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## INEQUALITIES FOR THE NUMERICAL RADIUS IN UNITAL NORMED ALGEBRAS

**Abstract.** In this paper, some inequalities between the numerical radius of an element from a unital normed algebra and certain semi-inner products involving that element and the unity are given.

### 1. Introduction

Let  $A$  be a unital normed algebra over the complex number field  $\mathbb{C}$  and let  $a \in A$ . Recall that the numerical radius of  $a$  is given by (see [2, p. 15])

$$(1.1) \quad v(a) = \sup\{|f(a)|, f \in A', \|f\| \leq 1 \text{ and } f(1) = 1\},$$

where  $A'$  denotes the dual space of  $A$ , i.e., the Banach space of all continuous linear functionals on  $A$ .

It is known that  $v(\cdot)$  is a norm on  $A$  that is equivalent to the given norm  $\|\cdot\|$ . More precisely, the following double inequality holds:

$$(1.2) \quad \frac{1}{e}\|a\| \leq v(a) \leq \|a\|$$

for any  $a \in A$ , where  $e = \exp(1)$ .

Following [2], we notice that this crucial result appears slightly hidden in Bohnenblust and Karlin [1, Theorem 1] together with the inequality  $\|x\| \leq e\Psi(x)$ , where  $\Psi(x) = \sup\{|\lambda|^{-1} \log \|e^{\lambda x}\|\}$  over  $\lambda$  complex,  $\lambda \neq 0$ , which occurs on page 219. A simpler proof was given by Lumer [5], though with the constant  $1/4$  in place of  $1/e$ . For a simple proof of (1.2) that borrows ideas from Lumer and from Glickfeld [6], see [2, p. 34].

A generalisation of (1.2) for powers has been obtained by M. J. Crabb [3] who proved that

$$(1.3) \quad \|a^n\| \leq n!\left(\frac{e}{n}\right)^n [v(a)]^n \quad n = 1, 2, \dots$$

for any  $a \in A$ .

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In this paper, some inequalities between the numerical radius of an element and the superior semi-inner product of that element and the unity in the normed algebra  $A$  are given via the celebrated representation result of Lumer from [5].

## 2. Some subsets in $A$

Let  $D(1) := \{f \in A' \mid \|f\| \leq 1 \text{ and } f(1) = 1\}$ . For  $\lambda \in \mathbb{C}$  and  $r > 0$ , we define the subset of  $A$  by

$$\bar{\Delta}(\lambda, r) := \{a \in A \mid |f(a) - \lambda| \leq r \text{ for each } f \in D(1)\}.$$

The following result holds.

**PROPOSITION 1.** *Let  $\lambda \in \mathbb{C}$  and  $r > 0$ . Then  $\bar{\Delta}(\lambda, r)$  is a closed convex subset of  $A$  and*

$$(2.1) \quad \bar{B}(\lambda, r) \subseteq \bar{\Delta}(\lambda, r),$$

where  $\bar{B}(\lambda, r) := \{a \in A \mid \|a - \lambda\| \leq r\}$ .

Now, for  $\gamma, \Gamma \in \mathbb{C}$ , define the set

$$\bar{U}(\gamma, \Gamma) := \{a \in A \mid \operatorname{Re}[(\Gamma - f(a))(\overline{f(a)} - \bar{\gamma})] \geq 0 \text{ for each } f \in D(1)\}.$$

The following representation result may be stated.

**PROPOSITION 2.** *For any  $\gamma, \Gamma \in \mathbb{C}$ ,  $\gamma \neq \Gamma$ , we have:*

$$(2.2) \quad \bar{U}(\gamma, \Gamma) = \bar{\Delta}\left(\frac{\gamma + \Gamma}{2}, \frac{1}{2}|\Gamma - \gamma|\right).$$

**Proof.** We observe that for any  $z \in \mathbb{C}$  we have the equivalence

$$\left|z - \frac{\gamma + \Gamma}{2}\right| \leq \frac{1}{2}|\Gamma - \gamma|$$

if and only if

$$\operatorname{Re}[(\Gamma - z)(\bar{z} - \bar{\gamma})] \geq 0.$$

This follows by the equality

$$\frac{1}{4}|\Gamma - \gamma|^2 - \left|z - \frac{\gamma + \Gamma}{2}\right|^2 = \operatorname{Re}[(\Gamma - z)(\bar{z} - \bar{\gamma})]$$

that holds for any  $z \in \mathbb{C}$ .

