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INEQUALITIES FOR THE WEIGHTED MEAN OF r–PREINVEX FUNCTIONS ON AN INVEX SET

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Abstract. In this paper, the inequalities for the weighted mean of weakly r-preinvex functions on an invex set are established. As applications, inequalities between the two-parameter mean of weakly r-preinvex functions and extended mean values are given.

1. Introduction

The concepts of means are very important notions in mathematics. For example, some definitions of norms are often special means and have explicit geometric meanings [17], and have been applied in fields of heat conduction, chemistry [20], electrostatics [14] and medicine [4].

Recall the power mean $M_r(x,y;\lambda)$ of order r of positive numbers x,y which is defined by

$$M_r(x,y;\lambda)=\begin{cases}(\lambda x^r+(1-\lambda)y^r)^{\frac{1}{r}}, & \text{if } r\neq 0,\\ x^{\lambda}y^{1-\lambda}, & \text{if } r=0,\end{cases}$$

see [7].

In [15, 16], Qi gave the following weighted mean values of a positive function f defined on the interval between x and y with two parameters $p,q\in R$ and nonnegative weight w, which is not equivalent 0, by

$$M_{w,f}(p,q;x,y)$$
$$=\begin{cases}\left(\int_x^y w(t)f^p(t)dt\Big/\int_x^y w(t)f^q(t)dt\right)^{\frac{1}{(p-q)}}, & \text{if } (p-q)(x-y)\neq 0,\\ \exp\left(\int_x^y w(t)f^q(t)\ln f(t)dt\Big/\int_x^y w(t)f^q(t)dt\right), & \text{if } p=q, x\neq y.\end{cases}$$

and $M_{w,f}(p,q;x,x)=f(x)$. Let $x,y,s\in R$, and w and f be positive and integrable functions on the closed interval $[x,y]$. The weighted mean of order s of the function f on $[x,y]$ with the weight w is defined in [8] as

$$M^{[s]}(f,w;x,y)=\begin{cases}\left(\int_x^y w(t)f^s(t)dt\Big/\int_x^y w(t)dt\right)^{\frac{1}{s}}, & \text{if } s\neq 0,\\ \exp\left(\int_x^y w(t)\ln f(t)dt\Big/\int_x^y w(t)dt\right), & \text{if } s=0.\end{cases}$$

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In addition, $M^{[s]}(f,w;x,x) = f(x)$. By taking $s = p-q, p,q \in R$, and replacing $w(t)$ by $w(t)f^q(t)$ in $M^{[s]}(f,w;x,y)$, we have that $M^{[p-q]}(f,wf^q;x,y) = M_{w,f}(p,q;x,y)$. It is obvious that the weighted mean $M^{[s]}(f,w;x,y)$ is equivalent to the generalized weighted mean values $M_{w,f}(p,q;x,y)$. Taking $w(t) \equiv 1$, the mean $M_{w,f}(p,q;x,y)$ reduces to the two-parameter mean $M_{p,q}(f;a,b)$ of a positive function f on $[a,b]$ which is given in [18].

The classical Hermite-Hadamard inequality for convex functions states that if $f: [a,b] \to R$ is convex, then

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(t)dt \leqslant \frac{f(a)+f(b)}{2}.$$

In [19], Sun and Yang extend the following right hand side of Hermite-Hadamard inequality to the weighted mean of order s of a positive r-convex function on an interval $[a,b]$. They obtain more extensive results than the main results in [5, 12, 13, 18].

THEOREM 1. *Let $f(t)$ be a positive and continuous function on the interval $[x,y]$ with continuous derivative $f'(t)$ on $[x,y]$, let $w(t)$ be a positive and continuous function on the range J of the function $f(t)$, and let $h(t) = t$. Then if f is r-convex,*

$$M^{[s]}(f,w\circ f;x,y) \leqslant M^{[s]}(h,wh^{r-1};f(x),f(y)) \tag{1.1}$$

for any real number s, and if f is r-concave, the inequality is reversed.

