



Research article

New generalized integral inequalities with applications

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Abstract: The authors have proved an identity for a generalized integral operator via differentiable function. By applying the established identity, the generalized trapezium type integral inequalities have been discovered. It is pointed out that the results of this research provide integral inequalities for almost all fractional integrals discovered in recent past decades. Various special cases have been identified. Some applications of presented results have been analyzed.

Keywords: Hermite-Hadamard inequality; general fractional integrals; Hölder’s inequality; power mean inequality

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1. Introduction

Theorem 1.1. Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be convex, $a_1, a_2 \in I$ and $a_1 < a_2$. Then

$$f\left(\frac{a_1 + a_2}{2}\right) \leq \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} f(x)dx \leq \frac{f(a_1) + f(a_2)}{2}. \tag{1.1}$$

This inequality (1.1) is called Hermite-Hadamard inequality.

Authors of recent decades have studied (1.1) in the premises of newly invented definitions due to motivation of convex function. Interested readers see the references [1–20]. It is important to summarize the study of fractional integrals as follow:

Definition 1.2. The k -gamma function, where $k \in \mathbb{R}^+$ and $x \in \mathbb{C}$, is defined by

$$\Gamma_k(x) = \lim_{n \rightarrow \infty} \frac{n!k^n(nk)^{\frac{x}{k}-1}}{(x)_{n,k}}. \tag{1.2}$$

Its integral representation is given by

$$\Gamma_k(\alpha) = \int_0^\infty t^{\alpha-1} e^{-\frac{t^k}{k}} dt. \quad (1.3)$$

One can note that

$$\Gamma_k(\alpha + k) = \alpha \Gamma_k(\alpha). \quad (1.4)$$

Definition 1.3. [11] Let $f \in L[a_1, a_2]$. Then k -fractional integrals of order $\alpha, k > 0$, where $a_1 \geq 0$ is given as

$$I_{a_1^+}^{\alpha,k} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_{a_1}^x (x-t)^{\frac{\alpha}{k}-1} f(t) dt, \quad x > a_1$$

and

$$I_{a_2^-}^{\alpha,k} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_x^{a_2} (t-x)^{\frac{\alpha}{k}-1} f(t) dt, \quad a_2 > x. \quad (1.5)$$

Let $\phi : [0, \infty) \rightarrow [0, \infty)$ and

$$\int_0^1 \frac{\phi(t)}{t} dt < \infty, \quad (1.6)$$

$$\frac{1}{\mathbf{A}} \leq \frac{\phi(s)}{\phi(r)} \leq \mathbf{A} \text{ for } \frac{1}{2} \leq \frac{s}{r} \leq 2 \quad (1.7)$$

$$\frac{\phi(r)}{r^2} \leq \mathbf{B} \frac{\phi(s)}{s^2} \text{ for } s \leq r \quad (1.8)$$

$$\left| \frac{\phi(r)}{r^2} - \frac{\phi(s)}{s^2} \right| \leq \mathbf{C} |r-s| \frac{\phi(r)}{r^2} \text{ for } \frac{1}{2} \leq \frac{s}{r} \leq 2 \quad (1.9)$$

where $\mathbf{A}, \mathbf{B}, \mathbf{C} > 0$ are independent of $r, s > 0$. If $\phi(r)r^\alpha$ is increasing for some $\alpha \geq 0$ and $\frac{\phi(r)}{r^\beta}$ is decreasing for some $\beta \geq 0$, then ϕ satisfies (1.6)–(1.9), see [15]. Therefore, the left-sided and right-sided generalized integral operators are defined as follows:

$$a_1^+ I_\phi f(x) = \int_{a_1}^x \frac{\phi(x-t)}{x-t} f(t) dt, \quad x > a_1, \quad (1.10)$$

$$a_2^- I_\phi f(x) = \int_x^{a_2} \frac{\phi(t-x)}{t-x} f(t) dt, \quad x < a_2. \quad (1.11)$$

For other feature of generalized integrals, see [14].

The main objective of this paper is to discover in Section 2, an interesting identity in order to study some new bounds regarding trapezium type integral inequalities. By using the established identity as an auxiliary result, some new estimates for trapezoidal type integral inequalities via generalized integrals are obtained. In Section 3, some applications are given. At the end, a briefly conclusion is provided as well.

