $S=Q$ and $V=P$, then we get

$$t\left(\frac{\operatorname{tr}(P\sharp_p Q)}{\operatorname{tr}(P)}-2\frac{\operatorname{tr}(P\sharp_{p/2}Q)}{\operatorname{tr}(P)}\frac{\operatorname{tr}\left(Q\sharp_{q/2}P\right)}{\operatorname{tr}(Q)}+\frac{\operatorname{tr}(Q\sharp_q P)}{\operatorname{tr}(Q)}\right)\le\frac{1}{p}\frac{\operatorname{tr}(P\sharp_p Q)}{\operatorname{tr}(P)}+\frac{1}{q}\frac{\operatorname{tr}(Q\sharp_q P)}{\operatorname{tr}(Q)}-1$$

$$\le T\left(\frac{\operatorname{tr}(P\sharp_p Q)}{\operatorname{tr}(P)}-2\frac{\operatorname{tr}(P\sharp_{p/2}Q)}{\operatorname{tr}(P)}\frac{\operatorname{tr}\left(Q\sharp_{q/2}P\right)}{\operatorname{tr}(Q)}+\frac{\operatorname{tr}(Q\sharp_q P)}{\operatorname{tr}(Q)}\right),$$

where $t=\min\left\{\frac{1}{p},\frac{1}{q}\right\}$ and $T=\max\left\{\frac{1}{p},\frac{1}{q}\right\}$.

3 Trace Inequalities Via Liao-Wu-Zhao and Zuo-Shi-Fujii Results

We consider the *Kantorovich's ratio* defined by

$$K(h):=\frac{(h+1)^2}{4h},\ h>0.$$

The function K is decreasing on $(0,1)$ and increasing on $[1,\infty)$, $K(h)\ge 1$ for any $h>0$ and $K(h)=K\left(\frac{1}{h}\right)$ for any $h>0$.

The following multiplicative refinement and reverse of Young inequality in terms of Kantorovich's ratio holds

$$K^r\left(\frac{a}{b}\right)a^{1-\nu}b^{\nu}\le(1-\nu)a+\nu b\le K^R\left(\frac{a}{b}\right)a^{1-\nu}b^{\nu}, \tag{3.1}$$

where $a, b>0$, $\nu\in[0,1]$, $r=\min\{1-\nu,\nu\}$ and $R=\max\{1-\nu,\nu\}$.

The first inequality in (3.1) was obtained by Zuo et al. in [34] while the second by Liao et al. [18].

We can give a simple direct proof for (3.1) as follows.

Indeed, if we write the inequality (2.3) for the convex function $\Phi(x) = -\ln x$, and for the positive numbers a and b we get

$$2\min\{\nu, 1-\nu\}\left[\ln\left(\frac{a+b}{2}\right) - \frac{\ln a + \ln b}{2}\right] \le \ln[\nu b + (1-\nu)a] - (1-\nu)\ln a - \nu \ln b$$
$$\le 2\max\{\nu, 1-\nu\}\left[\ln\left(\frac{a+b}{2}\right) - \frac{\ln a + \ln b}{2}\right]$$

that is equivalent to

$$\min\{\nu, 1-\nu\}\ln\left(\frac{a+b}{2\sqrt{ab}}\right)^2 \le \ln\left[\frac{\nu b + (1-\nu)a}{a^{1-\nu}b^{\nu}}\right]$$
$$\le \max\{\nu, 1-\nu\}\ln\left(\frac{a+b}{2\sqrt{ab}}\right)^2$$

and to (3.1), as stated.