In [9], Mohan et al. introduced the definitions of invex sets and preinvex functions. In [1, 2], Antczak investigated some interesting concept of r-invex and r-preinvex functions on an invex set and gave a new method to solve nonlinear mathematical programming problems. In [10], Noor gave some Hermite-Hadamard inequality for the preinvex and log-preinvex functions. Moreover, in [21], Wasim Ui-Haq and Javed Iqbal introduced the Hermite-Hadamard inequality for r-preinvex functions. Quite recently, in [6], Hwang and Dragomir investigated weakly r-preinvex functions on an invex set and established some Hermite-Hadamard's inequalities for a relation of two extended means.

Recall the following definitions of η-path on an invex set that were introduced by Antczak in [3]. Let $K \subset R^n$ be a nonempty set, $\eta : K \times K \to R^n$ and $u \in K$. Then the set K is said to be invex at u with respect to η, if

$$u + \lambda\eta(v,u) \in K$$

for every $v \in K$ and $\lambda \in [0,1]$. K is said to be an invex set with respect to η, if K is invex at each $u \in K$ with respect to the same function η. For $x \in K$, a closed and an open η-paths joining the points u and $x = u+\eta(v,u)$ are defined by the notation:

$$P_{ux} := \{u + \lambda\eta(v,u) : \lambda \in [0,1]\}$$

and

$$P_{ux}^0 := \{u + \lambda\eta(v,u) : \lambda \in (0,1)\},$$

respectively. We note that if $\eta(v,u) = v-u$, then the set $P_{ux} = P_{uv} = \{\lambda v + (1-\lambda)u : \lambda \in [0,1]\}$ is the line segment with the end points u and v.

Let $K \subset R^n$ be a nonempty invex set with respect to η. The class of r-preinvex functions with respect to η is introduced via power means given by Antczak in [1]. A function $f : K \to R^+$ is said to be r-preinvex with respect to η, if there is a vector-valued function $\eta : K \times K \to R^n$ such that

$$f(u+\lambda\eta(v,u)) \leqslant \begin{cases} (\lambda f(v)^r + (1-\lambda)f(u)^r)^{\frac{1}{r}}, & \text{if } r \neq 0, \\ f(v)^{\lambda} f(u)^{1-\lambda}, & \text{if } r = 0. \end{cases}$$

for every $v,u \in K$ and $\lambda \in [0,1]$. We note that 0-preinvex functions are logarithmic preinvex and 1-preinvex functions are preinvex functions. It is obvious that if f is r-preinvex, then f^r is a preinvex function for positive r.

A more natural idea of weakly r-preinvex with respect to η is investigated via power means given by Hwang and Dragomir, see [6]. Let $K \subset R^n$ be a nonempty invex set with respect to η. A function $f : K \to R^+$ is said to be weakly r-preinvex with respect to η, if there is a vector-valued function $\eta : K \times K \to R^n$ such that

$$f(u+\lambda\eta(v,u)) \leqslant M_r(f(u+\eta(v,u)), f(u); \lambda)$$

for every $v,u \in K$ and $\lambda \in [0,1]$. It is clear that if f is weakly r-preinvex, then f^r is weakly preinvex for positive r, if f is weakly 0-preinvex, then $log \circ f$ is weakly preinvex, and if f is weakly 1-preinvex, then f is weakly preinvex.

Let $K \subset R^n$ be a nonempty invex set with respect to $\eta : K \times K \to R^n$. A function $f : K \to R$ is invex with respect to the same η. If the inequality

$$f(u+\eta(v,u)) \leqslant f(v)$$

holds for any $u,v \in K$, we say that the function f satisfies the Condition D, see [22]. We note that, if f satisfies the Condition D, f is also an r-preinvex function. In [6], applying the definition of weakly r-preinvex function, Hwang and Dragomire extend the Hermite-Hadamard inequality that involves a mean of two-parameters for weakly r-preinvex functions on an invex set.

In this paper, we shall establish the Hermite-Hadamard inequality for the weighted mean of weakly r-preinvex functions on an invex set. As applications, some inequalities between the two-parameter mean of weakly r-preinvex functions and extended mean values are given. The results are not only to generalize the Hermite-Hadamard inequality given in [10, 21], but also to establish the weighted type inequality, given in [15, 19], for weakly r-preinvex functions on an invex set.