2. Main results

Let $a_1 < a_2$ and $m \in (0, 1]$ be a fixed number. Throughout this study, for brevity, we define

$$\Lambda_m(t) := \int_0^t \frac{\phi((a_2 - ma_1)u)}{u} du < \infty, \quad \forall t \in [0, 1], \quad \forall x \in P = [ma_1, a_2]. \quad (2.1)$$

The following lemma is crucial:

Lemma 2.1. *Let $f : P \rightarrow \mathbb{R}$ be a differentiable mapping on (ma_1, a_2) . If $f' \in L(P)$ and $\lambda \in (0, 1]$, then*

$$\begin{aligned} & f\left(ma_1 + \frac{\lambda}{2}(a_2 - ma_1)\right) - \frac{1}{2\Lambda_m\left(\frac{\lambda}{2}\right)} \\ & \times \left[(ma_1 + \frac{\lambda}{2}(a_2 - ma_1))^+ I_\phi f(ma_1(1 - \lambda) + a_2\lambda) + (ma_1 + \frac{\lambda}{2}(a_2 - ma_1))^- I_\phi f(ma_1) \right] \\ & = \frac{\lambda(a_2 - ma_1)}{2\Lambda_m\left(\frac{\lambda}{2}\right)} \times \left\{ \int_0^{\frac{\lambda}{2}} \Lambda_m(\lambda t) f'(ma_1 + (\lambda t)(a_2 - ma_1)) dt \right. \\ & \quad \left. - \int_{\frac{\lambda}{2}}^1 \Lambda_m((1 - t)\lambda) f'(ma_1 + (\lambda t)(a_2 - ma_1)) dt \right\}. \end{aligned} \quad (2.2)$$

We denote

$$\begin{aligned} T_{f, \Lambda_m}(\lambda; a_1, a_2) & := \frac{\lambda(a_2 - ma_1)}{2\Lambda_m\left(\frac{\lambda}{2}\right)} \times \left\{ \int_0^{\frac{\lambda}{2}} \Lambda_m(\lambda t) f'(ma_1 + (\lambda t)(a_2 - ma_1)) dt \right. \\ & \quad \left. - \int_{\frac{\lambda}{2}}^1 \Lambda_m((1 - t)\lambda) f'(ma_1 + (\lambda t)(a_2 - ma_1)) dt \right\}. \end{aligned} \quad (2.3)$$

Proof. Integrating by parts Eq (2.3), we get

$$\begin{aligned} T_{f, \Lambda_m}(\lambda; a_1, a_2) & = \frac{\lambda(a_2 - ma_1)}{2\Lambda_m\left(\frac{\lambda}{2}\right)} \times \left\{ \frac{\Lambda_m(\lambda t) f(ma_1 + (\lambda t)(a_2 - ma_1))}{\lambda(a_2 - ma_1)} \Big|_0^{\frac{\lambda}{2}} \right. \\ & \quad - \frac{1}{(a_2 - ma_1)} \times \int_0^{\frac{\lambda}{2}} \frac{\phi((a_2 - ma_1)(\lambda t))}{\lambda t} f(ma_1 + (\lambda t)(a_2 - ma_1)) dt \\ & \quad \left. - \frac{\Lambda_m((1 - t)\lambda) f(ma_1 + (\lambda t)(a_2 - ma_1))}{\lambda(a_2 - ma_1)} \Big|_{\frac{\lambda}{2}}^1 \right. \\ & \quad \left. - \frac{1}{(a_2 - ma_1)} \times \int_{\frac{\lambda}{2}}^1 \frac{\phi((a_2 - ma_1)(1 - t)\lambda)}{(1 - t)\lambda} f(ma_1 + (\lambda t)(a_2 - ma_1)) dt \right\} \\ & = \frac{\lambda(a_2 - ma_1)}{2\Lambda_m\left(\frac{\lambda}{2}\right)} \end{aligned}$$

$$\begin{aligned} & \times \left\{ \frac{\Lambda_m\left(\frac{\lambda}{2}\right) f\left(ma_1 + \frac{\lambda}{2}(a_2 - ma_1)\right)}{\lambda(a_2 - ma_1)} - \frac{1}{(a_2 - ma_1)} \times (ma_1 + \frac{\lambda}{2}(a_2 - ma_1))^{-} I_{\phi} f(ma_1) \right. \\ & \left. + \frac{\Lambda_m\left(\frac{\lambda}{2}\right) f\left(ma_1 + \frac{\lambda}{2}(a_2 - ma_1)\right)}{\lambda(a_2 - ma_1)} - \frac{1}{(a_2 - ma_1)} \times (ma_1 + \frac{\lambda}{2}(a_2 - ma_1))^{+} I_{\phi} f(ma_1(1 - \lambda) + a_2\lambda) \right\} \\ & = f\left(ma_1 + \frac{\lambda}{2}(a_2 - ma_1)\right) - \frac{1}{2\Lambda_m\left(\frac{\lambda}{2}\right)} \\ & \times \left[(ma_1 + \frac{\lambda}{2}(a_2 - ma_1))^{+} I_{\phi} f(ma_1(1 - \lambda) + a_2\lambda) + (ma_1 + \frac{\lambda}{2}(a_2 - ma_1))^{-} I_{\phi} f(ma_1) \right]. \end{aligned}$$