If $a \in [m_1, M_1]$ and $b \in [m_2, M_2]$ with $0 < m_1 < M_1$, $0 < m_2 < M_2$ then

$$\frac{m_1}{M_2} \le \frac{a}{b} \le \frac{M_1}{m_2}.$$

Denote

$$m =: \min_{(a,b)\in[m_1,M_1]\times[m_2,M_2]} K\left(\frac{a}{b}\right) \text{ and } M =: \max_{(a,b)\in[m_1,M_1]\times[m_2,M_2]} K\left(\frac{a}{b}\right).$$

Taking into account the properties of Kantorovich's ratio we have

$$m := \begin{cases} K\left(\frac{M_1}{m_2}\right) > 1 \text{ if } \frac{M_1}{m_2} < 1, \\ 1 \text{ if } \frac{m_1}{M_2} \le 1 \le \frac{M_1}{m_2}, \\ K\left(\frac{m_1}{M_2}\right) > 1 \text{ if } 1 < \frac{m_1}{M_2}, \end{cases} = \begin{cases} K\left(\frac{m_2}{M_1}\right) > 1 \text{ if } \frac{M_1}{m_2} < 1, \\ 1 \text{ if } \frac{m_1}{M_2} \le 1 \le \frac{M_1}{m_2}, \\ K\left(\frac{M_2}{m_1}\right) > 1 \text{ if } 1 < \frac{m_1}{M_2} \end{cases} \tag{3.2}$$

and

$$M := \begin{cases} K\left(\frac{m_1}{M_2}\right) > 1 \text{ if } \frac{M_1}{m_2} < 1, \\ \max\left\{K\left(\frac{m_1}{M_2}\right), K\left(\frac{M_1}{m_2}\right)\right\} > 1 \text{ if } \frac{m_1}{M_2} \le 1 \le \frac{M_1}{m_2}, \\ K\left(\frac{M_1}{m_2}\right) > 1 \text{ if } 1 < \frac{m_1}{M_2}, \end{cases} \tag{3.3}$$
$$= \begin{cases} K\left(\frac{M_2}{m_1}\right) > 1 \text{ if } \frac{M_1}{m_2} < 1, \\ \max\left\{K\left(\frac{M_2}{m_1}\right), K\left(\frac{M_1}{m_2}\right)\right\} > 1 \text{ if } \frac{m_1}{M_2} \le 1 \le \frac{M_1}{m_2}, \\ K\left(\frac{M_1}{m_2}\right) > 1 \text{ if } 1 < \frac{m_1}{M_2}. \end{cases}$$

We have the following result:

Theorem 2. *Let A, B be two operators such that*

$$0 < m_1 I \le A < M_1 I,\ 0 < m_2 I \le B \le M_2 I \tag{3.4}$$

and P, $Q \in \mathcal{B}_1(H)$ with P, $Q > 0$. Then for any $\nu \in [0, 1]$, we have for m, M as defined by (3.2) and (3.3) that

$$m^r \frac{\operatorname{tr}(PA^{1-\nu})}{\operatorname{tr}(P)}\frac{\operatorname{tr}(QB^{\nu})}{\operatorname{tr}(Q)} \le (1-\nu)\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} + \nu\frac{\operatorname{tr}(QB)}{\operatorname{tr}(Q)} \le M^R \frac{\operatorname{tr}(PA^{1-\nu})}{\operatorname{tr}(P)}\frac{\operatorname{tr}(QB^{\nu})}{\operatorname{tr}(Q)}, \tag{3.5}$$

where $r = \min\{1-\nu, \nu\}$ *and* $R = \max\{1-\nu, \nu\}$.

In particular, we have

$$m^{1/2}\frac{\operatorname{tr}\left(PA^{1/2}\right)}{\operatorname{tr}(P)}\frac{\operatorname{tr}\left(QB^{1/2}\right)}{\operatorname{tr}(Q)} \le \frac{1}{2}\left[\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}+\frac{\operatorname{tr}(QB)}{\operatorname{tr}(Q)}\right] \le M^{1/2}\frac{\operatorname{tr}\left(PA^{1/2}\right)}{\operatorname{tr}(P)}\frac{\operatorname{tr}\left(QB^{1/2}\right)}{\operatorname{tr}(Q)}.$$

Proof. From (3.1) we have

$$m^r a^{1-\nu}b^\nu \le (1-\nu)a+\nu b \le M^R a^{1-\nu}b^\nu, \tag{3.6}$$

where $a\in[m_1,M_1]$, $b\in[m_2,M_2]$ and $\nu\in[0,1]$.