2. Preliminary definition and lemma

In order to obtain our results, we shall introduce the following new definition related to a weighted mean for two-parameters on an invex set.

DEFINITION 1. Let $K \subset R^n$ be a nonempty invex set with respect to a vector-valued function $\eta : K \times K \to R^n$ and let $f, w : K \to R^+$ be integrable on the η-path P_{ux} for $x = u + \eta(v,u)$ where $v, u \in K$, $\lambda \in [0,1]$. Set $y(\lambda) = u + \lambda\eta(v,u)$. We define the weighted mean of the function $f(u+\lambda\eta(v,u))$ on $[0,1]$ with respect to λ by

$$M_{p,q}(f,w;u,u+\eta(v,u)) = \begin{cases} \left(\frac{\int_0^1 w(y(\lambda))f^p(y(\lambda))d\lambda}{\int_0^1 w(y(\lambda))f^q(y(\lambda))d\lambda}\right)^{\frac{1}{(p-q)}}, & \text{if } p \neq q, \\ \exp\left(\frac{\int_0^1 w(y(\lambda))f^q(y(\lambda))\ln f(y(\lambda))d\lambda}{\int_0^1 w(y(\lambda))f^q(y(\lambda))d\lambda}\right), & \text{if } p = q. \end{cases}$$

In the special case, $q = 0$, $M_{p,0}(f,w;u,u+\eta(v,u)) = M^{[p]}(f,w;u,u+\eta(v,u))$ is the weighted mean of order p of the function f on $[u, u+\eta(v,u)]$ with the weight w.

Let $K \subset R^n$ be a nonempty invex set with respect to $\eta : K \times K \to R^n$ and $v, u \in K$, $\lambda \in [0,1]$. We say that the function η satisfies the Condition C, see [9, 11], if the following two identities

(i) $\eta(u, u+\lambda\eta(v,u)) = -\lambda\eta(v,u)$

and

(ii) $\eta(v, u+\lambda\eta(v,u)) = (1-\lambda)\eta(v,u)$

hold.

In [6], Hwang and Dragomir have given the following lemma for weakly r-preinvex functions.

LEMMA 1. *Let $K \subset R^n$ be a nonempty invex set with respect to $\eta : K \times K \to R^n$ and suppose that η satisfies Condition C. Let $u \in K$ and let $f : P_{ux} \to R$ for every $v \in K$, $\lambda \in [0,1]$ and $x = u + \eta(v,u) \in K$. Suppose that f is continuous on P_{ux} and is twice-differentiable on P_{ux}^0 and $r \geqslant 0$. Then f is a weakly r-preinvex function with respect to η if and only if*

$$rf^{r-2}(u)\{(r-1)[\eta(v,u)^T\nabla f(u)]^2 + f(u)\eta(v,u)^T\nabla^2 f(u)\eta(v,u)\} \geqslant 0$$

for $r > 0$,

$$\{\eta(v,u)^T\nabla^2 f(u)\eta(v,u)f(u) - [\eta(v,u)^T\nabla f(u)]^2\}/f^2(u) \geqslant 0$$

for $r = 0$.

3. Main results

In this section, we assume that $K \subset R^n$ be a nonempty invex set with respect to a vector-valued function $\eta : K \times K \to R^n$. Applying the definition and lemma in section 2, we have the following theorem which is our main result.