The proof of Lemma 2.1 is completed. \square

Remark 2.2. Taking $\lambda = 1$ and $\phi(t) = t$ in Lemma 2.1, we have

$$\begin{aligned} T_f(a_1, a_2) & := f\left(\frac{ma_1 + a_2}{2}\right) - \frac{1}{(a_2 - ma_1)} \int_{ma_1}^{a_2} f(t) dt \\ & = (a_2 - ma_1) \left\{ \int_0^{\frac{1}{2}} t f'(ma_1 + t(a_2 - ma_1)) dt - \int_{\frac{1}{2}}^1 (1 - t) f'(ma_1 + t(a_2 - ma_1)) dt \right\}. \end{aligned} \quad (2.4)$$

Theorem 2.3. Let $f : P \rightarrow \mathbb{R}$ be a differentiable mapping on (ma_1, a_2) . If $|f'|^q$ is convex on P and $\lambda \in (0, 1]$ for $q > 1$ and $p^{-1} + q^{-1} = 1$, then

$$\begin{aligned} |T_{f, \Lambda_m}(\lambda; a_1, a_2)| & \leq \frac{\lambda(a_2 - ma_1)}{2\sqrt[q]{8}\Lambda_m\left(\frac{\lambda}{2}\right)} \sqrt[p]{B_{\Lambda_m}(\lambda; p)} \\ & \times \left\{ \sqrt[q]{(4 - \lambda)|f'(ma_1)|^q + \lambda|f'(a_2)|^q} + \sqrt[q]{(4 - 3\lambda)|f'(ma_1)|^q + 3\lambda|f'(a_2)|^q} \right\}, \end{aligned} \quad (2.5)$$

where

$$B_{\Lambda_m}(\lambda; p) := \int_0^{\frac{1}{2}} [\Lambda_m(\lambda t)]^p dt. \quad (2.6)$$

Proof. Using Lemma 2.1, convexity of $|f'|^q$ and Hölder inequality, we get

$$\begin{aligned} |T_{f, \Lambda_m}(\lambda; a_1, a_2)| & \leq \frac{\lambda(a_2 - ma_1)}{2\Lambda_m\left(\frac{\lambda}{2}\right)} \times \left\{ \int_0^{\frac{1}{2}} \Lambda_m(\lambda t) |f'(ma_1 + (\lambda t)(a_2 - ma_1))| dt \right. \\ & \left. + \int_{\frac{1}{2}}^1 \Lambda_m((1 - t)\lambda) |f'(ma_1 + (\lambda t)(a_2 - ma_1))| dt \right\} \\ & \leq \frac{\lambda(a_2 - ma_1)}{2\Lambda_m\left(\frac{\lambda}{2}\right)} \times \left\{ \left(\int_0^{\frac{1}{2}} [\Lambda_m(\lambda t)]^p dt \right)^{\frac{1}{p}} \left(\int_0^{\frac{1}{2}} |f'(ma_1 + (\lambda t)(a_2 - ma_1))|^q dt \right)^{\frac{1}{q}} \right. \\ & \left. + \left(\int_{\frac{1}{2}}^1 [\Lambda_m((1 - t)\lambda)]^p dt \right)^{\frac{1}{p}} \left(\int_{\frac{1}{2}}^1 |f'(ma_1 + (\lambda t)(a_2 - ma_1))|^q dt \right)^{\frac{1}{q}} \right\} \end{aligned}$$

$$\begin{aligned}
&\leq \frac{\lambda(a_2 - ma_1)}{2\Lambda_m\left(\frac{\lambda}{2}\right)} \sqrt[p]{B_{\Lambda_m}(\lambda; p)} \times \left\{ \left(\int_0^{\frac{1}{2}} [(1-\lambda t)|f'(ma_1)|^q + (\lambda t)|f'(a_2)|^q] dt \right)^{\frac{1}{q}} \right. \\
&\quad \left. + \left(\int_{\frac{1}{2}}^1 [(1-\lambda t)|f'(ma_1)|^q + (\lambda t)|f'(a_2)|^q] dt \right)^{\frac{1}{q}} \right\} \\
&= \frac{\lambda(a_2 - ma_1)}{2\sqrt[q]{8}\Lambda_m\left(\frac{\lambda}{2}\right)} \sqrt[p]{B_{\Lambda_m}(\lambda; p)} \\
&\quad \times \left\{ \sqrt[q]{(4-\lambda)|f'(ma_1)|^q + \lambda|f'(a_2)|^q} + \sqrt[q]{(4-3\lambda)|f'(ma_1)|^q + 3\lambda|f'(a_2)|^q} \right\}.
\end{aligned}$$

