Inequalities for the numerical radius in unital normed algebras

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INEQUALITIES FOR THE NUMERICAL RADIUS IN UNITAL NORMED ALGEBRAS

Abstract. In this paper, some inequalities between the numerical radius of an element from a unital normed algebra and certain semi-inner products involving that element and the unity are given.

1. Introduction

Let A be a unital normed algebra over the complex number field $\mathbb{C}$ and let $a \in A$. Recall that the numerical radius of a is given by (see [2, p. 15])

$$v(a) = \sup\{|f(a)|,\ f \in A',\ \|f\| \leq 1 \ \text{ and } \ f(1) = 1\}, \tag{1.1}$$

where A' denotes the dual space of A, i.e., the Banach space of all continuous linear functionals on A.

It is known that $v(\cdot)$ is a norm on A that is equivalent to the given norm $\|\cdot\|$. More precisely, the following double inequality holds:

$$\frac{1}{e}\|a\| \leq v(a) \leq \|a\| \tag{1.2}$$

for any $a \in A$, where $e = \exp(1)$.

Following [2], we notice that this crucial result appears slightly hidden in Bohnenblust and Karlin [1, Theorem 1] together with the inequality $\|x\| \leq e\Psi(x)$, where $\Psi(x) = \sup\{|\lambda|^{-1} \log \|e^{\lambda x}\|\}$ over λ complex, $\lambda \neq 0$, which occurs on page 219. A simpler proof was given by Lumer [5], though with the constant $1/4$ in place of $1/e$. For a simple proof of (1.2) that borrows ideas from Lumer and from Glickfeld [6], see [2, p. 34].

A generalisation of (1.2) for powers has been obtained by M. J. Crabb [3] who proved that

$$\|a^n\| \leq n!(\frac{e}{n})^n [v(a)]^n \qquad n = 1, 2, \ldots \tag{1.3}$$

for any $a \in A$.

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In this paper, some inequalities between the numerical radius of an element and the superior semi-inner product of that element and the unity in the normed algebra A are given via the celebrated representation result of Lumer from [5].

2. Some subsets in A

Let $D(1) := \{f \in A' | \|f\| \le 1 \text{ and } f(1) = 1\}$. For $\lambda \in \mathbb{C}$ and $r > 0$, we define the subset of A by

$$\bar{\Delta}(\lambda, r) := \{a \in A \mid |f(a) - \lambda| \le r \ \text{ for each } \ f \in D(1)\}.$$

The following result holds.

Proposition 1. *Let $\lambda \in \mathbb{C}$ and $r > 0$. Then $\bar{\Delta}(\lambda, r)$ is a closed convex subset of A and*

$$\bar{B}(\lambda, r) \subseteq \bar{\Delta}(\lambda, r), \tag{2.1}$$

where $\bar{B}(\lambda, r) := \{a \in A | \|a - \lambda\| \le r\}$.

Now, for $\gamma, \Gamma \in \mathbb{C}$, define the set

$$\bar{U}(\gamma, \Gamma) := \{a \in A \mid \mathrm{Re}[(\Gamma - f(a))(\overline{f(a)} - \bar{\gamma})] \ge 0 \ \text{ for each } \ f \in D(1)\}.$$

The following representation result may be stated.

Proposition 2. *For any $\gamma, \Gamma \in \mathbb{C}$, $\gamma \ne \Gamma$, we have:*

$$\bar{U}(\gamma, \Gamma) = \bar{\Delta}\left(\frac{\gamma + \Gamma}{2}, \frac{1}{2}|\Gamma - \gamma|\right). \tag{2.2}$$

Proof. We observe that for any $z \in \mathbb{C}$ we have the equivalence

$$\left| z - \frac{\gamma + \Gamma}{2} \right| \le \frac{1}{2}|\Gamma - \gamma|$$

if and only if

$$\mathrm{Re}[(\Gamma - z)(\bar{z} - \bar{\gamma})] \ge 0.$$

This follows by the equality

$$\frac{1}{4}|\Gamma - \gamma|^2 - \left| z - \frac{\gamma + \Gamma}{2} \right|^2 = \mathrm{Re}[(\Gamma - z)(\bar{z} - \bar{\gamma})]$$

that holds for any $z \in \mathbb{C}$.