THEOREM 2. *Let f be a weakly r-preinvex function on an invex set K with $r \geqslant 0$. Assume that f be a positive and continuous function on P_{ax} and twice-differentiable on P_{ax}^0 for every $a,b \in K$, $\lambda \in [0,1]$ and $a < x = a+\eta(b,a)$, and let η satisfy Condition C. Let m and M be the minimum and maximum of f on P_{ax}, respectively. Further, let w,h be positive and continuous on $[m,M]$ with $h(x)=x$, and let $g_1,g_2 : (0,\infty) \to R$ and suppose that g_2 is positive and integrable on $[m,M]$ and the ratio g_1/g_2 is integrable on $[m,M]$. If g_1/g_2 is increasing on $[m,M]$, then*

$$\frac{\int_0^1 w(f(a+\lambda\eta(b,a)))g_1(f(a+\lambda\eta(b,a)))d\lambda}{\int_0^1 w(f(a+\lambda\eta(b,a)))g_2(f(a+\lambda\eta(b,a)))d\lambda} \tag{3.1}$$

$$\leqslant \frac{\int_{f(a)}^{f(a+\eta(b,a))} w(x)h^{r-1}(x)g_1(h(x))dx}{\int_{f(a)}^{f(a+\eta(b,a))} w(x)h^{r-1}(x)g_2(h(x))dx}$$

for $f(a) \neq f(a+\eta(b,a))$; the right-hand side of (3.1) is defined by $g_1(f(a))/g_2(f(a))$ for $f(a) = f(a+\eta(b,a))$. If g_1/g_2 is decreasing, then the inequality (3.1) is reversed.

Proof. Let $\phi(\lambda) = f^r(a+\lambda\eta(b,a))$ for $r \neq 0$ and $\phi(\lambda) = \ln f(a+\lambda\eta(b,a))$ for $r=0$. We give only the proof in the case of $r>0$ and g_1/g_2 increasing. The proof in the other case is analogous. For convenience, let $\psi(\lambda) = f(a+\lambda\eta(b,a))$. Since f is weakly r-preinvex with respect to η, Lemma 1 gives that

$$\phi''(\lambda) = rf^{(r-2)}(a)\{(r-1)[\eta(b,a)^T\nabla f(a)]^2 + f(a)\eta(b,a)^T\nabla^2 f(a)\eta(b,a)\}$$

is positive.

When $f(a) \neq f(a+\eta(b,a))$, it is easy to see that inequality (3.1) is equivalent to

$$\frac{\int_0^1 w(\psi(\lambda))g_1(\psi(\lambda))d\lambda}{\int_0^1 w(\psi(\lambda))g_2(\psi(\lambda))d\lambda} \leqslant \frac{\int_0^1 w(\psi(\lambda))\psi^{r-1}(\lambda)g_1(\psi(\lambda))\psi'(\lambda)d\lambda}{\int_0^1 w(\psi(\lambda))\psi^{r-1}(\lambda)g_2(\psi(\lambda))\psi'(\lambda)d\lambda}. \tag{3.2}$$

Consider

$$\begin{aligned} I &= \int_0^1 w(\psi(\lambda))g_1(\psi(\lambda))d\lambda \int_0^1 w(\psi(\mu))\psi^{r-1}(\mu)g_2(\psi(\mu))\psi'(\mu)d\mu \qquad (3.3)\\ &\quad - \int_0^1 w(\psi(\lambda))g_2(\psi(\lambda))d\lambda \int_0^1 w(\psi(\mu))\psi^{r-1}(\mu)g_1(\psi(\mu))\psi'(\mu)d\mu \\ &= \int_0^1\int_0^1 w(\psi(\lambda))w(\psi(\mu))g_2(\psi(\lambda))g_2(\psi(\mu))\psi^{r-1}(\mu)\psi'(\mu) \\ &\quad \times \Big[\frac{g_1(\psi(\lambda))}{g_2(\psi(\lambda))} - \frac{g_1(\psi(\mu))}{g_2(\psi(\mu))}\Big]d\lambda d\mu. \end{aligned}$$

Replacing λ and μ by each other in (3.3) and adding the resulting equations we get

$$I = \frac{1}{2r}\int_0^1\int_0^1 w(\psi(\lambda))w(\psi(\mu))g_2(\psi(\lambda))g_2(\psi(\mu))\left[(\psi^r(\mu))' - (\psi^r(\lambda))'\right] \times \left[\frac{g_1(\psi(\lambda))}{g_2(\psi(\lambda))} - \frac{g_1(\psi(\mu))}{g_2(\psi(\mu))}\right]d\lambda d\mu. \tag{3.4}$$

