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Multiplicative inequalities for weighted arithmetic and harmonic operator means

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Abstract. In this paper we establish some multiplicative inequalities for weighted arithmetic and harmonic operator means under various assumption for the positive invertible operators A , B . Some applications when A , B are bounded above and below by positive constants are given as well.

1 Introduction

Throughout this paper A , B are positive invertible operators on a complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$. We use the following notations for operators

$$A\nabla_{\nu}B := (1 - \nu)A + \nu B,$$

the *weighted operator arithmetic mean*,

$$A\sharp_{\nu}B := A^{1/2} \left(A^{-1/2} B A^{-1/2} \right)^{\nu} A^{1/2},$$

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the *weighted operator geometric mean* and

$$A!_{\nu}B := \left((1 - \nu) A^{-1} + \nu B^{-1} \right)^{-1}$$

the *weighted operator harmonic mean*, where $\nu \in [0, 1]$.

When $\nu = \frac{1}{2}$, we write $A\nabla B$, $A\sharp B$ and $A!B$ for brevity, respectively.

The following fundamental inequality between the weighted arithmetic, geometric and harmonic operator means holds

$$A!_{\nu}B \leq A\sharp_{\nu}B \leq A\nabla_{\nu}B \quad (1)$$

for any $\nu \in [0, 1]$.

For various recent inequalities between these means we recommend the recent papers [3]-[6], [8]-[12] and the references therein.

The following additive double inequality has been obtained in the recent paper [7]:

$$\nu(1 - \nu) \frac{(b - a)^2}{\max\{b, a\}} \leq A_{\nu}(a, b) - H_{\nu}(a, b) \leq \nu(1 - \nu) \frac{(b - a)^2}{\min\{b, a\}}, \quad (2)$$

for any $a, b > 0$ and $\nu \in [0, 1]$, where $A_{\nu}(a, b)$ and $H_{\nu}(a, b)$ are the scalar weighted arithmetic mean and harmonic mean, respectively, namely

$$A_{\nu}(a, b) := (1 - \nu)a + \nu b \text{ and } H_{\nu}(a, b) := \frac{ab}{(1 - \nu)b + \nu a}.$$

In particular,

$$\frac{1}{4} \frac{(b - a)^2}{\max\{b, a\}} \leq A(a, b) - H(a, b) \leq \frac{1}{4} \frac{(b - a)^2}{\min\{b, a\}}, \quad (3)$$

where

$$A(a, b) := \frac{a + b}{2} \text{ and } H(a, b) := \frac{2ab}{b + a}.$$

We consider the *Kantorovich's constant* defined by

$$K(h) := \frac{(h + 1)^2}{4h}, \quad h > 0. \quad (4)$$

The function K is decreasing on $(0, 1)$ and increasing on $[1, \infty)$, $K(h) \geq 1$ for any $h > 0$ and $K(h) = K(\frac{1}{h})$ for any $h > 0$.

Observe that for any $h > 0$

$$\mathcal{K}(h) - 1 = \frac{(h-1)^2}{4h} = \mathcal{K}\left(\frac{1}{h}\right) - 1.$$

Observe that

$$\mathcal{K}\left(\frac{b}{a}\right) - 1 = \frac{(b-a)^2}{4ab} \text{ for } a, b > 0.$$

Since, obviously

$$ab = \min\{a, b\} \max\{a, b\} \text{ for } a, b > 0,$$

then we have the following version of (2):

$$\begin{aligned} 4\nu(1-\nu) \min\{a, b\} \left[\mathcal{K}\left(\frac{b}{a}\right) - 1 \right] &\leq A_\nu(a, b) - H_\nu(a, b) \\ &\leq 4\nu(1-\nu) \max\{a, b\} \left[\mathcal{K}\left(\frac{b}{a}\right) - 1 \right]. \end{aligned} \quad (5)$$

for any $a, b > 0$ and $\nu \in [0, 1]$.

