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Multiplicative inequalities for weighted arithmetic and harmonic operator means

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Abstract. In this paper we establish some multiplicative inequalities for weighted arithmetic and harmonic operator means under various assumption for the positive invertible operators A, B. Some applications when A, B are bounded above and below by positive constants are given as well.

1 Introduction

Throughout this paper A, B are positive invertible operators on a complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$. We use the following notations for operators

$$A \nabla_\nu B := (1-\nu) A + \nu B,$$

the *weighted operator arithmetic mean,*

$$A \sharp_\nu B := A^{1/2} \left(A^{-1/2} B A^{-1/2} \right)^\nu A^{1/2},$$

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the *weighted operator geometric mean* and

$$A!_\nu B := \left((1-\nu)A^{-1} + \nu B^{-1}\right)^{-1}$$

the *weighted operator harmonic mean,* where $\nu \in [0,1]$.

When $\nu = \frac{1}{2}$, we write $A\nabla B$, $A\sharp B$ and $A!B$ for brevity, respectively.

The following fundamental inequality between the weighted arithmetic, geometric and harmonic operator means holds

$$A!_\nu B \le A\sharp_\nu B \le A\nabla_\nu B \tag{1}$$

for any $\nu \in [0,1]$.

For various recent inequalities between these means we recommend the recent papers [3]-[6], [8]-[12] and the references therein.

The following additive double inequality has been obtained in the recent paper [7]:

$$\nu(1-\nu)\frac{(b-a)^2}{\max\{b,a\}} \le A_\nu(a,b) - H_\nu(a,b) \le \nu(1-\nu)\frac{(b-a)^2}{\min\{b,a\}}, \tag{2}$$

for any a, $b > 0$ and $\nu \in [0,1]$, where $A_\nu(a,b)$ and $H_\nu(a,b)$ are the scalar weighted arithmetic mean and harmonic mean, respectively, namely

$$A_\nu(a,b) := (1-\nu)a + \nu b \text{ and } H_\nu(a,b) := \frac{ab}{(1-\nu)b + \nu a}.$$

In particular,

$$\frac{1}{4}\frac{(b-a)^2}{\max\{b,a\}} \le A(a,b) - H(a,b) \le \frac{1}{4}\frac{(b-a)^2}{\min\{b,a\}}, \tag{3}$$

where

$$A(a,b) := \frac{a+b}{2} \text{ and } H(a,b) := \frac{2ab}{b+a}.$$

We consider the *Kantorovich's constant* defined by

$$K(h) := \frac{(h+1)^2}{4h}, \; h > 0. \tag{4}$$

The function K is decreasing on $(0,1)$ and increasing on $[1,\infty)$, $K(h) \ge 1$ for any $h > 0$ and $K(h) = K\left(\frac{1}{h}\right)$ for any $h > 0$.

Observe that for any $h > 0$

$$K(h) - 1 = \frac{(h-1)^2}{4h} = K\left(\frac{1}{h}\right) - 1.$$

Observe that

$$K\left(\frac{b}{a}\right) - 1 = \frac{(b-a)^2}{4ab} \text{ for } a,\ b > 0.$$

Since, obviously

$$ab = \min\{a,b\}\max\{a,b\} \text{ for } a,\ b > 0,$$

then we have the following version of (2):

$$\begin{aligned} 4\nu(1-\nu)\min\{a,b\}\left[K\left(\frac{b}{a}\right)-1\right] &\le A_\nu(a,b) - H_\nu(a,b) \\ &\le 4\nu(1-\nu)\max\{a,b\}\left[K\left(\frac{b}{a}\right)-1\right]. \end{aligned} \tag{5}$$

for any a, $b > 0$ and $\nu \in [0,1]$.

For positive invertible operators A, B we define

$$A\nabla_\infty B := \frac{1}{2}(A+B) + \frac{1}{2}A^{1/2}\left|A^{-1/2}(B-A)A^{-1/2}\right|A^{1/2}$$

and

$$A\nabla_{-\infty} B := \frac{1}{2}(A+B) - \frac{1}{2}A^{1/2}\left|A^{-1/2}(B-A)A^{-1/2}\right|A^{1/2}.$$

If we consider the continuous functions f_∞, $f_{-\infty} : [0,\infty) \to [0,\infty)$ defined by

$$f_\infty(x) = \max\{x,1\} = \frac{1}{2}(x+1) + \frac{1}{2}|x-1|$$

and

$$f_{-\infty}(x) = \max\{x,1\} = \frac{1}{2}(x+1) - \frac{1}{2}|x-1|,$$

then, obviously, we have

$$A\nabla_{\pm\infty}B = A^{1/2}f_{\pm\infty}\left(A^{-1/2}BA^{-1}\right)A^{1/2}.$$