The equality (2.2) is thus a simple conclusion of this fact. ■

Making use of some obvious properties in $\mathbb{C}$ and for continuous linear functionals, we can state the following corollary as well.

Corollary 1. *For any $\gamma, \Gamma \in \mathbb{C}$, we have*

$$\begin{aligned}(2.3)\quad \bar{U}(\gamma,\Gamma) &= \{a \in A \mid \operatorname{Re}[f(\Gamma - a)\overline{f(a-\gamma)}] \geq 0 \text{ for each } f \in D(1)\} \\ &= \{a \in A \mid (\operatorname{Re}\Gamma - \operatorname{Re} f(a))(\operatorname{Re} f(a) - \operatorname{Re}\gamma) \\ &\quad + (\operatorname{Im}\Gamma - \operatorname{Im} f(a))(\operatorname{Im} f(a) - \operatorname{Im}\gamma) \geq 0 \text{ for each } f \in D(1)\}.\end{aligned}$$

Now, if we assume that $\operatorname{Re}(\Gamma) \geq \operatorname{Re}(\gamma)$ and $\operatorname{Im}(\Gamma) \geq \operatorname{Im}(\gamma)$, then we can define the following subset of A :

$$\begin{aligned}(2.4)\quad \bar{S}(\gamma,\Gamma) := \{a \in A \mid\ & \operatorname{Re}(\Gamma) \geq \operatorname{Re} f(a) \geq \operatorname{Re}(\gamma) \text{ and} \\ & \operatorname{Im}(\Gamma) \geq \operatorname{Im} f(a) \geq \operatorname{Im}(\gamma) \text{ for each } f \in D(1)\}.\end{aligned}$$

One can easily observe that $\bar{S}(\gamma,\Gamma)$ is closed, convex and

$$\bar{S}(\gamma,\Gamma) \subseteq \bar{U}(\gamma,\Gamma). \tag{2.5}$$

3. Semi-inner products and Lumer's theorem

Let $(X, \|\cdot\|)$ be a normed linear space over the real of complex number field $\mathbb{K}$. The mapping $f : X \to \mathbb{R}$, $f(x) = \frac{1}{2}\|x\|^2$ is obviously convex and then there exist the following limits:

$$\langle x, y\rangle_i = \lim_{t\to 0^-} \frac{\|y+tx\|^2 - \|y\|^2}{2t},$$
$$\langle x, y\rangle_s = \lim_{t\to 0^+} \frac{\|y+tx\|^2 - \|y\|^2}{2t}$$

for every two elements $x, y \in X$. The mapping $\langle\cdot,\cdot\rangle_s$ ($\langle\cdot,\cdot\rangle_i$) will be called the *superior semi-inner product* (*the interior semi-inner product*) associated to the norm $\|\cdot\|$.

We list some properties of these semi-inner products that can be easily derived from the definition (see for instance [4]). If $p, q \in \{s, i\}$ and $p \neq q$, then:

(i) $\langle x, x\rangle_p = \|x\|^2$; $\langle ix, x\rangle_p = \langle x, ix\rangle_p = 0$, $x \in X$;
(ii) $\langle \lambda x, y\rangle_p = \lambda\langle x, y\rangle_p$; $\langle x, \lambda y\rangle_p = \lambda\langle x, y\rangle_p$ for $\lambda \geq 0$, $x, y \in X$;
(iii) $\langle \lambda x, y\rangle_p = \lambda\langle x, y\rangle_q$; $\langle x, \lambda y\rangle_p = \lambda\langle x, y\rangle_q$ for $\lambda < 0$, $x, y \in X$;
(iv) $\langle ix, y\rangle_p = -\langle x, iy\rangle_p$; $\langle \alpha x, \beta y\rangle = \alpha\beta\langle x, y\rangle$ if $\alpha\beta \geq 0$, $x, y \in X$;
(v) $\langle -x, y\rangle_p = \langle x, -y\rangle_p = -\langle x, y\rangle_q$, $x, y \in X$;
(vi) $|\langle x, y\rangle_p| \leq \|x\|\|y\|$, $x, y \in X$;
(vii) $\langle x_1 + x_2, y\rangle_{s(i)} \leq (\geq) \langle x_1, y\rangle_{s(i)} + \langle x_2, y\rangle_{s(i)}$ for $x_1, x_2, y \in X$;
(ix) $\langle \alpha x + y, x\rangle_p = \alpha\|x\|^2 + \langle y, x\rangle_p$, $\alpha \in \mathbb{R}$, $x, y \in X$;
(x) $|\langle y + z, x\rangle_p - \langle z, x\rangle_p| \leq \|y\|\|x\|$, $x, y, z \in X$;
(xi) the mapping $\langle\cdot, x\rangle_p$ is continuous on $(X, \|\cdot\|)$ for each $x \in X$.