For positive invertible operators A, B we define

$$A\nabla_\infty B := \frac{1}{2}(A+B) + \frac{1}{2}A^{1/2} \left| A^{-1/2}(B-A)A^{-1/2} \right| A^{1/2}$$

and

$$A\nabla_{-\infty} B := \frac{1}{2}(A+B) - \frac{1}{2}A^{1/2} \left| A^{-1/2}(B-A)A^{-1/2} \right| A^{1/2}.$$

If we consider the continuous functions $f_\infty, f_{-\infty} : [0, \infty) \rightarrow [0, \infty)$ defined by

$$f_\infty(x) = \max\{x, 1\} = \frac{1}{2}(x+1) + \frac{1}{2}|x-1|$$

and

$$f_{-\infty}(x) = \max\{x, 1\} = \frac{1}{2}(x+1) - \frac{1}{2}|x-1|,$$

then, obviously, we have

$$A\nabla_{\pm\infty} B = A^{1/2} f_{\pm\infty} \left(A^{-1/2} B A^{-1/2} \right) A^{1/2}.$$

If A and B are commutative, then

$$A\nabla_{\pm\infty} B = \frac{1}{2}(A+B) \pm \frac{1}{2}|B-A| = B\nabla_{\pm\infty} A.$$

The following additive inequality between the weighted arithmetic and harmonic operator means holds [7]:

Theorem 1 Let A, B be positive invertible operators and $M > m > 0$ such that the condition

$$mA \leq B \leq MA \quad (6)$$

holds. Then we have

$$\begin{aligned} 4\nu(1-\nu)g(m, M)A\nabla_{-\infty}B &\leq A\nabla_{\nu}B - A!_{\nu}B \\ &\leq 4\nu(1-\nu)G(m, M)A\nabla_{\infty}B, \end{aligned} \quad (7)$$

where

$$g(m, M) := \begin{cases} K(M) - 1 & \text{if } M < 1, \\ 0 & \text{if } m \leq 1 \leq M, \\ K(m) - 1 & \text{if } 1 < m \end{cases}$$

and

$$G(m, M) := \begin{cases} K(m) - 1 & \text{if } M < 1, \\ \max\{K(m), K(M)\} - 1 & \text{if } m \leq 1 \leq M, \\ K(M) - 1 & \text{if } 1 < m. \end{cases}$$

In particular,

$$g(m, M)A\nabla_{-\infty}B \leq A\nabla B - A!B \leq G(m, M)A\nabla_{\infty}B. \quad (8)$$

Motivated by the above facts, we establish in this paper some multiplicative inequalities for weighted arithmetic and harmonic operator means under various assumption for the positive invertible operators A, B . Some applications when A, B are bounded above and below by positive constants are given as well.

2 Multiplicative inequalities

The following result is of interest in itself:

Lemma 1 For any $a, b > 0$ and $\nu \in [0, 1]$ we have

$$\nu(1-\nu) \left(1 - \frac{\min\{a, b\}}{\max\{a, b\}}\right)^2 \leq \frac{A_{\nu}(a, b)}{H_{\nu}(a, b)} - 1 \leq \nu(1-\nu) \left(\frac{\max\{a, b\}}{\min\{a, b\}} - 1\right)^2. \quad (9)$$

In particular,

$$\frac{1}{4} \left(1 - \frac{\min\{a, b\}}{\max\{a, b\}}\right)^2 \leq \frac{A(a, b)}{H(a, b)} - 1 \leq \frac{1}{4} \left(\frac{\max\{a, b\}}{\min\{a, b\}} - 1\right)^2. \quad (10)$$

Proof. We have for any $a, b > 0$ and $\nu \in [0, 1]$ that

$$\begin{aligned} \frac{A_\nu(a, b)}{H_\nu(a, b)} &= \frac{[(1-\nu)a + \nu b][(1-\nu)b + \nu a]}{ab} \\ &= \frac{(1-\nu)^2 ab + \nu(1-\nu)b^2 + \nu(1-\nu)a^2 + \nu^2 ab}{ab} \\ &= \frac{\nu(1-\nu)(b^2 + a^2) + (1-2\nu + 2\nu^2)ab}{ab}, \end{aligned}$$

which is equivalent with

$$\frac{A_\nu(a, b)}{H_\nu(a, b)} - 1 = \nu(1-\nu) \frac{(b-a)^2}{ab} \quad (11)$$

for any $a, b > 0$ and $\nu \in [0, 1]$.