Using the functional calculus for the operator A, we have

$$m^r b^\nu\left\langle A^{1-\nu}x,x\right\rangle \le (1-\nu)\langle Ax,x\rangle+\nu b\|x\|^2 \le M^R b^\nu\left\langle A^{1-\nu}x,x\right\rangle, \tag{3.7}$$

for any $x\in H$, $b\in[m_2,M_2]$ and $\nu\in[0,1]$.

Using the functional calculus for B we get from (3.7) that

$$m^r\left\langle A^{1-\nu}x,x\right\rangle\left\langle B^\nu y,y\right\rangle \le (1-\nu)\langle Ax,x\rangle\|y\|^2+\nu\|x\|^2\langle By,y\rangle \le M^R\left\langle A^{1-\nu}x,x\right\rangle\left\langle B^\nu y,y\right\rangle, \tag{3.8}$$

for any $x,y\in H$ and $\nu\in[0,1]$.

This is an inequality of interest in itself as well.

Further, let $x=P^{1/2}e$, $y=Q^{1/2}f$ where $e,f\in H$. Then by (3.8) we have

$$m^r\left\langle P^{1/2}A^{1-\nu}P^{1/2}e,e\right\rangle\left\langle Q^{1/2}B^\nu Q^{1/2}f,f\right\rangle \le (1-\nu)\left\langle P^{1/2}AP^{1/2}e,e\right\rangle\langle Qf,f\rangle+\nu\langle Pe,e\rangle\left\langle Q^{1/2}BQ^{1/2}f,f\right\rangle \le M^R\left\langle P^{1/2}A^{1-\nu}P^{1/2}e,e\right\rangle\left\langle Q^{1/2}B^\nu Q^{1/2}f,f\right\rangle,$$

for any $e,f\in H$ and $\nu\in[0,1]$.

Now, on making use of a similar argument as in the proof of Theorem 1, we get the desired result (3.5). □

Remark 3. *Let* A, B *be two operators such that the condition (3.4) is valid and* $P\in\mathcal{B}_1(H)$ *with* $P>0$. *Then for any* $\nu\in[0,1]$, *we have for* m, M *as defined by (3.2) and (3.3) that*

$$m^r\frac{\operatorname{tr}\left(PA^{1-\nu}\right)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PB^\nu)}{\operatorname{tr}(P)} \le \frac{\operatorname{tr}(P[(1-\nu)A+\nu B])}{\operatorname{tr}(P)} \le M^R\frac{\operatorname{tr}\left(PA^{1-\nu}\right)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PB^\nu)}{\operatorname{tr}(P)},$$

where $r=\min\{1-\nu,\nu\}$ *and* $R=\max\{1-\nu,\nu\}$.

In particular, we have

$$m^{1/2}\frac{\operatorname{tr}\left(PA^{1/2}\right)}{\operatorname{tr}(P)}\frac{\operatorname{tr}\left(PB^{1/2}\right)}{\operatorname{tr}(P)} \le \frac{\operatorname{tr}\left(P\left(\frac{A+B}{2}\right)\right)}{\operatorname{tr}(P)} \le M^{1/2}\frac{\operatorname{tr}\left(PA^{1/2}\right)}{\operatorname{tr}(P)}\frac{\operatorname{tr}\left(PB^{1/2}\right)}{\operatorname{tr}(P)}.$$

For $0<m_1<M_1$, $0<m_2<M_2$ and $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ we define

$$m_{p,q} := \begin{cases} K\left(\frac{M_1^p}{m_2^q}\right)>1 \text{ if } \frac{M_1^p}{m_2^q}<1, \\ 1 \text{ if } \frac{m_1^p}{M_2^q}\le 1\le\frac{M_1^p}{m_2^q}, \\ K\left(\frac{M_2^q}{m_1^p}\right)>1 \text{ if } 1<\frac{m_1^p}{M_2^q} \end{cases} \tag{3.9}$$

and

$$M_{p,q} := \begin{cases} K\left(\frac{M_2^q}{m_1^p}\right) > 1 \text{ if } \frac{M_1^p}{m_2^q} < 1, \\ \max\left\{K\left(\frac{M_2^q}{m_1^p}\right), K\left(\frac{M_1^p}{m_2^q}\right)\right\} > 1 \text{ if } \frac{m_1^p}{M_2^q} \le 1 \le \frac{M_1^p}{m_2^q}, \\ K\left(\frac{M_1^p}{m_2^q}\right) > 1 \text{ if } 1 < \frac{m_1^p}{M_2^q}. \end{cases} \tag{3.10}$$