If A and B are commutative, then

$$A\nabla_{\pm\infty}B = \frac{1}{2}(A+B) \pm \frac{1}{2}|B-A| = B\nabla_{\pm\infty}A.$$

The following additive inequality between the weighted arithmetic and harmonic operator means holds [7]:

Theorem 1 *Let* A, B *be positive invertible operators and* $M > m > 0$ *such that the condition*

$$mA \leq B \leq MA \tag{6}$$

holds. Then we have

$$\begin{aligned} 4\nu(1-\nu)\, g(m,M)\, A\nabla_{-\infty}B &\leq A\nabla_\nu B - A!_\nu B \\ &\leq 4\nu(1-\nu)\, G(m,M)\, A\nabla_{\infty}B, \end{aligned} \tag{7}$$

where

$$g(m,M) := \begin{cases} K(M) - 1 \text{ \textit{if} } M < 1, \\ 0 \text{ \textit{if} } m \leq 1 \leq M, \\ K(m) - 1 \text{ \textit{if} } 1 < m \end{cases}$$

and

$$G(m,M) := \begin{cases} K(m) - 1 \text{ \textit{if} } M < 1, \\ \max\{K(m), K(M)\} - 1 \text{ \textit{if} } m \leq 1 \leq M, \\ K(M) - 1 \text{ \textit{if} } 1 < m. \end{cases}$$

In particular,

$$g(m,M)\, A\nabla_{-\infty}B \leq A\nabla B - A!B \leq G(m,M)\, A\nabla_{\infty}B. \tag{8}$$

Motivated by the above facts, we establish in this paper some multiplicative inequalities for weighted arithmetic and harmonic operator means under various assumption for the positive invertible operators A, B. Some applications when A, B are bounded above and below by positive constants are given as well.

2 Multiplicative inequalities

The following result is of interest in itself:

Lemma 1 *For any* a, $b > 0$ *and* $\nu \in [0,1]$ *we have*

$$\nu(1-\nu)\left(1 - \frac{\min\{a,b\}}{\max\{a,b\}}\right)^2 \leq \frac{A_\nu(a,b)}{H_\nu(a,b)} - 1 \leq \nu(1-\nu)\left(\frac{\max\{a,b\}}{\min\{a,b\}} - 1\right)^2. \tag{9}$$

In particular,

$$\frac{1}{4}\left(1 - \frac{\min\{a,b\}}{\max\{a,b\}}\right)^2 \leq \frac{A(a,b)}{H(a,b)} - 1 \leq \frac{1}{4}\left(\frac{\max\{a,b\}}{\min\{a,b\}} - 1\right)^2. \tag{10}$$

Proof. We have for any $a,\ b>0$ and $\nu\in[0,1]$ that

$$\begin{aligned}\frac{A_\nu(a,b)}{H_\nu(a,b)}&=\frac{[(1-\nu)a+\nu b][(1-\nu)b+\nu a]}{ab}\\&=\frac{(1-\nu)^2ab+\nu(1-\nu)b^2+\nu(1-\nu)a^2+\nu^2ab}{ab}\\&=\frac{\nu(1-\nu)\left(b^2+a^2\right)+\left(1-2\nu+2\nu^2\right)ab}{ab},\end{aligned}$$

which is equivalent with

$$\frac{A_\nu(a,b)}{H_\nu(a,b)}-1=\nu(1-\nu)\frac{(b-a)^2}{ab} \tag{11}$$

for any $a,\ b>0$ and $\nu\in[0,1]$.

Since $\min^2\{a,b\}\le ab\le\max^2\{a,b\}$ hence

$$\begin{aligned}\nu(1-\nu)\frac{(b-a)^2}{ab}&\le\nu(1-\nu)\frac{(b-a)^2}{\min^2\{a,b\}}\\&=\nu(1-\nu)\left(\frac{\max\{a,b\}}{\min\{a,b\}}-1\right)^2\end{aligned}$$

and

$$\begin{aligned}\nu(1-\nu)\frac{(b-a)^2}{ab}&\ge\nu(1-\nu)\frac{(b-a)^2}{\max^2\{a,b\}}\\&=\nu(1-\nu)\left(1-\frac{\min\{a,b\}}{\max\{a,b\}}\right)^2\end{aligned}$$