Since $\min^2\{a, b\} \leq ab \leq \max^2\{a, b\}$ hence

$$\begin{aligned} \nu(1-\nu) \frac{(b-a)^2}{ab} &\leq \nu(1-\nu) \frac{(b-a)^2}{\min^2\{a, b\}} \\ &= \nu(1-\nu) \left(\frac{\max\{a, b\}}{\min\{a, b\}} - 1 \right)^2 \end{aligned}$$

and

$$\begin{aligned} \nu(1-\nu) \frac{(b-a)^2}{ab} &\geq \nu(1-\nu) \frac{(b-a)^2}{\max^2\{a, b\}} \\ &= \nu(1-\nu) \left(1 - \frac{\min\{a, b\}}{\max\{a, b\}} \right)^2 \end{aligned}$$

and by (11) we get the desired result (9). \square

We observe that the inequality (9) can be written in an equivalent form as

$$\begin{aligned} &\left[\nu(1-\nu) \left(1 - \frac{\min\{a, b\}}{\max\{a, b\}} \right)^2 + 1 \right] H_\nu(a, b) \\ &\leq A_\nu(a, b) \\ &\leq \left[\nu(1-\nu) \left(\frac{\max\{a, b\}}{\min\{a, b\}} - 1 \right)^2 + 1 \right] H_\nu(a, b) \end{aligned} \quad (12)$$

for any $a, b > 0$ and $\nu \in [0, 1]$, while (10) as

$$\begin{aligned} & \left[\frac{1}{4} \left(1 - \frac{\min\{a, b\}}{\max\{a, b\}} \right)^2 + 1 \right] H(a, b) \\ & \leq A(a, b) \\ & \leq \left[\frac{1}{4} \left(\frac{\max\{a, b\}}{\min\{a, b\}} - 1 \right)^2 + 1 \right] H(a, b) \end{aligned} \quad (13)$$

for any $a, b > 0$.

Corollary 1 For any $a, b \in [k, K] \subset (0, \infty)$ and $\nu \in [0, 1]$ we have

$$\frac{A_\nu(a, b)}{H_\nu(a, b)} - 1 \leq \nu(1 - \nu) \left(\frac{K}{k} - 1 \right)^2. \quad (14)$$

In particular,

$$\frac{A(a, b)}{H(a, b)} - 1 \leq \frac{1}{4} \left(\frac{K}{k} - 1 \right)^2. \quad (15)$$

We have the following multiplicative inequality between the weighted arithmetic and harmonic operator means:

Theorem 2 Let A, B be positive invertible operators and $M > m > 0$ such that the condition (6) holds. Then we have

$$\begin{aligned} & \left[\nu(1 - \nu) \left(1 - \frac{\min\{M, 1\}}{\max\{m, 1\}} \right)^2 + 1 \right] A!_\nu B \\ & \leq A\nabla_\nu B \\ & \leq \left[\nu(1 - \nu) \left(\frac{\max\{M, 1\}}{\min\{m, 1\}} - 1 \right)^2 + 1 \right] A!_\nu B \end{aligned} \quad (16)$$

for any $\nu \in [0, 1]$.