The equality (2.2) is thus a simple conclusion of this fact. ■

Making use of some obvious properties in  $\mathbb{C}$  and for continuous linear functionals, we can state the following corollary as well.

**COROLLARY 1.** For any  $\gamma, \Gamma \in \mathbb{C}$ , we have

$$(2.3) \quad \begin{aligned} \bar{U}(\gamma, \Gamma) &= \{a \in A \mid \operatorname{Re}[f(\Gamma - a)\overline{f(a - \gamma)}] \geq 0 \text{ for each } f \in D(1)\} \\ &= \{a \in A \mid (\operatorname{Re} \Gamma - \operatorname{Re} f(a))(\operatorname{Re} f(a) - \operatorname{Re} \gamma) \\ &\quad + (\operatorname{Im} \Gamma - \operatorname{Im} f(a))(\operatorname{Im} f(a) - \operatorname{Im} \gamma) \geq 0 \text{ for each } f \in D(1)\}. \end{aligned}$$

Now, if we assume that  $\operatorname{Re}(\Gamma) \geq \operatorname{Re}(\gamma)$  and  $\operatorname{Im}(\Gamma) \geq \operatorname{Im}(\gamma)$ , then we can define the following subset of  $A$ :

$$(2.4) \quad \begin{aligned} \bar{S}(\gamma, \Gamma) &:= \{a \in A \mid \operatorname{Re}(\Gamma) \geq \operatorname{Re} f(a) \geq \operatorname{Re}(\gamma) \text{ and} \\ &\quad \operatorname{Im}(\Gamma) \geq \operatorname{Im} f(a) \geq \operatorname{Im}(\gamma) \text{ for each } f \in D(1)\}. \end{aligned}$$

One can easily observe that  $\bar{S}(\gamma, \Gamma)$  is closed, convex and

$$(2.5) \quad \bar{S}(\gamma, \Gamma) \subseteq \bar{U}(\gamma, \Gamma).$$

### 3. Semi-inner products and Lumer's theorem

Let  $(X, \|\cdot\|)$  be a normed linear space over the real or complex number field  $\mathbb{K}$ . The mapping  $f : X \rightarrow \mathbb{R}$ ,  $f(x) = \frac{1}{2}\|x\|^2$  is obviously convex and then there exist the following limits:

$$\begin{aligned} \langle x, y \rangle_i &= \lim_{t \rightarrow 0^-} \frac{\|y + tx\|^2 - \|y\|^2}{2t}, \\ \langle x, y \rangle_s &= \lim_{t \rightarrow 0^+} \frac{\|y + tx\|^2 - \|y\|^2}{2t} \end{aligned}$$

for every two elements  $x, y \in X$ . The mapping  $\langle \cdot, \cdot \rangle_s$  ( $\langle \cdot, \cdot \rangle_i$ ) will be called the *superior semi-inner product* (the *interior semi-inner product*) associated to the norm  $\|\cdot\|$ .

We list some properties of these semi-inner products that can be easily derived from the definition (see for instance [4]). If  $p, q \in \{s, i\}$  and  $p \neq q$ , then:

- (i)  $\langle x, x \rangle_p = \|x\|^2$ ;  $\langle ix, x \rangle_p = \langle x, ix \rangle_p = 0$ ,  $x \in X$ ;
- (ii)  $\langle \lambda x, y \rangle_p = \lambda \langle x, y \rangle_p$ ;  $\langle x, \lambda y \rangle_p = \lambda \langle x, y \rangle_p$  for  $\lambda \geq 0$ ,  $x, y \in X$ ;
- (iii)  $\langle \lambda x, y \rangle_p = \lambda \langle x, y \rangle_q$ ;  $\langle x, \lambda y \rangle_p = \lambda \langle x, y \rangle_q$  for  $\lambda < 0$ ,  $x, y \in X$ ;
- (iv)  $\langle ix, y \rangle_p = -\langle x, iy \rangle_p$ ;  $\langle \alpha x, \beta y \rangle_p = \alpha \beta \langle x, y \rangle_p$  if  $\alpha \beta \geq 0$ ,  $x, y \in X$ ;
- (v)  $\langle -x, y \rangle_p = \langle x, -y \rangle_p = -\langle x, y \rangle_q$ ,  $x, y \in X$ ;
- (vi)  $|\langle x, y \rangle_p| \leq \|x\| \|y\|$ ,  $x, y \in X$ ;
- (vii)  $\langle x_1 + x_2, y \rangle_{s(i)} \leq (\geq) \langle x_1, y \rangle_{s(i)} + \langle x_2, y \rangle_{s(i)}$  for  $x_1, x_2, y \in X$ ;
- (ix)  $\langle \alpha x + y, x \rangle_p = \alpha \|x\|^2 + \langle y, x \rangle_p$ ,  $\alpha \in \mathbb{R}$ ,  $x, y \in X$ ;
- (x)  $|\langle y + z, x \rangle_p - \langle z, x \rangle_p| \leq \|y\| \|x\|$ ,  $x, y, z \in X$ ;
- (xi) the mapping  $\langle \cdot, x \rangle_p$  is continuous on  $(X, \|\cdot\|)$  for each  $x \in X$ .