and by (11) we get the desired result (9). □

We observe that the inequality (9) can be written in an equivalent form as

$$\begin{aligned}&\left[\nu(1-\nu)\left(1-\frac{\min\{a,b\}}{\max\{a,b\}}\right)^2+1\right]H_\nu(a,b)\\&\le A_\nu(a,b)\\&\le\left[\nu(1-\nu)\left(\frac{\max\{a,b\}}{\min\{a,b\}}-1\right)^2+1\right]H_\nu(a,b)\end{aligned} \tag{12}$$

for any $a,\ b>0$ and $\nu\in[0,1]$, while (10) as

$$\left[\frac{1}{4}\left(1-\frac{\min\{a,b\}}{\max\{a,b\}}\right)^{2}+1\right]H(a,b)\tag{13}$$
$$\leq A(a,b)$$
$$\leq\left[\frac{1}{4}\left(\frac{\max\{a,b\}}{\min\{a,b\}}-1\right)^{2}+1\right]H(a,b)$$

for any $a,\ b>0$.

Corollary 1 *For any* $a,\ b\in[k,K]\subset(0,\infty)$ *and* $\nu\in[0,1]$ *we have*

$$\frac{A_{\nu}(a,b)}{H_{\nu}(a,b)}-1\leq\nu(1-\nu)\left(\frac{K}{k}-1\right)^{2}.\tag{14}$$

In particular,

$$\frac{A(a,b)}{H(a,b)}-1\leq\frac{1}{4}\left(\frac{K}{k}-1\right)^{2}.\tag{15}$$

We have the following multiplicative inequality between the weighted arithmetic and harmonic operator means:

Theorem 2 *Let* $A,\ B$ *be positive invertible operators and* $M>m>0$ *such that the condition* (6) *holds. Then we have*

$$\left[\nu(1-\nu)\left(1-\frac{\min\{M,1\}}{\max\{m,1\}}\right)^{2}+1\right]A!_{\nu}B\tag{16}$$
$$\leq A\nabla_{\nu}B$$
$$\leq\left[\nu(1-\nu)\left(\frac{\max\{M,1\}}{\min\{m,1\}}-1\right)^{2}+1\right]A!_{\nu}B$$

for any $\nu\in[0,1]$.

In particular,

$$\left[\frac{1}{4}\left(1-\frac{\min\{M,1\}}{\max\{m,1\}}\right)^{2}+1\right]A!B\tag{17}$$
$$\leq A\nabla B$$
$$\leq\left[\frac{1}{4}\left(\frac{\max\{M,1\}}{\min\{m,1\}}-1\right)^{2}+1\right]A!B.$$

Proof. If we write the inequality (12) for $a = 1$ and $b = x \in (0, \infty)$ then we have

$$\begin{aligned} &\left[\nu(1-\nu)\left(1-\frac{\min\{1,x\}}{\max\{1,x\}}\right)^2+1\right]\left(1-\nu+\nu x^{-1}\right)^{-1} \\ &\le 1-\nu+\nu x \\ &\le \left[\nu(1-\nu)\left(\frac{\max\{1,x\}}{\min\{1,x\}}-1\right)^2+1\right]\left(1-\nu+\nu x^{-1}\right)^{-1}. \end{aligned} \tag{18}$$

for any $\nu \in [0, 1]$.

If $x \in [m, M] \subset (0, \infty)$, then $\max\{m, 1\} \le \max\{x, 1\} \le \max\{M, 1\}$ and $\min\{m, 1\} \le \min\{x, 1\} \le \min\{M, 1\}$.

We have

$$\left(\frac{\max\{1,x\}}{\min\{1,x\}}-1\right)^2 \le \left(\frac{\max\{M,1\}}{\min\{m,1\}}-1\right)^2$$

and

$$\left(1-\frac{\min\{M,1\}}{\max\{m,1\}}\right)^2 \le \left(1-\frac{\min\{1,x\}}{\max\{1,x\}}\right)^2$$

for any $x \in [m, M] \subset (0, \infty)$.

Therefore, by (18) we have

$$\begin{aligned} &\left[\nu(1-\nu)\left(1-\frac{\min\{M,1\}}{\max\{m,1\}}\right)^2+1\right]\left(1-\nu+\nu x^{-1}\right)^{-1} \\ &\le 1-\nu+\nu x \\ &\le \left[\nu(1-\nu)\left(\frac{\max\{M,1\}}{\min\{m,1\}}-1\right)^2+1\right]\left(1-\nu+\nu x^{-1}\right)^{-1}, \end{aligned} \tag{19}$$

for any $x \in [m, M]$ and for any $\nu \in [0, 1]$.