In particular,

$$\begin{aligned} & \left[\frac{1}{4} \left(1 - \frac{\min\{M, 1\}}{\max\{m, 1\}} \right)^2 + 1 \right] A!B \\ & \leq A\nabla B \\ & \leq \left[\frac{1}{4} \left(\frac{\max\{M, 1\}}{\min\{m, 1\}} - 1 \right)^2 + 1 \right] A!B. \end{aligned} \quad (17)$$

Proof. If we write the inequality (12) for $\mathbf{a} = 1$ and $\mathbf{b} = x \in (0, \infty)$ then we have

$$\begin{aligned} & \left[\nu(1-\nu) \left(1 - \frac{\min\{1, x\}}{\max\{1, x\}} \right)^2 + 1 \right] \left(1 - \nu + \nu x^{-1} \right)^{-1} \\ & \leq 1 - \nu + \nu x \\ & \leq \left[\nu(1-\nu) \left(\frac{\max\{1, x\}}{\min\{1, x\}} - 1 \right)^2 + 1 \right] \left(1 - \nu + \nu x^{-1} \right)^{-1}. \end{aligned} \quad (18)$$

for any $\nu \in [0, 1]$.

If $x \in [m, M] \subset (0, \infty)$, then $\max\{m, 1\} \leq \max\{x, 1\} \leq \max\{M, 1\}$ and $\min\{m, 1\} \leq \min\{x, 1\} \leq \min\{M, 1\}$.

We have

$$\left(\frac{\max\{1, x\}}{\min\{1, x\}} - 1 \right)^2 \leq \left(\frac{\max\{M, 1\}}{\min\{m, 1\}} - 1 \right)^2$$

and

$$\left(1 - \frac{\min\{M, 1\}}{\max\{m, 1\}} \right)^2 \leq \left(1 - \frac{\min\{1, x\}}{\max\{1, x\}} \right)^2$$

for any $x \in [m, M] \subset (0, \infty)$.

Therefore, by (18) we have

$$\begin{aligned} & \left[\nu(1-\nu) \left(1 - \frac{\min\{M, 1\}}{\max\{m, 1\}} \right)^2 + 1 \right] \left(1 - \nu + \nu x^{-1} \right)^{-1} \\ & \leq 1 - \nu + \nu x \\ & \leq \left[\nu(1-\nu) \left(\frac{\max\{M, 1\}}{\min\{m, 1\}} - 1 \right)^2 + 1 \right] \left(1 - \nu + \nu x^{-1} \right)^{-1}, \end{aligned} \quad (19)$$

for any $x \in [m, M]$ and for any $\nu \in [0, 1]$.

If we use the continuous functional calculus for the positive invertible operator X with $mI \leq X \leq MI$, then we have from (19) that

$$\begin{aligned} & \left[\nu(1-\nu) \left(1 - \frac{\min\{M, 1\}}{\max\{m, 1\}} \right)^2 + 1 \right] \left((1-\nu)I + \nu X^{-1} \right)^{-1} \\ & \leq (1-\nu)I + \nu X \\ & \leq \left[\nu(1-\nu) \left(\frac{\max\{M, 1\}}{\min\{m, 1\}} - 1 \right)^2 + 1 \right] \left((1-\nu)I + \nu X^{-1} \right)^{-1}, \end{aligned} \quad (20)$$

for any $\nu \in [0, 1]$.

If we multiply (6) both sides by $A^{-1/2}$ we get $MI \geq A^{-1/2}BA^{-1/2} \geq mI$.

By writing the inequality (20) for $X = A^{-1/2}BA^{-1/2}$ we obtain

$$\begin{aligned} & \left[\nu(1-\nu) \left(1 - \frac{\min\{M, 1\}}{\max\{m, 1\}} \right)^2 + 1 \right] \\ & \times \left((1-\nu)I + \nu \left(A^{-1/2}BA^{-1/2} \right)^{-1} \right)^{-1} \\ & \leq (1-\nu)I + \nu A^{-1/2}BA^{-1/2} \\ & \leq \left[\nu(1-\nu) \left(\frac{\max\{M, 1\}}{\min\{m, 1\}} - 1 \right)^2 + 1 \right] \\ & \times \left((1-\nu)I + \nu \left(A^{-1/2}BA^{-1/2} \right)^{-1} \right)^{-1}, \end{aligned} \quad (21)$$

for any $\nu \in [0, 1]$.