The following result essentially due to Lumer [5] (see [2, p. 17]) can be stated.

**THEOREM 1.** Let  $A$  be a unital normed algebra over  $\mathbb{K}$  ( $\mathbb{K} = \mathbb{C}, \mathbb{R}$ ). For each  $a \in A$ ,

$$(3.1) \quad \max\{\operatorname{Re} \lambda | \lambda \in V(a)\} = \inf_{\alpha > 0} \frac{1}{\alpha} [\|1 + \alpha a\| - 1] = \lim_{\alpha \rightarrow 0^+} \frac{1}{\alpha} [\|1 + \alpha a\| - 1],$$

where  $V(a)$  is the numerical range of  $a$  (see for instance [2, p. 15]).

**REMARK 1.** In terms of semi-inner products, the above identity can be stated as:

$$(3.2) \quad \max\{\operatorname{Re} f(a) | f \in D(1)\} = \langle a, 1 \rangle_s.$$

The following result that provides more information may be stated.

**THEOREM 2.** For any  $a \in A$ , we have:

$$(3.3) \quad \langle a, 1 \rangle_{v,s} = \langle a, 1 \rangle_s,$$

where

$$\langle a, b \rangle_{v,s} := \lim_{t \rightarrow 0^+} \frac{v^2(b + ta) - v^2(b)}{2t}$$

is the superior semi-inner product associated with the numerical radius.

**Proof.** Since  $v(a) \leq \|a\|$ , we have:

$$\begin{aligned} \langle a, 1 \rangle_{v,s} &= \lim_{t \rightarrow 0^+} \frac{v^2(1 + ta) - v^2(1)}{2t} = \lim_{t \rightarrow 0^+} \frac{v^2(1 + ta) - 1}{2t} \\ &\leq \lim_{t \rightarrow 0^+} \frac{\|1 + ta\|^2 - 1}{2t} = \langle a, 1 \rangle_s. \end{aligned}$$

Now, let  $f \in D(1)$ . Then, for each  $\alpha > 0$ ,

$$f(a) = \frac{1}{\alpha} [f(1 + \alpha a) - f(1)] = \frac{1}{\alpha} [f(1 + \alpha a) - 1],$$

giving

$$\begin{aligned} \operatorname{Re} f(a) &= \frac{1}{\alpha} [\operatorname{Re} f(1 + \alpha a) - f(1)] \leq \frac{1}{\alpha} [|f(1 + \alpha a)| - 1] \\ &\leq \frac{1}{\alpha} [v(1 + \alpha a) - 1]. \end{aligned}$$

Taking the infimum over  $\alpha > 0$ , we deduce that

$$\begin{aligned} (3.4) \quad \operatorname{Re} f(a) &\leq \inf_{\alpha > 0} \left[ \frac{1}{\alpha} [v(1 + \alpha a) - 1] \right] = \lim_{\alpha \rightarrow 0^+} \left[ \frac{v^2(1 + \alpha a) - 1}{2\alpha} \right] \\ &= \lim_{\alpha \rightarrow 0^+} \frac{v(1 + \alpha a) - 1}{\alpha} = \langle a, 1 \rangle_{v,s}. \end{aligned}$$

If we now take the supremum over  $f \in D(1)$  in (3.4), we obtain:

$$\sup\{\operatorname{Re} f(a) | f \in D(1)\} \leq \langle a, 1 \rangle_{v,s}$$

which, by Lumer's identity, implies that  $\langle a, 1 \rangle_s \leq \langle a, 1 \rangle_{v,s}$ . ■

**COROLLARY 2.** *The following inequality holds*

$$(3.5) \quad |\langle a, 1 \rangle_s| \leq v(a) \quad (\leq \|a\|).$$

**Proof.** Schwarz's inequality for the norm  $v(\cdot)$  gives that

$$|\langle a, 1 \rangle_{v,s}| \leq v(a)v(1) = v(a),$$

and by (3.3), the inequality (3.5) is proved. ■

#### 4. Reverse inequalities for the numerical radius

Utilising the inequality (3.5) we observe that for any complex number  $\beta$  located in the closed disc centered in 0 and with radius 1 we have  $|\langle \beta a, 1 \rangle_s|$  as a lower bound for the numerical radius  $v(a)$ . Therefore, it is a natural question to ask how far these quantities are from each other under various assumptions for the element  $a$  in the unital normed algebra  $A$  and the scalar  $\beta$ . A number of results answering this question are incorporated in the following theorems.