The following result essentially due to Lumer [5] (see [2, p. 17]) can be stated.

Theorem 1. *Let A be a unital normed algebra over $\mathbb{K}$ $(\mathbb{K} = \mathbb{C}, \mathbb{R})$. For each $a \in A$,*

$$\max\{\operatorname{Re}\lambda | \lambda \in V(a)|\} = \inf_{\alpha>0} \frac{1}{\alpha}[\|1+\alpha a\| - 1] = \lim_{\alpha\to 0^+} \frac{1}{\alpha}[\|1+\alpha a\| - 1], \tag{3.1}$$

where $V(a)$ is the numerical range of a (see for instance [2, p. 15]*).*

Remark 1. In terms of semi-inner products, the above identity can be stated as:

$$\max\{\operatorname{Re} f(a) | f \in D(1)\} = \langle a, 1\rangle_s. \tag{3.2}$$

The following result that provides more information may be stated.

Theorem 2. *For any $a \in A$, we have:*

$$\langle a, 1\rangle_{v,s} = \langle a, 1\rangle_s, \tag{3.3}$$

where

$$\langle a, b\rangle_{v,s} := \lim_{t\to 0^+} \frac{v^2(b+ta) - v^2(b)}{2t}$$

is the superior semi-inner product associated with the numerical radius.

Proof. Since $v(a) \le \|a\|$, we have:

$$\begin{aligned}\langle a, 1\rangle_{v,s} &= \lim_{t\to 0^+} \frac{v^2(1+ta) - v^2(1)}{2t} = \lim_{t\to 0^+} \frac{v^2(1+ta) - 1}{2t} \\ &\le \lim_{t\to 0^+} \frac{\|1+ta\|^2 - 1}{2t} = \langle a, 1\rangle_s.\end{aligned}$$

Now, let $f \in D(1)$. Then, for each $\alpha > 0$,

$$f(a) = \frac{1}{\alpha}[f(1+\alpha a) - f(1)] = \frac{1}{\alpha}[f(1+\alpha a) - 1],$$

giving

$$\begin{aligned}\operatorname{Re} f(a) &= \frac{1}{\alpha}[\operatorname{Re} f(1+\alpha a) - f(1)] \le \frac{1}{\alpha}[|f(1+\alpha a)| - 1] \\ &\le \frac{1}{\alpha}[v(1+\alpha a) - 1].\end{aligned}$$

Taking the infimum over $\alpha > 0$, we deduce that

$$\begin{aligned}\operatorname{Re} f(a) &\le \inf_{\alpha>0}\left[\frac{1}{\alpha}[v(1+\alpha a) - 1]\right] = \lim_{\alpha\to 0^+}\left[\frac{v^2(1+\alpha a) - 1}{2\alpha}\right] \\ &= \lim_{\alpha\to 0^+} \frac{v(1+\alpha a) - 1}{\alpha} = \langle a, 1\rangle_{v,s}.\end{aligned} \tag{3.4}$$

If we now take the supremum over $f \in D(1)$ in (3.4), we obtain:

$$\sup\{\operatorname{Re} f(a) | f \in D(1)\} \le \langle a, 1\rangle_{v,s}$$

which, by Lumer's identity, implies that $\langle a, 1\rangle_s \le \langle a, 1\rangle_{v,s}$. ∎

Corollary 2. *The following inequality holds*

$$|\langle a,1\rangle_s| \le v(a) \quad (\le \|a\|). \tag{3.5}$$

Proof. Schwarz's inequality for the norm $v(.)$ gives that

$$|\langle a,1\rangle_{v,s}| \le v(a)v(1) = v(a),$$

and by (3.3), the inequality (3.5) is proved. ■

4. Reverse inequalities for the numerical radius

Utilising the inequality (3.5) we observe that for any complex number β located in the closed disc centered in 0 and with radius 1 we have $|\langle \beta a,1\rangle_s|$ as a lower bound for the numerical radius $v(a)$. Therefore, it is a natural question to ask how far these quantities are from each other under various assumptions for the element a in the unital normed algebra A and the scalar β. A number of results answering this question are incorporated in the following theorems.