If the derivative $\phi'(\lambda) = (\psi^r(\lambda))' \geqslant 0$ for all $\lambda \in (0,1)$, from $\phi''(\lambda) = (\psi^r(\lambda))'' \geqslant 0$, we always have

$$\frac{1}{r}\left[(\psi^r(\mu))' - (\psi^r(\lambda))'\right]\left[\frac{g_1(\psi(\lambda))}{g_2(\psi(\lambda))} - \frac{g_1(\psi(\mu))}{g_2(\psi(\mu))}\right] \leqslant 0.$$

From (3.4), we get $I \leqslant 0$. This implies that the inequality (3.2) holds and then (3.1) holds. If the derivative $\phi'(\lambda) = (\psi^r(\lambda))' \leqslant 0$ for all $\lambda \in (0,1)$, a similar argument gives $I \geqslant 0$ and again the inequality (3.1) holds.

Now suppose that $\phi'(\lambda) = (\psi^r(\lambda))'$ changes sign and $\phi(0) < \phi(1)$. Then $\psi^r(0) \leqslant \psi^r(1)$ and there exists a point $\alpha \in (0,1)$ such that $\phi'(\alpha) = (\psi^r(\alpha))' = 0$ and $(\psi^r(\lambda))' \leqslant 0$ for all $\lambda \in [0,\alpha]$ and $(\psi^r(\lambda))' \geqslant 0$ for all $\lambda \in [\alpha,1]$. Therefore, there exists a point $\beta \in (\alpha,1)$ such that $\psi(0) = \psi(\beta)$. Thus

$$\int_0^\beta w(\psi(\lambda))\psi^{r-1}(\lambda)g_1(\psi(\lambda))\psi'(\lambda)d\lambda$$

$$= \int_{\psi(0)}^{\psi(\alpha)} w(\psi(\lambda))x^{r-1}g_1(x)dx + \int_{\psi(\alpha)}^{\psi(\beta)} w(\psi(\lambda))x^{r-1}g_1(x)dx = 0,$$

and, similarly,

$$\int_0^\beta w(\psi(\lambda))\psi^{r-1}(\lambda)g_2(\psi(\lambda))\psi'(\lambda)d\lambda = 0.$$

Consequently, the inequality (3.1) is equivalent to

$$\frac{\int_0^1 w(\psi(\lambda))g_1(\psi(\lambda))d\lambda}{\int_0^1 w(\psi(\lambda))g_2(\psi(\lambda))d\lambda} \leqslant \frac{\int_\beta^1 w(\psi(\lambda))\psi^{r-1}(\lambda)g_1(\psi(\lambda))\psi'(\lambda)d\lambda}{\int_\beta^1 w(\psi(\lambda))\psi^{r-1}(\lambda)g_2(\psi(\lambda))\psi'(\lambda)d\lambda}. \tag{3.5}$$

Consider

$$\begin{aligned} I_2 &= \int_0^1 w(\psi(\lambda))g_1(\psi(\lambda))d\lambda \int_\beta^1 w(\psi(\mu))\psi^{r-1}(\mu)g_2(\psi(\mu))\psi'(\mu)d\mu \\ &\quad - \int_0^1 w(\psi(\lambda))g_2(\psi(\lambda))d\lambda \int_\beta^1 w(\psi(\mu))\psi^{r-1}(\mu)g_1(\psi(\mu))\psi'(\mu)d\mu \\ &= \frac{1}{r}\int_0^1\int_\beta^1 w(\psi(\lambda))w(\psi(\mu))g_2(\psi(\lambda))g_2(\psi(\mu))\psi^{r-1}(\mu)\psi'(\mu) \\ &\quad \times \left[\frac{g_1(\psi(\lambda))}{g_2(\psi(\lambda))} - \frac{g_1(\psi(\mu))}{g_2(\psi(\mu))}\right]d\lambda d\mu. \end{aligned}$$