The proof of Theorem 2.3 is completed. \square

Corollary 2.4. For $p = q = 2$ in Theorem 2.3, we get

$$\begin{aligned}
|T_{f,\Lambda_m}(\lambda; a_1, a_2)| &\leq \frac{\lambda(a_2 - ma_1)}{4\sqrt{2}\Lambda_m\left(\frac{\lambda}{2}\right)} \sqrt{B_{\Lambda_m}(\lambda; 2)} \\
&\quad \times \left\{ \sqrt{(4-\lambda)|f'(ma_1)|^2 + \lambda|f'(a_2)|^2} + \sqrt{(4-3\lambda)|f'(ma_1)|^2 + 3\lambda|f'(a_2)|^2} \right\}.
\end{aligned} \tag{2.7}$$

Corollary 2.5. For $\phi(t) = t$ in Theorem 2.3, we have

$$\begin{aligned}
|T_{f,\Lambda_m}(\lambda; a_1, a_2)| &\leq \frac{\lambda(a_2 - ma_1)}{\sqrt[q]{8} \sqrt[p]{2^{p+1}(p+1)}} \\
&\quad \times \left\{ \sqrt[q]{(4-\lambda)|f'(ma_1)|^q + \lambda|f'(a_2)|^q} + \sqrt[q]{(4-3\lambda)|f'(ma_1)|^q + 3\lambda|f'(a_2)|^q} \right\}.
\end{aligned} \tag{2.8}$$

Remark 2.6. For $\lambda = 1$ in Corollary 2.5, we obtain

$$\begin{aligned}
|T_f(a_1, a_2)| &\leq \frac{(a_2 - ma_1)}{\sqrt[q]{8} \sqrt[p]{2^{p+1}(p+1)}} \\
&\quad \times \left\{ \sqrt[q]{3|f'(ma_1)|^q + |f'(a_2)|^q} + \sqrt[q]{|f'(ma_1)|^q + 3|f'(a_2)|^q} \right\}.
\end{aligned} \tag{2.9}$$

Corollary 2.7. For $\phi(t) = \frac{t^\alpha}{\Gamma(\alpha)}$ in Theorem 2.3, we get

$$\begin{aligned}
|T_{f,\Lambda_m}(\lambda; a_1, a_2)| &\leq \frac{2^{\alpha-1}\lambda(a_2 - ma_1)}{\sqrt[q]{8} \sqrt[p]{2^{p\alpha+1}(p\alpha+1)}} \\
&\quad \times \left\{ \sqrt[q]{(4-\lambda)|f'(ma_1)|^q + \lambda|f'(a_2)|^q} + \sqrt[q]{(4-3\lambda)|f'(ma_1)|^q + 3\lambda|f'(a_2)|^q} \right\}.
\end{aligned} \tag{2.10}$$

Corollary 2.8. For $\phi(t) = \frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ in Theorem 2.3, we have

$$\begin{aligned}
|T_{f,\Lambda_m}(\lambda; a_1, a_2)| &\leq \frac{2^{\frac{\alpha}{k}-1}\lambda(a_2 - ma_1)}{\sqrt[q]{8} \sqrt[p]{2^{\frac{p\alpha}{k}+1}\left(\frac{p\alpha}{k}+1\right)}} \\
&\quad \times \left\{ \sqrt[q]{(4-\lambda)|f'(ma_1)|^q + \lambda|f'(a_2)|^q} + \sqrt[q]{(4-3\lambda)|f'(ma_1)|^q + 3\lambda|f'(a_2)|^q} \right\}.
\end{aligned} \tag{2.11}$$