Theorem 3. *Let* $\lambda \in \mathbb{C}\setminus\{0\}$ *and* $r>0$. *If* $a \in \bar{\Delta}(\lambda, r)$, *then*

$$v(a) \le \left\langle \frac{\bar{\lambda}}{|\lambda|}a, 1\right\rangle_s + \frac{1}{2}\cdot\frac{r^2}{|\lambda|}. \tag{4.1}$$

Proof. Since $a \in \bar{\Delta}(\lambda, 1)$, we have $|f(a)-\lambda|^2 \le r^2$, giving that

$$|f(a)|^2 + |\lambda|^2 \le 2\,\mathrm{Re}[f(\bar{\lambda}a)] + r^2 \tag{4.2}$$

for each $f \in D(1)$.

Taking the supremum over $f \in D(1)$ in (4.2) and utilising the representation (3.2), we deduce that

$$v^2(a) + |\lambda|^2 \le 2\langle \bar{\lambda}a, 1\rangle_s + r^2 \tag{4.3}$$

which is an inequality of interest in and of itself.

On the other hand, we have the elementary inequality

$$2v(a)|\lambda| \le v^2(a) + |\lambda|^2, \tag{4.4}$$

which, together with (4.3) implies the desired result (4.1). ■

Remark 2. Notice that, by the inclusion (2.1), a sufficient condition for (4.1) to hold is that $a \in \bar{B}(\lambda, r)$.

Corollary 3. *Let* $\gamma, \Gamma \in \mathbb{C}$ *with* $\Gamma \ne \pm\gamma$. *If* $a \in \bar{U}(\gamma, \Gamma)$, *then*

$$v(a) \le \left\langle \frac{\bar{\Gamma}+\bar{\gamma}}{|\Gamma+\gamma|}a, 1\right\rangle_s + \frac{1}{4}\cdot\frac{|\Gamma-\gamma|^2}{|\Gamma+\gamma|}. \tag{4.5}$$

Remark 3. If $M > m \ge 0$ and $a \in \bar{U}(m, M)$, then

$$(0 \le)\ v(a) - \langle a,1\rangle_s \le \frac{1}{4}\cdot\frac{(M-m)^2}{m+M}. \tag{4.6}$$

Observe that, due to the inclusion (2.5), a sufficient condition for (4.6) to hold is that $M \geq \operatorname{Re} f(a), \operatorname{Im} f(a) \geq m$ for any $f \in D(1)$.

The following result may be stated as well.

Theorem 4. *Let $\lambda \in \mathbb{C}$ and $r > 0$ with $|\lambda| > r$. If $a \in \bar{\Delta}(\lambda, r)$, then*

$$v(a) \leq \left\langle \frac{\bar{\lambda}}{\sqrt{|\lambda|^2 - r^2}} a, 1 \right\rangle_s \tag{4.7}$$

and, equivalently,

$$v^2(a) \leq \left\langle \frac{\bar{\lambda}}{|\lambda|} a, 1 \right\rangle_s^2 + \frac{r^2}{|\lambda|^2} \cdot v^2(a). \tag{4.8}$$

Proof. Since $|\lambda| > r$, we have $\sqrt{|\lambda|^2 - r^2} > 0$, hence the inequality (4.3) divided by this quantity becomes

$$\frac{v^2(a)}{\sqrt{|\lambda|^2 - r^2}} + \sqrt{|\lambda|^2 - r^2} \leq \frac{2}{\sqrt{|\lambda|^2 - r^2}} \langle \bar{\lambda} a, 1 \rangle_s. \tag{4.9}$$

On the other hand, we also have

$$2v(a) \leq \frac{v^2(a)}{\sqrt{|\lambda|^2 - r^2}} + \sqrt{|\lambda|^2 - r^2},$$

which, together with (4.9), gives

$$v(a) \leq \frac{1}{\sqrt{|\lambda|^2 - r^2}} \langle \bar{\lambda} a, 1 \rangle_s. \tag{4.10}$$