**THEOREM 3.** *Let  $\lambda \in \mathbb{C} \setminus \{0\}$  and  $r > 0$ . If  $a \in \bar{\Delta}(\lambda, r)$ , then*

$$(4.1) \quad v(a) \leq \left\langle \frac{\bar{\lambda}}{|\lambda|} a, 1 \right\rangle_s + \frac{1}{2} \cdot \frac{r^2}{|\lambda|}.$$

**Proof.** Since  $a \in \bar{\Delta}(\lambda, 1)$ , we have  $|f(a) - \lambda|^2 \leq r^2$ , giving that

$$(4.2) \quad |f(a)|^2 + |\lambda|^2 \leq 2 \operatorname{Re}[f(\bar{\lambda}a)] + r^2$$

for each  $f \in D(1)$ .

Taking the supremum over  $f \in D(1)$  in (4.2) and utilising the representation (3.2), we deduce that

$$(4.3) \quad v^2(a) + |\lambda|^2 \leq 2 \langle \bar{\lambda}a, 1 \rangle_s + r^2$$

which is an inequality of interest in and of itself.

On the other hand, we have the elementary inequality

$$(4.4) \quad 2v(a)|\lambda| \leq v^2(a) + |\lambda|^2,$$

which, together with (4.3) implies the desired result (4.1). ■

**REMARK 2.** Notice that, by the inclusion (2.1), a sufficient condition for (4.1) to hold is that  $a \in \bar{B}(\lambda, r)$ .

**COROLLARY 3.** *Let  $\gamma, \Gamma \in \mathbb{C}$  with  $\Gamma \neq \pm\gamma$ . If  $a \in \bar{U}(\gamma, \Gamma)$ , then*

$$(4.5) \quad v(a) \leq \left\langle \frac{\bar{\Gamma} + \bar{\gamma}}{|\Gamma + \gamma|} a, 1 \right\rangle_s + \frac{1}{4} \cdot \frac{|\Gamma - \gamma|^2}{|\Gamma + \gamma|}.$$

**REMARK 3.** If  $M > m \geq 0$  and  $a \in \bar{U}(m, M)$ , then

$$(4.6) \quad (0 \leq) v(a) - \langle a, 1 \rangle_s \leq \frac{1}{4} \cdot \frac{(M - m)^2}{m + M}.$$

Observe that, due to the inclusion (2.5), a sufficient condition for (4.6) to hold is that  $M \geq \operatorname{Re} f(a), \operatorname{Im} f(a) \geq m$  for any  $f \in D(1)$ .

The following result may be stated as well.

**THEOREM 4.** *Let  $\lambda \in \mathbb{C}$  and  $r > 0$  with  $|\lambda| > r$ . If  $a \in \bar{\Delta}(\lambda, r)$ , then*

$$(4.7) \quad v(a) \leq \left\langle \frac{\bar{\lambda}}{\sqrt{|\lambda|^2 - r^2}} a, 1 \right\rangle_s$$

and, equivalently,

$$(4.8) \quad v^2(a) \leq \left\langle \frac{\bar{\lambda}}{|\lambda|} a, 1 \right\rangle_s^2 + \frac{r^2}{|\lambda|^2} \cdot v^2(a).$$

**Proof.** Since  $|\lambda| > r$ , we have  $\sqrt{|\lambda|^2 - r^2} > 0$ , hence the inequality (4.3) divided by this quantity becomes

$$(4.9) \quad \frac{v^2(a)}{\sqrt{|\lambda|^2 - r^2}} + \sqrt{|\lambda|^2 - r^2} \leq \frac{2}{\sqrt{|\lambda|^2 - r^2}} \langle \bar{\lambda} a, 1 \rangle_s.$$