Corollary 2.9. For $\phi(t) = t(a_2 - t)^{\alpha-1}$ for $\alpha \in (0, 1)$ in Theorem 2.3, we obtain

$$|T_{f,\Lambda_m}(\lambda; a_1, a_2)| \leq \frac{\lambda\alpha(a_2 - ma_1)}{2\sqrt[q]{8[a_2^\alpha - ((ma_1 - a_2)\frac{\lambda}{2} + a_2)^\alpha]}} \sqrt[p]{B_{\Lambda_m}^*(\lambda; p)} \quad (2.12)$$

$$\times \left\{ \sqrt[q]{(4-\lambda)|f'(ma_1)|^q + \lambda|f'(a_2)|^q} + \sqrt[q]{(4-3\lambda)|f'(ma_1)|^q + 3\lambda|f'(a_2)|^q} \right\},$$

where

$$B_{\Lambda_m}^*(\lambda; p) := \frac{1}{\lambda\alpha^p(a_2 - ma_1)} \int_{(ma_1 - a_2)\frac{\lambda}{2} + a_2}^{a_2} (a_2^\alpha - t^\alpha)^p dt. \quad (2.13)$$

Corollary 2.10. For $\phi(t) = \frac{t}{\alpha} \exp\left[-\frac{1-\alpha}{\alpha}t\right]$ for $\alpha \in (0, 1)$ in Theorem 2.3, we get

$$|T_{f,\Lambda_m}(\lambda; a_1, a_2)| \leq \frac{\lambda(\alpha-1)(a_2 - ma_1)}{2\sqrt[q]{8\left\{\exp\left[-\frac{1-\alpha}{\alpha}(a_2 - ma_1)\frac{\lambda}{2}\right] - 1\right\}}} \sqrt[p]{B_{\Lambda_m}^\circ(\lambda; p)} \quad (2.14)$$

$$\times \left\{ \sqrt[q]{(4-\lambda)|f'(ma_1)|^q + \lambda|f'(a_2)|^q} + \sqrt[q]{(4-3\lambda)|f'(ma_1)|^q + 3\lambda|f'(a_2)|^q} \right\},$$

where

$$B_{\Lambda_m}^\circ(\lambda; p) := \frac{\alpha}{\lambda(\alpha-1)^{p+1}(a_2 - ma_1)} \int_0^{\exp\left[-\frac{1-\alpha}{\alpha}(a_2 - ma_1)\frac{\lambda}{2}\right] - 1} \frac{t^p}{t+1} dt. \quad (2.15)$$

Theorem 2.11. Let $f : P \rightarrow \mathbb{R}$ be a differentiable mapping on (ma_1, a_2) . If $|f'|^q$ is convex on P and $\lambda \in (0, 1]$ for $q \geq 1$, then

$$|T_{f,\Lambda_m}(\lambda; a_1, a_2)| \leq \frac{\lambda(a_2 - ma_1)}{2\Lambda_m\left(\frac{\lambda}{2}\right)} (B_{\Lambda_m}(\lambda; 1))^{1-\frac{1}{q}} \quad (2.16)$$

$$\times \left\{ \sqrt[q]{[B_{\Lambda_m}(\lambda; 1) - \lambda C_{\Lambda_m}(\lambda)]|f'(ma_1)|^q + \lambda C_{\Lambda_m}(\lambda)|f'(a_2)|^q} \right.$$

$$\left. + \sqrt[q]{[(1-\lambda)B_{\Lambda_m}(\lambda; 1) - \lambda C_{\Lambda_m}(\lambda)]|f'(ma_1)|^q + \lambda[B_{\Lambda_m}(\lambda; 1) - C_{\Lambda_m}(\lambda)]|f'(a_2)|^q} \right\},$$

where

$$C_{\Lambda_m}(\lambda) := \int_0^{\frac{\lambda}{2}} t\Lambda_m(\lambda t) dt \quad (2.17)$$

and $B_{\Lambda_m}(\lambda; 1)$ is defined as in Theorem 2.3.

Proof. Using Lemma 2.1, convexity of $|f'|^q$ and power mean inequality, we get

$$|T_{f,\Lambda_m}(\lambda; a_1, a_2)| \leq \frac{\lambda(a_2 - ma_1)}{2\Lambda_m\left(\frac{\lambda}{2}\right)} \times \left\{ \int_0^{\frac{\lambda}{2}} \Lambda_m(\lambda t) |f'(ma_1 + (\lambda t)(a_2 - ma_1))| dt \right.$$

$$\left. + \int_{\frac{\lambda}{2}}^1 \Lambda_m((1-t)\lambda) |f'(ma_1 + (\lambda t)(a_2 - ma_1))| dt \right\}$$