Corollary 3. *Let A, B be two operators such that (3.4) is valid and $P, Q \in \mathcal{B}_1(H)$ with $P, Q > 0$. Then for any $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ we have for $m_{p,q}$, $M_{p,q}$ as defined by (3.9) and (3.10) that*

$$\begin{aligned} m_{p,q}^t \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} \frac{\operatorname{tr}(QB)}{\operatorname{tr}(Q)} &\le \frac{1}{p}\frac{\operatorname{tr}(PA^p)}{\operatorname{tr}(P)} + \frac{1}{q}\frac{\operatorname{tr}(QB^q)}{\operatorname{tr}(Q)} \\ &\le M_{p,q}^T \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} \frac{\operatorname{tr}(QB)}{\operatorname{tr}(Q)}, \end{aligned} \tag{3.11}$$

where $t = \min\left\{\frac{1}{p}, \frac{1}{q}\right\}$ and $T = \max\left\{\frac{1}{p}, \frac{1}{q}\right\}$.

Proof. From (3.4) we have

$$0 < m_1^p I \le A^p < M_1^p I,\ 0 < m_2^q I \le B^q \le M_2^q I.$$

By replacing A by A^p, B by B^q and $\nu = \frac{1}{q}$ in (3.5) then we get the desired result (3.11). □

Remark 4. *If we take $Q = P$ in (3.11), then we get*

$$\begin{aligned} m_{p,q}^t \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} \frac{\operatorname{tr}(PB)}{\operatorname{tr}(P)} &\le \frac{\operatorname{tr}\left[P\left(\frac{1}{p}A^p + \frac{1}{q}B^q\right)\right]}{\operatorname{tr}(P)} \\ &\le M_{p,q}^T \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} \frac{\operatorname{tr}(PB)}{\operatorname{tr}(P)}. \end{aligned}$$

For $p = q = 2$ we consider

$$\tilde{m}_2 := \begin{cases} K\left[\left(\frac{M_1}{m_2}\right)^2\right] > 1 \text{ if } \frac{M_1}{m_2} < 1, \\ 1 \text{ if } \frac{m_1}{M_2} \le 1 \le \frac{M_1}{m_2}, \\ K\left[\left(\frac{M_2}{m_1}\right)^2\right] > 1 \text{ if } 1 < \frac{m_1}{M_2} \end{cases} \tag{3.12}$$

and

$$\tilde{M}_2 := \begin{cases} K\left[\left(\frac{M_2}{m_1}\right)^2\right] > 1 \text{ if } \frac{M_1}{m_2} < 1, \\ \max\left\{K\left[\left(\frac{M_2}{m_1}\right)^2\right], K\left[\left(\frac{M_1}{m_2}\right)^2\right]\right\} > 1 \text{ if } \frac{m_1}{M_2} \le 1 \le \frac{M_1}{m_2}, \\ K\left[\left(\frac{M_1}{m_2}\right)^2\right] > 1 \text{ if } 1 < \frac{m_1}{M_2}. \end{cases} \tag{3.13}$$

Corollary 4. *Let A, B be two operators such that (3.4) is valid and $P, Q \in \mathcal{B}_1(H)$ with $P, Q > 0$. Then for $\tilde{m}_2$, $\tilde{M}_2$ as defined by (3.12) and (3.13) we have that*

$$\begin{aligned} \tilde{m}_2^{1/2} \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} \frac{\operatorname{tr}(QB)}{\operatorname{tr}(Q)} &\le \frac{1}{p}\frac{\operatorname{tr}(PA^2)}{\operatorname{tr}(P)} + \frac{1}{q}\frac{\operatorname{tr}(QB^2)}{\operatorname{tr}(Q)} \\ &\le \tilde{M}_2^{1/2} \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} \frac{\operatorname{tr}(QB)}{\operatorname{tr}(Q)}. \end{aligned}$$