Split the double integral I_2 into two parts

$$I_{21} = \frac{1}{r}\int_0^{\beta}\int_{\beta}^{1} w(\psi(\lambda))w(\psi(\mu))g_2(\psi(\lambda))g_2(\psi(\mu))\psi^{r-1}(\mu)\psi'(\mu) \times \Big[\frac{g_1(\psi(\lambda))}{g_2(\psi(\lambda))} - \frac{g_1(\psi(\mu))}{g_2(\psi(\mu))}\Big] d\lambda d\mu,$$

and

$$I_{22} = \frac{1}{r}\int_{\beta}^{1}\int_{\beta}^{1} w(\psi(\lambda))w(\psi(\mu))g_2(\psi(\lambda))g_2(\psi(\mu))\psi^{r-1}(\mu)\psi'(\mu) \times \Big[\frac{g_1(\psi(\lambda))}{g_2(\psi(\lambda))} - \frac{g_1(\psi(\mu))}{g_2(\psi(\mu))}\Big] d\lambda d\mu.$$

When $(\lambda,\mu) \in [0,\beta]\times[\beta,1]$, we have $\lambda \leqslant \mu$ and $(\psi^r(\mu))' = r\psi^{r-1}(\mu)\psi'(\mu) \geqslant 0$ for all $\mu \in (\beta,1)$. Thus $\psi'(\mu) \geqslant 0$ for all $\mu \in (\beta,1)$ and

$$\frac{g_1(\psi(\lambda))}{g_2(\psi(\lambda))} \leqslant \frac{g_1(\psi(\beta))}{g_2(\psi(\beta))} \leqslant \frac{g_1(\psi(\mu))}{g_2(\psi(\mu))}.$$

Therefore we have that $I_{21} \leqslant 0$. By the result proved in case of $\phi'(\lambda) = (\psi^r(\lambda))' \geqslant 0$, we can get $I_{22} \leqslant 0$. Therefore, $I_2 = I_{21} + I_{22} \leqslant 0$. It follows that (3.5) and also (3.1) holds. Finally, if the sign of the derivative $\phi'(\lambda) = (\psi^r(\lambda))'$ changes and $\psi(0) \geqslant \psi(1)$ a similar proof again shows that (3.1) holds.

When $f(a) = f(a+\eta(b,a))$, $\psi(0) = \psi(1)$, and so $\phi(0) = \phi(1)$. Since $\phi'' = (\psi^r(\lambda))'' \geqslant 0$, we see that $\phi' = (\psi^r(\lambda))'$ is continuous and increasing for $\lambda \in (0,1)$. There exists a point $\alpha \in (0,1)$ such that $(\psi^r(\alpha))' = 0$ and $(\psi^r(\lambda))' \leqslant 0$ for all $\lambda \in (0,\alpha)$, and $(\psi^r(\lambda))' \geqslant 0$ for all $\lambda \in (\alpha,1)$. Hence

$$\frac{g_1(\psi(\lambda))}{g_2(\psi(\lambda))} \leqslant \frac{g_1(\psi(1))}{g_2(\psi(1))},$$

for all $\lambda \in (0,1)$. It follows that

$$\int_0^1 w(\psi(\lambda))g_1(\psi(\lambda))d\lambda \leqslant \frac{g_1(\psi(1))}{g_2(\psi(1))}\int_0^1 w(\psi(\lambda))g_2(\psi(\lambda))d\lambda.$$

Therefore, the inequality (3.1) is valid. This completes the proof of Theorem 2. □

If we take $g_1(x) = x^p$, $g_2(x) = x^q$ for real numbers p,q in Theorem 2, we get the following weighted type of the Hermite-Hadamard inequality for weakly r-preinvex functions on an invex set.