If we use the continuous functional calculus for the positive invertible operator X with $mI \le X \le MI$, then we have from (19) that

$$\begin{aligned} &\left[\nu(1-\nu)\left(1-\frac{\min\{M,1\}}{\max\{m,1\}}\right)^2+1\right]\left((1-\nu)I+\nu X^{-1}\right)^{-1} \\ &\le (1-\nu)I+\nu X \\ &\le \left[\nu(1-\nu)\left(\frac{\max\{M,1\}}{\min\{m,1\}}-1\right)^2+1\right]\left((1-\nu)I+\nu X^{-1}\right)^{-1}, \end{aligned} \tag{20}$$

for any $\nu \in [0, 1]$.

If we multiply (6) both sides by $A^{-1/2}$ we get $MI \geq A^{-1/2}BA^{-1/2} \geq mI$.

By writing the inequality (20) for $X = A^{-1/2}BA^{-1/2}$ we obtain

$$\begin{aligned} &\left[\nu(1-\nu)\left(1-\frac{\min\{M,1\}}{\max\{m,1\}}\right)^2+1\right] \\ &\times\left((1-\nu)I+\nu\left(A^{-1/2}BA^{-1/2}\right)^{-1}\right)^{-1} \\ &\leq (1-\nu)I+\nu A^{-1/2}BA^{-1/2} \\ &\leq \left[\nu(1-\nu)\left(\frac{\max\{M,1\}}{\min\{m,1\}}-1\right)^2+1\right] \\ &\times\left((1-\nu)I+\nu\left(A^{-1/2}BA^{-1/2}\right)^{-1}\right)^{-1}, \end{aligned} \tag{21}$$

for any $\nu \in [0, 1]$.

If we multiply the inequality (21) both sides with $A^{1/2}$, then we get

$$\begin{aligned} &\left[\nu(1-\nu)\left(1-\frac{\min\{M,1\}}{\max\{m,1\}}\right)^2+1\right] \\ &\times A^{1/2}\left((1-\nu)I+\nu\left(A^{-1/2}BA^{-1/2}\right)^{-1}\right)^{-1}A^{1/2} \\ &\leq (1-\nu)A+\nu B \\ &\leq \left[\nu(1-\nu)\left(\frac{\max\{M,1\}}{\min\{m,1\}}-1\right)^2+1\right] \\ &\times A^{1/2}\left((1-\nu)I+\nu\left(A^{-1/2}BA^{-1/2}\right)^{-1}\right)^{-1}A^{1/2}, \end{aligned} \tag{22}$$

for any $\nu \in [0, 1]$.

Since

$$\begin{aligned} &A^{1/2}\left((1-\nu)I+\nu\left(A^{-1/2}BA^{-1/2}\right)^{-1}\right)^{-1}A^{1/2} \\ &= A^{1/2}\left((1-\nu)I+\nu A^{1/2}B^{-1}A^{1/2}\right)^{-1}A^{1/2} \\ &= A^{1/2}\left(A^{1/2}\left((1-\nu)A^{-1}+\nu B^{-1}\right)A^{1/2}\right)^{-1}A^{1/2} \end{aligned}$$

$$= A^{1/2}\left(A^{-1/2}\left((1-\nu)A^{-1}+\nu B^{-1}\right)^{-1}A^{-1/2}\right)A^{1/2}$$
$$= \left((1-\nu)A^{-1}+\nu B^{-1}\right)^{-1} = A!_\nu B$$

hence by (22) we get the desired result (16). □

We also have:

Theorem 3 *Let* A, B *be positive invertible operators and* $M > m > 0$ *such that the condition (6) holds. Then we have*

$$d_\nu(m,M)\,A!_\nu B \le A\nabla_\nu B \le D_\nu(m,M)\,A!_\nu B \tag{23}$$

for any $\nu \in [0,1]$, *where*

$$d_\nu(m,M) := 4\left[\left(\nu-\frac{1}{2}\right)^2 + \nu(1-\nu)\times\begin{cases} K(M) & \text{if } M<1, \\ 1 & \text{if } m\le 1\le M, \\ K(m) & \text{if } 1<m \end{cases}\right]$$

and

$$D_\nu(m,M)$$
$$:= 4\left[\left(\nu-\frac{1}{2}\right)^2 + \nu(1-\nu)\times\begin{cases} K(m) & \text{if } M<1, \\ \max\{K(m),K(M)\} & \text{if } m\le 1\le M, \\ K(M) & \text{if } 1<m. \end{cases}\right]$$