If we multiply the inequality (21) both sides with $A^{1/2}$, then we get

$$\begin{aligned} & \left[\nu(1-\nu) \left(1 - \frac{\min\{M, 1\}}{\max\{m, 1\}} \right)^2 + 1 \right] \\ & \times A^{1/2} \left((1-\nu)I + \nu \left(A^{-1/2}BA^{-1/2} \right)^{-1} \right)^{-1} A^{1/2} \\ & \leq (1-\nu)A + \nu B \\ & \leq \left[\nu(1-\nu) \left(\frac{\max\{M, 1\}}{\min\{m, 1\}} - 1 \right)^2 + 1 \right] \\ & \times A^{1/2} \left((1-\nu)I + \nu \left(A^{-1/2}BA^{-1/2} \right)^{-1} \right)^{-1} A^{1/2}, \end{aligned} \quad (22)$$

for any $\nu \in [0, 1]$.

Since

$$\begin{aligned} & A^{1/2} \left((1-\nu)I + \nu \left(A^{-1/2}BA^{-1/2} \right)^{-1} \right)^{-1} A^{1/2} \\ & = A^{1/2} \left((1-\nu)I + \nu A^{1/2}B^{-1}A^{1/2} \right)^{-1} A^{1/2} \\ & = A^{1/2} \left(A^{1/2} \left((1-\nu)A^{-1} + \nu B^{-1} \right) A^{1/2} \right)^{-1} A^{1/2} \end{aligned}$$

$$\begin{aligned}
&= A^{1/2} \left(A^{-1/2} \left((1-\nu) A^{-1} + \nu B^{-1} \right)^{-1} A^{-1/2} \right) A^{1/2} \\
&= \left((1-\nu) A^{-1} + \nu B^{-1} \right)^{-1} = A!_{\nu} B
\end{aligned}$$

hence by (22) we get the desired result (16). \square

We also have:

Theorem 3 *Let A, B be positive invertible operators and $M > m > 0$ such that the condition (6) holds. Then we have*

$$d_{\nu}(m, M) A!_{\nu} B \leq A \nabla_{\nu} B \leq D_{\nu}(m, M) A!_{\nu} B \quad (23)$$

for any $\nu \in [0, 1]$, where

$$d_{\nu}(m, M) := 4 \left[\left(\nu - \frac{1}{2} \right)^2 + \nu(1-\nu) \times \begin{cases} K(M) & \text{if } M < 1, \\ 1 & \text{if } m \leq 1 \leq M, \\ K(m) & \text{if } 1 < m \end{cases} \right]$$

and

$$\begin{aligned}
&D_{\nu}(m, M) \\
&:= 4 \left[\left(\nu - \frac{1}{2} \right)^2 + \nu(1-\nu) \times \begin{cases} K(m) & \text{if } M < 1, \\ \max\{K(m), K(M)\} & \text{if } m \leq 1 \leq M, \\ K(M) & \text{if } 1 < m. \end{cases} \right]
\end{aligned}$$

In particular, we have

$$d(m, M) A!B \leq A \nabla B \leq D(m, M) A!B \quad (24)$$

where

$$d(m, M) := \begin{cases} K(M) & \text{if } M < 1, \\ 1 & \text{if } m \leq 1 \leq M, \\ K(m) & \text{if } 1 < m \end{cases}$$

and

$$D(m, M) := \begin{cases} K(m) & \text{if } M < 1, \\ \max\{K(m), K(M)\} & \text{if } m \leq 1 \leq M, \\ K(M) & \text{if } 1 < m. \end{cases}$$

Proof. From (11) we have for any $x \in (0, \infty)$ and for any $\nu \in [0, 1]$ that

$$\frac{A_{\nu}(1, x)}{H_{\nu}(1, x)} - 1 = \nu(1-\nu) \frac{(x-1)^2}{x}. \quad (25)$$