$$\begin{aligned}
&\leq \frac{\lambda(a_2 - ma_1)}{2\Lambda_m\left(\frac{\lambda}{2}\right)} \times \left\{ \left(\int_0^{\frac{1}{2}} \Lambda_m(\lambda t) dt \right)^{1-\frac{1}{q}} \left(\int_0^{\frac{1}{2}} \Lambda_m(\lambda t) |f'(ma_1 + (\lambda t)(a_2 - ma_1))|^q dt \right)^{\frac{1}{q}} \right. \\
&\quad \left. + \left(\int_{\frac{1}{2}}^1 \Lambda_m((1-t)\lambda) dt \right)^{1-\frac{1}{q}} \left(\int_{\frac{1}{2}}^1 \Lambda_m((1-t)\lambda) |f'(ma_1 + (\lambda t)(a_2 - ma_1))|^q dt \right)^{\frac{1}{q}} \right\} \\
&\leq \frac{\lambda(a_2 - ma_1)}{2\Lambda_m\left(\frac{\lambda}{2}\right)} (B_{\Lambda_m}(\lambda; 1))^{1-\frac{1}{q}} \times \left\{ \left(\int_0^{\frac{1}{2}} \Lambda_m(\lambda t) [(1-\lambda t)|f'(ma_1)|^q + (\lambda t)|f'(a_2)|^q] dt \right)^{\frac{1}{q}} \right. \\
&\quad \left. + \left(\int_{\frac{1}{2}}^1 \Lambda_m((1-t)\lambda) [(1-\lambda t)|f'(ma_1)|^q + (\lambda t)|f'(a_2)|^q] dt \right)^{\frac{1}{q}} \right\} \\
&= \frac{\lambda(a_2 - ma_1)}{2\Lambda_m\left(\frac{\lambda}{2}\right)} (B_{\Lambda_m}(\lambda; 1))^{1-\frac{1}{q}} \\
&\quad \times \left\{ \sqrt[q]{[B_{\Lambda_m}(\lambda; 1) - \lambda C_{\Lambda_m}(\lambda)]|f'(ma_1)|^q + \lambda C_{\Lambda_m}(\lambda)|f'(a_2)|^q} \right. \\
&\quad \left. + \sqrt[q]{[(1-\lambda)B_{\Lambda_m}(\lambda; 1) - \lambda C_{\Lambda_m}(\lambda)]|f'(ma_1)|^q + \lambda[B_{\Lambda_m}(\lambda; 1) - C_{\Lambda_m}(\lambda)]|f'(a_2)|^q} \right\}.
\end{aligned}$$

The proof of Theorem 2.11 is completed. \square

Corollary 2.12. For $q = 1$ in Theorem 2.11, we get

$$|T_{f,\Lambda_m}(\lambda; a_1, a_2)| \leq \frac{\lambda(a_2 - ma_1)}{2\Lambda_m\left(\frac{\lambda}{2}\right)} \quad (2.18)$$

$$\times \left\{ [(2-\lambda)B_{\Lambda_m}(\lambda; 1) - 2\lambda C_{\Lambda_m}(\lambda)]|f'(ma_1)| + \lambda B_{\Lambda_m}(\lambda; 1)|f'(a_2)| \right\}.$$

Corollary 2.13. For $\phi(t) = t$ in Theorem 2.11, we have

$$|T_{f,\Lambda_m}(\lambda; a_1, a_2)| \leq \frac{\lambda(a_2 - ma_1)}{8\sqrt[q]{3}} \quad (2.19)$$

$$\times \left\{ \sqrt[q]{(3-\lambda)|f'(ma_1)|^q + \lambda|f'(a_2)|^q} + \sqrt[q]{(4\lambda-3)|f'(ma_1)|^q + 2\lambda|f'(a_2)|^q} \right\}.$$

Remark 2.14. For $\lambda = 1$ in Corollary 2.13, we obtain

$$|T_f(a_1, a_2)| \leq \frac{(a_2 - ma_1)}{8\sqrt[q]{3}} \quad (2.20)$$

$$\times \left\{ \sqrt[q]{2|f'(ma_1)|^q + |f'(a_2)|^q} + \sqrt[q]{|f'(ma_1)|^q + 2|f'(a_2)|^q} \right\}.$$

Corollary 2.15. For $\phi(t) = \frac{t^\alpha}{\Gamma(\alpha)}$ in Theorem 2.11, we get

$$|T_{f,\Lambda_m}(\lambda; a_1, a_2)| \leq \frac{\lambda(a_2 - ma_1) \Gamma(\alpha + 1)}{4 \Gamma(\alpha + 2)} \tag{2.21}$$

$$\times \left\{ \sqrt[q]{\left(1 - \frac{\lambda(\alpha + 1)}{2(\alpha + 2)}\right) |f'(ma_1)|^q + \frac{\lambda(\alpha + 1)}{2(\alpha + 2)} |f'(a_2)|^q} \right.$$

$$\left. + \sqrt[q]{\left((1 - \lambda) - \frac{\lambda(\alpha + 1)}{2(\alpha + 2)}\right) |f'(ma_1)|^q + \frac{\lambda(\alpha + 3)}{2(\alpha + 2)} |f'(a_2)|^q} \right\}.$$