In particular, we have

$$d(m,M)\,A!B \le A\nabla B \le D(m,M)\,A!B \tag{24}$$

where

$$d(m,M) := \begin{cases} K(M) & \text{if } M<1, \\ 1 & \text{if } m\le 1\le M, \\ K(m) & \text{if } 1<m \end{cases}$$

and

$$D(m,M) := \begin{cases} K(m) & \text{if } M<1, \\ \max\{K(m),K(M)\} & \text{if } m\le 1\le M, \\ K(M) & \text{if } 1<m. \end{cases}$$

Proof. From (11) we have for any $x \in (0,\infty)$ and for any $\nu \in [0,1]$ that

$$\frac{A_\nu(1,x)}{H_\nu(1,x)} - 1 = \nu(1-\nu)\frac{(x-1)^2}{x}. \tag{25}$$

Since $K(x)-1=\frac{(x-1)^2}{4x}$, $x>0$, then by (25) we have

$$\begin{aligned}\frac{A_\nu(1,x)}{H_\nu(1,x)} &= 1+4\nu(1-\nu)[K(x)-1]\\ &= 4\nu(1-\nu)K(x)+4\left(\nu-\frac{1}{2}\right)^2\\ &= 4\left[\nu(1-\nu)K(x)+\left(\nu-\frac{1}{2}\right)^2\right]\end{aligned}$$

or, equivalently,

$$A_\nu(1,x)=4\left[\nu(1-\nu)K(x)+\left(\nu-\frac{1}{2}\right)^2\right]H_\nu(1,x) \tag{26}$$

for any $x\in(0,\infty)$ and for any $\nu\in[0,1]$.

From (26) we then have for any $x\in[m,M]\subset(0,\infty)$ that

$$\begin{aligned}&4\left[\nu(1-\nu)\min_{x\in[m,M]}K(x)+\left(\nu-\frac{1}{2}\right)^2\right]H_\nu(1,x)\\ &\leq A_\nu(1,x)\leq 4\left[\nu(1-\nu)\max_{x\in[m,M]}K(x)+\left(\nu-\frac{1}{2}\right)^2\right]H_\nu(1,x).\end{aligned} \tag{27}$$

Since

$$\min_{x\in[m,M]}K(x)=\begin{cases}K(M) \text{ if } M<1,\\ 1 \text{ if } m\leq 1\leq M,\\ K(m) \text{ if } 1<m\end{cases}$$

and

$$\max_{x\in[m,M]}K(x)=\begin{cases}K(m) \text{ if } M<1,\\ \max\{K(m),K(M)\} \text{ if } m\leq 1\leq M,\\ K(M) \text{ if } 1<m,\end{cases}$$

then by (27) we have

$$\begin{aligned}d_\nu(m,M)\left(1-\nu+\nu x^{-1}\right)^{-1} &\leq 1-\nu+\nu x\\ &\leq D_\nu(m,M)\left(1-\nu+\nu x^{-1}\right)^{-1}\end{aligned} \tag{28}$$

for any $x\in[m,M]$ and for any $\nu\in[0,1]$.

By a similar argument to the one from Theorem 2 we deduce the desired operator inequality (23). The details are omitted. □

3 Some particular cases

Let A, B positive invertible operators and positive real numbers m, m', M, M' such that the condition $0 < mI \le A \le m'I < M'I \le B \le MI$ holds.

Put $h := \frac{M}{m}$ and $h' := \frac{M'}{m'}$, then we have

$$A < h'A = \frac{M'}{m'}A \le B \le \frac{M}{m}A = hA.$$

By (16) we get

$$\left[\nu(1-\nu)\left(\frac{h'-1}{h'}\right)^2 + 1\right] A!_\nu B \le A\nabla_\nu B \le \left[\nu(1-\nu)(h-1)^2 + 1\right] A!_\nu B \tag{29}$$

for any $\nu \in [0,1]$.

By (23) we get

$$4\left[\left(\nu - \frac{1}{2}\right)^2 + \nu(1-\nu)K(h')\right] A!_\nu B \le A\nabla_\nu B \le 4\left[\left(\nu - \frac{1}{2}\right)^2 + \nu(1-\nu)K(h)\right] A!_\nu B \tag{30}$$

for any $\nu \in [0,1]$.

If $0 < mI \le B \le m'I < M'I \le A \le MI$, then for $h := \frac{M}{m}$ and $h' := \frac{M'}{m'}$ we also have

$$\frac{1}{h}A \le B \le \frac{1}{h'}A < A.$$

Finally, by (16) we get (29) while from (23) we get (30) as well.

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