On the other hand, we also have

$$2v(a) \leq \frac{v^2(a)}{\sqrt{|\lambda|^2 - r^2}} + \sqrt{|\lambda|^2 - r^2},$$

which, together with (4.9), gives

$$(4.10) \quad v(a) \leq \frac{1}{\sqrt{|\lambda|^2 - r^2}} \langle \bar{\lambda} a, 1 \rangle_s.$$

Taking the square in (4.10), we have

$$v^2(a)(|\lambda|^2 - r^2) \leq \langle \bar{\lambda} a, 1 \rangle_s^2,$$

which is clearly equivalent to (4.8). ■

**COROLLARY 4.** *Let  $\gamma, \Gamma \in \mathbb{C}$  with  $\operatorname{Re}(\Gamma\bar{\gamma}) > 0$ . If  $a \in \bar{U}(\gamma, \Gamma)$ , then,*

$$(4.11) \quad v(a) \leq \left\langle \frac{\bar{\Gamma} + \bar{\gamma}}{2\sqrt{\operatorname{Re}(\Gamma\bar{\gamma})}} a, 1 \right\rangle_s.$$

**REMARK 4.** If  $M \geq m > 0$  and  $a \in \bar{U}(m, M)$ , then

$$(4.12) \quad v(a) \leq \frac{M + m}{2\sqrt{mM}} \langle a, 1 \rangle_s,$$

or, equivalently,

$$(0 \leq) v(a) - \langle a, 1 \rangle_s \leq \frac{(\sqrt{M} - \sqrt{m})^2}{2\sqrt{mM}} \langle a, 1 \rangle_s \quad \left( \leq \frac{(\sqrt{M} - \sqrt{m})^2}{2\sqrt{mM}} \|a\| \right).$$

The following result may be stated as well.

**THEOREM 5.** Let  $\lambda \in \mathbb{C} \setminus \{0\}$  and  $r > 0$  with  $|\lambda| > r$ . If  $a \in \bar{\Delta}(\lambda, r)$ , then

$$(4.13) \quad v^2(a) \leq \left\langle \frac{\bar{\lambda}}{|\lambda|} a, 1 \right\rangle_s^2 + 2(|\lambda| - \sqrt{|\lambda|^2 - r^2}) \left\langle \frac{\bar{\lambda}}{|\lambda|} a, 1 \right\rangle_s.$$

**Proof.** Since (by (4.2))  $\operatorname{Re}[f(\bar{\lambda}a)] > 0$ , dividing by it in (4.2) gives:

$$\frac{|f(a)|^2}{\operatorname{Re}[f(\bar{\lambda}a)]} + \frac{|\lambda|^2}{\operatorname{Re}[f(\bar{\lambda}a)]} \leq 2 + \frac{r^2}{\operatorname{Re}[f(\bar{\lambda}a)]},$$

which is clearly equivalent to:

$$(4.14) \quad \frac{|f(a)|^2}{\operatorname{Re}[f(\bar{\lambda}a)]} - \frac{\operatorname{Re}[f(\bar{\lambda}a)]}{|\lambda|^2} \leq 2 + \frac{r^2}{\operatorname{Re}[f(\bar{\lambda}a)]} - \frac{\operatorname{Re}[f(\bar{\lambda}a)]}{|\lambda|^2} - \frac{|\lambda|^2}{\operatorname{Re}[f(\bar{\lambda}a)]} =: I.$$

Further we have

$$(4.15) \quad \begin{aligned} I &= 2 - \frac{\operatorname{Re}[f(\bar{\lambda}a)]}{|\lambda|^2} - \frac{(|\lambda|^2 - r^2)}{\operatorname{Re}[f(\bar{\lambda}a)]} \\ &= 2 - 2 \frac{\sqrt{|\lambda|^2 - r^2}}{|\lambda|} - \left[ \frac{\sqrt{\operatorname{Re}[f(\bar{\lambda}a)]}}{|\lambda|} - \frac{\sqrt{|\lambda|^2 - r^2}}{\sqrt{\operatorname{Re}[f(\bar{\lambda}a)]}} \right]^2 \\ &\leq 2 \left( 1 - \sqrt{1 - \left( \frac{r}{|\lambda|} \right)^2} \right). \end{aligned}$$

Hence by (4.14) and (4.15) we have

$$(4.16) \quad |f(a)|^2 \leq \frac{(\operatorname{Re}[f(\bar{\lambda}a)])^2}{|\lambda|^2} + 2 \left( 1 - \sqrt{1 - \left( \frac{r}{|\lambda|} \right)^2} \right) \operatorname{Re}[f(\bar{\lambda}a)].$$