Corollary 2.16. For $\phi(t) = \frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ in Theorem 2.11, we have

$$|T_{f,\Lambda_m}(\lambda; a_1, a_2)| \leq \frac{\lambda(a_2 - ma_1) \Gamma_k(\alpha + k)}{4 \Gamma_k(\alpha + k + 1)} \tag{2.22}$$

$$\times \left\{ \sqrt[q]{\left(1 - \frac{\lambda(\frac{\alpha}{k} + 1)}{2(\frac{\alpha}{k} + 2)}\right) |f'(ma_1)|^q + \frac{\lambda(\frac{\alpha}{k} + 1)}{2(\frac{\alpha}{k} + 2)} |f'(a_2)|^q} \right.$$

$$\left. + \sqrt[q]{\left((1 - \lambda) - \frac{\lambda(\frac{\alpha}{k} + 1)}{2(\frac{\alpha}{k} + 2)}\right) |f'(ma_1)|^q + \frac{\lambda(\frac{\alpha}{k} + 3)}{2(\frac{\alpha}{k} + 2)} |f'(a_2)|^q} \right\}.$$

Corollary 2.17. For $\phi(t) = t(a_2 - t)^{\alpha-1}$ for $\alpha \in (0, 1)$ in Theorem 2.11, we obtain

$$|T_{f,\Lambda_m}(\lambda; a_1, a_2)| \leq \frac{\lambda\alpha(a_2 - ma_1)}{2[a_2^\alpha - ((ma_1 - a_2)\frac{\lambda}{2} + a_2)^\alpha]} (B_{\Lambda_m}^*(\lambda; 1))^{1-\frac{1}{q}} \tag{2.23}$$

$$\times \left\{ \sqrt[q]{[B_{\Lambda_m}^*(\lambda; 1) - \lambda C_{\Lambda_m}^*(\lambda)] |f'(ma_1)|^q + \lambda C_{\Lambda_m}^*(\lambda) |f'(a_2)|^q} \right.$$

$$\left. + \sqrt[q]{[(1 - \lambda)B_{\Lambda_m}^*(\lambda; 1) - \lambda C_{\Lambda_m}^*(\lambda)] |f'(ma_1)|^q + \lambda [B_{\Lambda_m}^*(\lambda; 1) - C_{\Lambda_m}^*(\lambda)] |f'(a_2)|^q} \right\},$$

where

$$C_{\Lambda_m}^*(\lambda) := \frac{1}{\alpha} \int_0^{\frac{1}{2}} t [a_2^\alpha - ((ma_1 - a_2)\lambda t + a_2)^\alpha] dt \tag{2.24}$$

and $B_{\Lambda_m}^*(\lambda; 1)$ is defined by Eq (2.13) for $p = 1$.

Corollary 2.18. For $\phi(t) = \frac{t}{\alpha} \exp\left[-\frac{1-\alpha}{\alpha}t\right]$ for $\alpha \in (0, 1)$ in Theorem 2.11, we get

$$|T_{f,\Lambda_m}(\lambda; a_1, a_2)| \leq \frac{\lambda(\alpha - 1)(a_2 - ma_1)}{2\left\{\exp\left[-\frac{1-\alpha}{\alpha}(a_2 - ma_1)\frac{\lambda}{2}\right] - 1\right\}} \sqrt[p]{B_{\Lambda_m}^\circ(\lambda; 1)} \tag{2.25}$$

$$\times \left\{ \sqrt[q]{[B_{\Lambda_m}^\circ(\lambda; 1) - \lambda C_{\Lambda_m}^\circ(\lambda)] |f'(ma_1)|^q + \lambda C_{\Lambda_m}^\circ(\lambda) |f'(a_2)|^q} \right.$$

$$+ \sqrt[q]{[(1-\lambda)B_{\Lambda_m}^\circ(\lambda; 1) - \lambda C_{\Lambda_m}^\circ(\lambda)]|f'(ma_1)|^q + \lambda[B_{\Lambda_m}^\circ(\lambda; 1) - C_{\Lambda_m}^\circ(\lambda)]|f'(a_2)|^q},$$

where

$$C_{\Lambda_m}^\circ(\lambda) := \frac{1}{(\alpha-1)} \int_0^{\frac{1}{2}} \left\{ \exp \left[\left(-\frac{1-\alpha}{\alpha} \right) (a_2 - ma_1) \lambda t \right] - 1 \right\} dt. \quad (2.26)$$

and $B_{\Lambda_m}^\circ(\lambda; 1)$ is defined by eq (2.15) for $p = 1$.