In particular,

$$\tilde{m}_2^{1/2}\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PB)}{\operatorname{tr}(P)} \le \frac{\operatorname{tr}\left[P\left(\frac{A^2+B^2}{2}\right)\right]}{\operatorname{tr}(P)} \le \tilde{M}_2^{1/2}\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PB)}{\operatorname{tr}(P)}.$$

Corollary 5. *If P, Q, S, V are positive invertible operators with $P, Q, S, V \in \mathcal{B}_1(H)$ and for $0 < m_1 < M_1$, $0 < m_2 < M_2$,*

$$0 < m_1 P \le S \le M_1 P,\ 0 < m_2 Q \le V \le M_2 Q. \tag{3.14}$$

Then for any $\nu \in [0,1]$, we have for m, M as defined by (3.2) and (3.3) that

$$\begin{aligned} m^r \frac{\operatorname{tr}(P\sharp_{1-\nu}S)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(Q\sharp_\nu V)}{\operatorname{tr}(Q)} &\le (1-\nu)\frac{\operatorname{tr}(S)}{\operatorname{tr}(P)} + \nu\frac{\operatorname{tr}(V)}{\operatorname{tr}(Q)} \\ &\le M^R \frac{\operatorname{tr}(P\sharp_{1-\nu}S)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(Q\sharp_\nu V)}{\operatorname{tr}(Q)}, \end{aligned} \tag{3.15}$$

where $r = \min\{1-\nu, \nu\}$ and $R = \max\{1-\nu,\nu\}$.

In particular, we have

$$\begin{aligned} m^{1/2}\frac{\operatorname{tr}(P\sharp S)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(Q\sharp V)}{\operatorname{tr}(Q)} &\le \frac{1}{2}\left[\frac{\operatorname{tr}(S)}{\operatorname{tr}(P)} + \frac{\operatorname{tr}(V)}{\operatorname{tr}(Q)}\right] \\ &\le M^{1/2}\frac{\operatorname{tr}(P\sharp S)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(Q\sharp V)}{\operatorname{tr}(Q)}. \end{aligned}$$

Proof. From (3.14) we have

$$0 < m_1 \le P^{-1/2}SP^{-1/2} \le M_1,\ 0 < m_2 \le Q^{-1/2}VQ^{-1/2} \le M_2.$$

If we use the inequality (3.5) for $A = P^{-1/2}SP^{-1/2}$ and $B = Q^{-1/2}VQ^{-1/2}$ then

$$\begin{aligned} m^r \frac{\operatorname{tr}\left(P\left(P^{-1/2}SP^{-1/2}\right)^{1-\nu}\right)}{\operatorname{tr}(P)}\frac{\operatorname{tr}\left(Q\left(Q^{-1/2}VQ^{-1/2}\right)^{\nu}\right)}{\operatorname{tr}(Q)} &\le (1-\nu)\frac{\operatorname{tr}\left(PP^{-1/2}SP^{-1/2}\right)}{\operatorname{tr}(P)} + \nu\frac{\operatorname{tr}\left(QQ^{-1/2}VQ^{-1/2}\right)}{\operatorname{tr}(Q)} \\ &\le M^R \frac{\operatorname{tr}\left(P\left(P^{-1/2}SP^{-1/2}\right)^{1-\nu}\right)}{\operatorname{tr}(P)}\frac{\operatorname{tr}\left(Q\left(Q^{-1/2}VQ^{-1/2}\right)^{\nu}\right)}{\operatorname{tr}(Q)}, \end{aligned}$$

which, by the properties of trace, is equivalent to (3.15). □

Remark 5. *If P, S, V are positive invertible operators with $P, S, V \in \mathcal{B}_1(H)$ and for $0 < m_1 < M_1$, $0 < m_2 < M_2$,*

$$0 < m_1 P \le S \le M_1 P,\ 0 < m_2 P \le V \le M_2 P,$$

then for any $\nu \in [0,1]$, we have for m, M as defined by (3.2) and (3.3) that

$$\begin{aligned} m^r \frac{\operatorname{tr}(P\sharp_{1-\nu}S)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(P\sharp_\nu V)}{\operatorname{tr}(P)} &\le \frac{\operatorname{tr}((1-\nu)S + \nu V)}{\operatorname{tr}(P)} \\ &\le M^R \frac{\operatorname{tr}(P\sharp_{1-\nu}S)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(P\sharp_\nu V)}{\operatorname{tr}(P)}, \end{aligned}$$

where $r = \min\{1-\nu,\nu\}$ and $R = \max\{1-\nu,\nu\}$.