Taking the supremum in  $f \in D(1)$  and utilising Lumer's result, we deduce the desired inequality (4.13). ■

**COROLLARY 5.** Let  $\gamma, \Gamma \in \mathbb{C}$  with  $\operatorname{Re}(\Gamma\bar{\gamma}) > 0$ . If  $a \in \bar{U}(\gamma, \Gamma)$ , then

$$v^2(a) \leq \left\langle \frac{\bar{\Gamma} + \bar{\gamma}}{|\Gamma + \gamma|} a, 1 \right\rangle_s^2 + 2 \left( \left| \frac{\gamma + \Gamma}{2} \right| - \sqrt{\operatorname{Re}(\Gamma\bar{\gamma})} \right) \left\langle \frac{\bar{\Gamma} + \bar{\gamma}}{|\Gamma + \gamma|} a, 1 \right\rangle_s.$$

**REMARK 5.** If  $M > m \geq 0$  and  $a \in \bar{U}(m, M)$ , then

$$(0 \leq) v^2(a) - \langle a, 1 \rangle_s^2 \leq (\sqrt{M} - \sqrt{m})^2 \langle a, 1 \rangle_s (\leq (\sqrt{M} - \sqrt{m})^2 \|a\|).$$

Finally, the following result can be stated as well.



**THEOREM 6.** Let  $\lambda \in \mathbb{C}$  and  $r > 0$  with  $|\lambda| > r$ . If  $a \in \bar{\Delta}(\lambda, r)$ , then

$$(4.17) \quad v(a) \leq (|\lambda| + \sqrt{|\lambda|^2 - r^2}) \langle \frac{\bar{\lambda}}{r^2} a, 1 \rangle_s + \frac{|\lambda|(|\lambda| + \sqrt{|\lambda|^2 - r^2})(|\lambda| - 2\sqrt{|\lambda|^2 - r^2})}{2r^2}.$$

**Proof.** From the proof of Theorem 3 above, we have

$$|f(a)|^2 + |\lambda|^2 \leq 2 \operatorname{Re}[f(\bar{\lambda}a)] + r^2$$

which is equivalent with

$$\begin{aligned} & |f(a)|^2 + (|\lambda| + \sqrt{|\lambda|^2 - r^2})^2 \\ & \leq 2 \operatorname{Re}[f(\bar{\lambda}a)] + r^2 - |\lambda|^2 + (|\lambda| - \sqrt{|\lambda|^2 - r^2})^2 \\ & = 2 \operatorname{Re}[f(\bar{\lambda}a)] + |\lambda|^2 - 2|\lambda|\sqrt{|\lambda|^2 - r^2}. \end{aligned}$$

Taking the supremum in this formula over  $f \in D(1)$  and utilising Lumer's representation theorem, we get:

$$(4.18) \quad v^2(a) + (|\lambda| - \sqrt{|\lambda|^2 - r^2})^2 \leq 2 \langle \bar{\lambda}a, 1 \rangle_s + |\lambda|(|\lambda| - 2\sqrt{|\lambda|^2 - r^2}).$$

Since  $r \neq 0$ , then  $|\lambda| - \sqrt{|\lambda|^2 - r^2} > 0$ , giving

$$(4.19) \quad 2(|\lambda| - \sqrt{|\lambda|^2 - r^2})v(a) \leq v^2(a) + (|\lambda| - \sqrt{|\lambda|^2 - r^2})^2.$$

Now, utilising (4.18) and (4.19), we deduce

$$v(a) \leq \frac{1}{|\lambda| - \sqrt{|\lambda|^2 - r^2}} \langle \bar{\lambda}a, 1 \rangle_s + \frac{|\lambda|(|\lambda| - 2\sqrt{|\lambda|^2 - r^2})}{2(|\lambda| - \sqrt{|\lambda|^2 - r^2})},$$

which is clearly equivalent with the desired result (4.17). ■

**REMARK 6.** If  $M > m \geq 0$  and  $a \in \bar{U}(m, M)$ , then

$$v(a) \leq \frac{M + m}{(\sqrt{M} - \sqrt{m})^2} \left[ \langle a, 1 \rangle_s + \frac{1}{2} \left( \frac{m + M}{2} - 2\sqrt{mM} \right) \right].$$

In particular, if  $a \in \bar{U}(0, \delta)$  with  $\delta > 0$ , then we have the following reverse inequality as well

$$(0 \leq) v(a) - \langle a, 1 \rangle_s \leq \frac{1}{4}\delta.$$

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