Taking the square in (4.10), we have

$$v^2(a)(|\lambda|^2 - r^2) \leq \langle \bar{\lambda} a, 1 \rangle_s^2,$$

which is clearly equivalent to (4.8). ■

Corollary 4. *Let $\gamma, \Gamma \in \mathbb{C}$ with $\operatorname{Re}(\Gamma \bar{\gamma}) > 0$. If $a \in \bar{U}(\gamma, \Gamma)$, then,*

$$v(a) \leq \left\langle \frac{\bar{\Gamma} + \bar{\gamma}}{2\sqrt{\operatorname{Re}(\Gamma \bar{\gamma})}} a, 1 \right\rangle_s. \tag{4.11}$$

Remark 4. If $M \geq m > 0$ and $a \in \bar{U}(m, M)$, then

$$v(a) \leq \frac{M + m}{2\sqrt{mM}} \langle a, 1 \rangle_s, \tag{4.12}$$

or, equivalently,

$$(0 \leq)\ v(a) - \langle a, 1 \rangle_s \leq \frac{(\sqrt{M} - \sqrt{m})^2}{2\sqrt{mM}} \langle a, 1 \rangle_s \quad \left(\leq \frac{(\sqrt{M} - \sqrt{m})^2}{2\sqrt{mM}} \|a\| \right).$$

The following result may be stated as well.

Theorem 5. *Let $\lambda \in \mathbb{C}\backslash\{0\}$ and $r > 0$ with $|\lambda| > r$. If $a \in \bar{\Delta}(\lambda, r)$, then*

$$v^2(a) \le \left\langle \frac{\bar{\lambda}}{|\lambda|}a, 1\right\rangle_s^2 + 2(|\lambda| - \sqrt{|\lambda|^2 - r^2})\left\langle \frac{\bar{\lambda}}{|\lambda|}a, 1\right\rangle_s. \tag{4.13}$$

Proof. Since (by (4.2)) $\mathrm{Re}[f(\bar{\lambda}a)] > 0$, dividing by it in (4.2) gives:

$$\frac{|f(a)|^2}{\mathrm{Re}[f(\bar{\lambda}a)]} + \frac{|\lambda|^2}{\mathrm{Re}[f(\bar{\lambda}a)]} \le 2 + \frac{r^2}{\mathrm{Re}[f(\bar{\lambda}a)]},$$

which is clearly equivalent to:

$$\begin{aligned} &\frac{|f(a)|^2}{\mathrm{Re}[f(\bar{\lambda}a)]} - \frac{\mathrm{Re}[f(\bar{\lambda}a)]}{|\lambda|^2} \\ &\qquad \le 2 + \frac{r^2}{\mathrm{Re}[f(\bar{\lambda}a)]} - \frac{\mathrm{Re}[f(\bar{\lambda}a)]}{|\lambda|^2} - \frac{|\lambda|^2}{\mathrm{Re}[f(\bar{\lambda}a)]} =: I. \end{aligned} \tag{4.14}$$

Further we have

$$\begin{aligned} I &= 2 - \frac{\mathrm{Re}[f(\bar{\lambda}a)]}{|\lambda|^2} - \frac{(|\lambda|^2 - r^2)}{\mathrm{Re}[f(\bar{\lambda}a)]} \\ &= 2 - 2\frac{\sqrt{|\lambda|^2 - r^2}}{|\lambda|} - \left[\frac{\sqrt{\mathrm{Re}[f(\bar{\lambda}a)]}}{|\lambda|} - \frac{\sqrt{|\lambda|^2 - r^2}}{\sqrt{\mathrm{Re}[f(\bar{\lambda}a)]}}\right]^2 \\ &\le 2\left(1 - \sqrt{1 - \left(\frac{r}{|\lambda|}\right)^2}\right). \end{aligned} \tag{4.15}$$

Hence by (4.14) and (4.15) we have

$$|f(a)|^2 \le \frac{(\mathrm{Re}[f(\bar{\lambda}a)])^2}{|\lambda|^2} + 2\left(1 - \sqrt{1 - \left(\frac{r}{|\lambda|}\right)^2}\right)\mathrm{Re}[f(\bar{\lambda}a)]. \tag{4.16}$$

Taking the supremum in $f \in D(1)$ and utilising Lumer's result, we deduce the desired inequality (4.13). ∎