In particular, we have

$$m^{1/2}\frac{\operatorname{tr}(P\sharp S)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(P\sharp V)}{\operatorname{tr}(P)} \le \frac{\operatorname{tr}\left(\frac{S+V}{2}\right)}{\operatorname{tr}(P)} \le M^{1/2}\frac{\operatorname{tr}(P\sharp S)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(P\sharp V)}{\operatorname{tr}(P)}.$$

4 Trace Inequalities Via Tominaga and Furuichi Results

We recall that *Specht's ratio* is defined by [28]

$$S(h) := \begin{cases} \dfrac{h^{\frac{1}{h-1}}}{e\ln\left(h^{\frac{1}{h-1}}\right)} \text{ if } h \in (0,1)\cup(1,\infty) \\ \\ 1 \text{ if } h = 1. \end{cases}$$

It is well known that $\lim_{h\to 1} S(h) = 1$, $S(h) = S\left(\frac{1}{h}\right) > 1$ for $h > 0$, $h \neq 1$. The function is decreasing on $(0,1)$ and increasing on $(1,\infty)$.

The following inequality provides a refinement and a multiplicative reverse for Young's inequality

$$S\left(\left(\frac{a}{b}\right)^r\right) a^{1-\nu}b^{\nu} \le (1-\nu)a + \nu b \le S\left(\frac{a}{b}\right) a^{1-\nu}b^{\nu}, \tag{4.1}$$

where $a, b > 0$, $\nu \in [0,1]$, $r = \min\{1-\nu, \nu\}$.

The second inequality in (4.1) is due to Tominaga [29] while the first one is due to Furuichi [11].

If $a \in [m_1, M_1]$ and $b \in [m_2, M_2]$ with $0 < m_1 < M_1$, $0 < m_2 < M_2$ then

$$\frac{m_1}{M_2} \le \frac{a}{b} \le \frac{M_1}{m_2}.$$

Denote, for $r \in (0,1)$

$$\check{m}_r =: \min_{(a,b)\in[m_1,M_1]\times[m_2,M_2]} S\left(\left(\frac{a}{b}\right)^r\right) \text{ and } \check{M} =: \max_{(a,b)\in[m_1,M_1]\times[m_2,M_2]} S\left(\frac{a}{b}\right).$$

Taking into account the properties of Specht's ratio we have

$$\check{m}_r := \begin{cases} S\left(\left(\frac{M_1}{m_2}\right)^r\right) > 1 \text{ if } \frac{M_1}{m_2} < 1, \\ \\ 1 \text{ if } \frac{m_1}{M_2} \le 1 \le \frac{M_1}{m_2}, \\ \\ S\left(\left(\frac{M_2}{m_1}\right)^r\right) > 1 \text{ if } 1 < \frac{m_1}{M_2}, \end{cases} \tag{4.2}$$

and

$$\check{M} := \begin{cases} S\left(\frac{M_2}{m_1}\right) > 1 \text{ if } \frac{M_1}{m_2} < 1, \\ \\ \max\left\{S\left(\frac{M_2}{m_1}\right), S\left(\frac{M_1}{m_2}\right)\right\} > 1 \text{ if } \frac{m_1}{M_2} \le 1 \le \frac{M_1}{m_2}, \\ \\ S\left(\frac{M_1}{m_2}\right) > 1 \text{ if } 1 < \frac{m_1}{M_2}. \end{cases} \tag{4.3}$$

We have the following result:

Theorem 3. *Let A, B be two operators such that*

$$0 < m_1 I \le A < M_1 I,\ 0 < m_2 I \le B \le M_2 I$$

and P, $Q \in \mathcal{B}_1(H)$ with $P, Q > 0$. Then for any $\nu \in [0,1]$, we have for $\check{m}_r$, $\check{M}$ as defined by (4.2) and (4.3) that

$$\begin{aligned} \check{m}_r \frac{\operatorname{tr}\left(PA^{1-\nu}\right)}{\operatorname{tr}(P)} \frac{\operatorname{tr}(QB^{\nu})}{\operatorname{tr}(Q)} &\le (1-\nu)\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} + \nu\frac{\operatorname{tr}(QB)}{\operatorname{tr}(Q)} \\ &\le \check{M} \frac{\operatorname{tr}\left(PA^{1-\nu}\right)}{\operatorname{tr}(P)} \frac{\operatorname{tr}(QB^{\nu})}{\operatorname{tr}(Q)}, \end{aligned} \tag{4.4}$$

where $r = \min\{1-\nu, \nu\}$ *and* $R = \max\{1-\nu, \nu\}$.

In particular, we have

$$\check{m}_{1/2}\frac{\operatorname{tr}\left(PA^{1/2}\right)}{\operatorname{tr}(P)}\frac{\operatorname{tr}\left(QB^{1/2}\right)}{\operatorname{tr}(Q)} \le \frac{1}{2}\left[\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}+\frac{\operatorname{tr}(QB)}{\operatorname{tr}(Q)}\right] \le \check{M}\frac{\operatorname{tr}\left(PA^{1/2}\right)}{\operatorname{tr}(P)}\frac{\operatorname{tr}\left(QB^{1/2}\right)}{\operatorname{tr}(Q)}.$$

Proof. From (3.1) we have

$$\check{m}_r a^{1-\nu}b^{\nu} \le (1-\nu)a + \nu b \le \check{M}a^{1-\nu}b^{\nu},$$

where $a \in [m_1, M_1]$, $b \in [m_2, M_2]$ and $\nu \in [0,1]$.

Now, on making use of a similar argument as in the proof of Theorem 2, we get the desired result (4.4). □

For $0 < m_1 < M_1$, $0 < m_2 < M_2$ and $p, q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$ we define for $r \in (0,1)$

$$\check{m}_{r,p,q} := \begin{cases} S\left(\left(\frac{M_1^p}{m_2^q}\right)^r\right) > 1 \text{ if } \frac{M_1^p}{m_2^q} < 1, \\ 1 \text{ if } \frac{m_1^p}{M_2^q} \le 1 \le \frac{M_1^p}{m_2^q}, \\ S\left(\left(\frac{M_2^q}{m_1^p}\right)^r\right) > 1 \text{ if } 1 < \frac{m_1^p}{M_2^q} \end{cases} \tag{4.5}$$

and

$$\check{M}_{p,q} := \begin{cases} S\left(\frac{M_2^q}{m_1^p}\right) > 1 \text{ if } \frac{M_1^p}{m_2^q} < 1, \\ \max\left\{S\left(\frac{M_2^q}{m_1^p}\right), S\left(\frac{M_1^p}{m_2^q}\right)\right\} > 1 \text{ if } \frac{m_1^p}{M_2^q} \le 1 \le \frac{M_1^p}{m_2^q}, \\ S\left(\frac{M_1^p}{m_2^q}\right) > 1 \text{ if } 1 < \frac{m_1^p}{M_2^q}. \end{cases} \tag{4.6}$$

Corollary 6. *Let A, B be two operators such that (3.4) is valid and $P, Q \in \mathcal{B}_1(H)$ with $P, Q > 0$. Then for any $p, q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$ we have for $\check{m}_{t,p,q}$, $\check{M}_{p,q}$ as defined by (4.5) and (4.6) that*

$$\check{m}_{t,p,q}\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(QB)}{\operatorname{tr}(Q)} \le \frac{1}{p}\frac{\operatorname{tr}\left(PA^p\right)}{\operatorname{tr}(P)}+\frac{1}{q}\frac{\operatorname{tr}\left(QB^q\right)}{\operatorname{tr}(Q)} \le \check{M}_{p,q}\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(QB)}{\operatorname{tr}(Q)},$$

where $t = \min\left\{\frac{1}{p}, \frac{1}{q}\right\}$.

The interested reader may write similar inequalities to those in the previous section, however we do not present them here.

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