Since $K(x) - 1 = \frac{(x-1)^2}{4x}$, $x > 0$, then by (25) we have

$$\begin{aligned} \frac{A_\nu(1, x)}{H_\nu(1, x)} &= 1 + 4\nu(1 - \nu)[K(x) - 1] \\ &= 4\nu(1 - \nu)K(x) + 4\left(\nu - \frac{1}{2}\right)^2 \\ &= 4\left[\nu(1 - \nu)K(x) + \left(\nu - \frac{1}{2}\right)^2\right] \end{aligned}$$

or, equivalently,

$$A_\nu(1, x) = 4\left[\nu(1 - \nu)K(x) + \left(\nu - \frac{1}{2}\right)^2\right]H_\nu(1, x) \quad (26)$$

for any $x \in (0, \infty)$ and for any $\nu \in [0, 1]$.

From (26) we then have for any $x \in [m, M] \subset (0, \infty)$ that

$$\begin{aligned} 4\left[\nu(1 - \nu)\min_{x \in [m, M]} K(x) + \left(\nu - \frac{1}{2}\right)^2\right]H_\nu(1, x) & \quad (27) \\ \leq A_\nu(1, x) \leq 4\left[\nu(1 - \nu)\max_{x \in [m, M]} K(x) + \left(\nu - \frac{1}{2}\right)^2\right]H_\nu(1, x). \end{aligned}$$

Since

$$\min_{x \in [m, M]} K(x) = \begin{cases} K(M) & \text{if } M < 1, \\ 1 & \text{if } m \leq 1 \leq M, \\ K(m) & \text{if } 1 < m \end{cases}$$

and

$$\max_{x \in [m, M]} K(x) = \begin{cases} K(m) & \text{if } M < 1, \\ \max\{K(m), K(M)\} & \text{if } m \leq 1 \leq M, \\ K(M) & \text{if } 1 < m, \end{cases}$$

then by (27) we have

$$\begin{aligned} d_\nu(m, M)\left(1 - \nu + \nu x^{-1}\right)^{-1} & \leq 1 - \nu + \nu x & (28) \\ & \leq D_\nu(m, M)\left(1 - \nu + \nu x^{-1}\right)^{-1} \end{aligned}$$

for any $x \in [m, M]$ and for any $\nu \in [0, 1]$.

By a similar argument to the one from Theorem 2 we deduce the desired operator inequality (23). The details are omitted. \square

3 Some particular cases

Let A, B positive invertible operators and positive real numbers m, m', M, M' such that the condition $0 < mI \leq A \leq m'I < M'I \leq B \leq MI$ holds.

Put $h := \frac{M}{m}$ and $h' := \frac{M'}{m'}$, then we have

$$A < h'A = \frac{M'}{m'}A \leq B \leq \frac{M}{m}A = hA.$$

By (16) we get

$$\begin{aligned} \left[\nu(1-\nu) \left(\frac{h'-1}{h'} \right)^2 + 1 \right] A!_{\nu}B &\leq A\nabla_{\nu}B \\ &\leq \left[\nu(1-\nu)(h-1)^2 + 1 \right] A!_{\nu}B \end{aligned} \quad (29)$$

for any $\nu \in [0, 1]$.

By (23) we get

$$\begin{aligned} 4 \left[\left(\nu - \frac{1}{2} \right)^2 + \nu(1-\nu)K(h') \right] A!_{\nu}B & \\ \leq A\nabla_{\nu}B \leq 4 \left[\left(\nu - \frac{1}{2} \right)^2 + \nu(1-\nu)K(h) \right] A!_{\nu}B & \end{aligned} \quad (30)$$

for any $\nu \in [0, 1]$.

If $0 < mI \leq B \leq m'I < M'I \leq A \leq MI$, then for $h := \frac{M}{m}$ and $h' := \frac{M'}{m'}$ we also have

$$\frac{1}{h}A \leq B \leq \frac{1}{h'}A < A.$$

Finally, by (16) we get (29) while from (23) we get (30) as well.

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