COROLLARY 1. *Let f be a weakly r-preinvex function on an invex set K with $r \geqslant 0$. Assume that f be a positive and continuous function on P_{ax} and twice-differentiable on P_{ax}^0 for every $a,b \in K$, $\lambda \in [0,1]$ and $a < x = a+\eta(b,a)$, and let η satisfy Condition C. Let m and M be the minimum and maximum of f on P_{ax}, respectively. Further,* *let w,h be positive and continuous on $[m,M]$ with $h(x)=x$, and let p and q be real number. If $p-q\geqslant 0$, then*

$$M_{p,q}(f,w\circ f;a,a+\eta(b,a))\leqslant M_{p,q}(h,wh^{r-1};f(a),f(a+\eta(b,a))) \tag{3.6}$$

for $f(a)\neq f(a+\eta(b,a))$; the right-hand side of (3.6) is defined by $f(a)^{p-q}$for $f(a)=f(a+\eta(b,a))$. If $p-q\leqslant 0$, then the inequality (3.6) is reversed.

Obviously, the following corollary holds if we take $q=0$ in corollary 1.

COROLLARY 2. *Suppose that the assumptions in corollary 1 hold. If the real number $p\geqslant 0$, then*

$$M^{[p]}(f,w\circ f;a,a+\eta(b,a))\leqslant M^{[p]}(h,wh^{r-1};f(a),f(a+\eta(b,a))) \tag{3.7}$$

for $f(a)\neq f(a+\eta(b,a))$; the right-hand side of (3.7) is defined by $f(a)^p$ for $f(a)=f(a+\eta(b,a))$. If $p\leqslant 0$, then the inequality (3.7) is reversed.

REMARK 1. Taking $p=1$ in (3.7), gives

$$\frac{\int_a^{a+\eta(b,a)}w(f(x))f(x)dx}{\int_a^{a+\eta(b,a)}w(f(x))dx}\leqslant\frac{\int_{f(a)}^{f(a+\eta(b,a))}w(x)x^r dx}{\int_{f(a)}^{f(a+\eta(b,a))}w(x)x^{r-1}dx}. \tag{3.8}$$

Taking $w\equiv 1$, the inequality (3.8) reduces to the inequality given by Ui-Haq and Iqbal in [21]. Further, taking $r=1$ or $r=0$, the inequality (3.8) reduces to the inequality given by Noor in [10]. So the inequality (3.1) is a greater generalization of the Hermite-Hadamard inequality for weakly r-preinvex functions on an invex set.

REMARK 2. When $\eta(b,a)=b-a$ in Corollary 1, it is clear that the set K is convex, Condition C is satisfied and the function f is r-convex. If $p-q\geqslant 0$, we have

$$M_{p,q}(f,w\circ f;a,b))\leqslant M_{p,q}(h,wh^{r-1};f(a),f(b)) \tag{3.9}$$

for $f(a)\neq f(b)$; the right-hand side of (3.9) is defined by $f(a)^p$ for $f(a)=f(b)$, while if $p-q\leqslant 0$ the inequality (3.9) is reversed. We note that the (3.9) is equivalent to the following inequality

$$M_{w\circ f,f}(p,q;a,b))\leqslant M_{wh^{r-1},h}(p,q;f(a),f(b)).$$

Taking $q=0$ in (3.9), the inequality (3.9) reduces to (1.1) in Theorem 1. So inequality (3.1) is also more extensive than the results in [5, 12, 13, 18]

The following corollary holds if we take $w\equiv 1$ in Theorem 2.

COROLLARY 3. *Suppose that the assumptions in theorem 2 hold and $w\equiv 1$. If g_1/g_2 is increasing on $[m,M]$, then*

$$\frac{\int_0^1 g_1(f(a+\lambda\eta(b,a)))d\lambda}{\int_0^1 g_2(f(a+\lambda\eta(b,a)))d\lambda}\leqslant\frac{\int_{f(a)}^{f(a+\eta(b,a))}x^{r-1}g_1(x)dx}{\int_{f(a)}^{f(a+\eta(b,a))}x^{r-1}g_2(x)dx} \tag{3.10}$$

for $f(a) \neq f(a+\eta(b,a))$*, the right-hand side of* (3.10) *is defined by* $g_1(f(a))/g_2(f(a))$ *for* $f(a) = f(a+\eta(b,a))$*, while if* g_1/g_2 *is decreasing, the inequality* (3.10) *is reversed.*

REMARK 3. The inequality (3.10) has been given in [6]. It is clear that inequality (3.1) is a weighted type of inequality (3.10).

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