Corollary 5. *Let $\gamma, \Gamma \in \mathbb{C}$ with $\mathrm{Re}(\Gamma\bar{\gamma}) > 0$. If $a \in \bar{U}(\gamma, \Gamma)$, then*

$$v^2(a) \le \left\langle \frac{\bar{\Gamma} + \bar{\gamma}}{|\Gamma + \gamma|}a, 1\right\rangle_s^2 + 2\left(\left|\frac{\gamma + \Gamma}{2}\right| - \sqrt{\mathrm{Re}(\Gamma\bar{\gamma})}\right)\left\langle \frac{\bar{\Gamma} + \bar{\gamma}}{|\Gamma + \gamma|}a, 1\right\rangle_s.$$

Remark 5. If $M > m \ge 0$ and $a \in \bar{U}(m, M)$, then

$$(0 \le) v^2(a) - \langle a, 1\rangle_s^2 \le (\sqrt{M} - \sqrt{m})^2\langle a, 1\rangle_s (\le (\sqrt{M} - \sqrt{m})^2\|a\|).$$

Finally, the following result can be stated as well.

Theorem 6. *Let $\lambda \in \mathbb{C}$ and $r > 0$ with $|\lambda| > r$. If $a \in \bar{\Delta}(\lambda, r)$, then*

$$v(a) \le (|\lambda| + \sqrt{|\lambda|^2 - r^2})\langle \frac{\bar{\lambda}}{r^2} a, 1\rangle_s + \frac{|\lambda|(|\lambda| + \sqrt{|\lambda|^2 - r^2})(|\lambda| - 2\sqrt{|\lambda|^2 - r^2})}{2r^2}. \tag{4.17}$$

Proof. From the proof of Theorem 3 above, we have

$$|f(a)|^2 + |\lambda|^2 \le 2\,\mathrm{Re}[f(\bar{\lambda}a)] + r^2$$

which is equivalent with

$$\begin{aligned} &|f(a)|^2 + (|\lambda| + \sqrt{|\lambda|^2 - r^2})^2 \\ &\le 2\,\mathrm{Re}[f(\bar{\lambda}a)] + r^2 - |\lambda|^2 + (|\lambda| - \sqrt{|\lambda|^2 - r^2})^2 \\ &= 2\,\mathrm{Re}[f(\bar{\lambda}a)] + |\lambda|^2 - 2|\lambda|\sqrt{|\lambda|^2 - r^2}. \end{aligned}$$

Taking the supremum in this formula over $f \in D(1)$ and utilising Lumer's representation theorem, we get:

$$v^2(a) + (|\lambda| - \sqrt{|\lambda|^2 - r^2})^2 \le 2\langle \bar{\lambda}a, 1\rangle_s + |\lambda|(|\lambda| - 2\sqrt{|\lambda|^2 - r^2}). \tag{4.18}$$

Since $r \neq 0$, then $|\lambda| - \sqrt{|\lambda|^2 - r^2} > 0$, giving

$$2(|\lambda| - \sqrt{|\lambda|^2 - r^2})v(a) \le v^2(a) + (|\lambda| - \sqrt{|\lambda|^2 - r^2})^2. \tag{4.19}$$

Now, utilising (4.18) and (4.19), we deduce

$$v(a) \le \frac{1}{|\lambda| - \sqrt{|\lambda|^2 - r^2}}\langle \bar{\lambda}a, 1\rangle_s + \frac{|\lambda|(|\lambda| - 2\sqrt{|\lambda|^2 - r^2})}{2(|\lambda| - \sqrt{|\lambda|^2 - r^2})},$$

which is clearly equivalent with the desired result (4.17). ∎

Remark 6. If $M > m \ge 0$ and $a \in \bar{U}(m, M)$, then

$$v(a) \le \frac{M + m}{(\sqrt{M} - \sqrt{m})^2}\left[\langle a, 1\rangle_s + \frac{1}{2}\left(\frac{m + M}{2} - 2\sqrt{mM}\right)\right].$$

In particular, if $a \in \bar{U}(0, \delta)$ with $\delta > 0$, then we have the following reverse inequality as well

$$(0 \le)\ v(a) - \langle a, 1\rangle_s \le \frac{1}{4}\delta.$$

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