3. Applications

1.

$$A := A(\wp_1, \wp_2) = \frac{\wp_1 + \wp_2}{2},$$

2.

$$H := H(\wp_1, \wp_2) = \frac{2}{\frac{1}{\wp_1} + \frac{1}{\wp_2}},$$

3.

$$L := L(\wp_1, \wp_2) = \frac{\wp_2 - \wp_1}{\ln |\wp_2| - \ln |\wp_1|},$$

4.

$$L_r := L_r(\wp_1, \wp_2) = \left[\frac{\wp_2^{r+1} - \wp_1^{r+1}}{(r+1)(\wp_2 - \wp_1)} \right]^{\frac{1}{r}}; \quad r \in \mathbb{Z} \setminus \{-1, 0\}.$$

From Section 2, we obtain:

Proposition 3.1. Let $m \in (0, 1]$ and $a_1, a_2 \in \mathbb{R} \setminus \{0\}$, where $a_1 < a_2$. Then for $r \in \mathbb{N}$ and $r \geq 2$, where $q > 1$ and $p^{-1} + q^{-1} = 1$,

$$\left| A^r(ma_1, a_2) - L_r^r(ma_1, a_2) \right| \leq \frac{r(a_2 - ma_1)}{\sqrt[q]{4} \sqrt[p]{2^{p+1}(p+1)}} \quad (3.1)$$

$$\times \left\{ \sqrt[q]{A(3|ma_1|^{q(r-1)}, |a_2|^{q(r-1)})} + \sqrt[q]{A(|ma_1|^{q(r-1)}, 3|a_2|^{q(r-1)})} \right\}.$$

Proof. Taking $\lambda = 1$, $f(t) = t^r$ and $\phi(t) = t$, in Theorem 2.3. □

Proposition 3.2. Let $m \in (0, 1]$ and $a_1, a_2 \in \mathbb{R} \setminus \{0\}$, where $a_1 < a_2$. Then for $q > 1$ and $p^{-1} + q^{-1} = 1$,

$$\left| \frac{1}{A(ma_1, a_2)} - \frac{1}{L(ma_1, a_2)} \right| \leq \sqrt[q]{\frac{3}{4}} \frac{(a_2 - ma_1)}{\sqrt[p]{2^{p+1}(p+1)}} \quad (3.2)$$

$$\times \left\{ \frac{1}{\sqrt[q]{H(3|ma_1|^{2q}, |a_2|^{2q})}} + \frac{1}{\sqrt[q]{H(|ma_1|^{2q}, 3|a_2|^{2q})}} \right\}.$$

Proof. Taking $\lambda = 1$, $f(t) = \frac{1}{t}$ and $\phi(t) = t$, in Theorem 2.3. □

Proposition 3.3. Let $m \in (0, 1]$ and $a_1, a_2 \in \mathbb{R} \setminus \{0\}$, where $a_1 < a_2$. Then for $r \in \mathbb{N}$ and $r \geq 2$, where $q \geq 1$,

$$\begin{aligned} & \left| A^r(ma_1, a_2) - L_r^r(ma_1, a_2) \right| \leq \sqrt[q]{\frac{2}{3}} \frac{r(a_2 - ma_1)}{8} \\ & \times \left\{ \sqrt[q]{A(2|ma_1|^{q(r-1)}, |a_2|^{q(r-1)})} + \sqrt[q]{A(|ma_1|^{q(r-1)}, 2|a_2|^{q(r-1)})} \right\}. \end{aligned} \quad (3.3)$$

Proof. Taking $\lambda = 1$, $f(t) = t^r$ and $\phi(t) = t$, in Theorem 2.11. \square

Proposition 3.4. Let $m \in (0, 1]$ and $a_1, a_2 \in \mathbb{R} \setminus \{0\}$, where $a_1 < a_2$. Then for $q \geq 1$, the following inequality hold:

$$\begin{aligned} & \left| \frac{1}{A(ma_1, a_2)} - \frac{1}{L(ma_1, a_2)} \right| \leq \sqrt[q]{\frac{4}{3}} \frac{(a_2 - ma_1)}{8} \\ & \times \left\{ \frac{1}{\sqrt[q]{H(2|ma_1|^{2q}, |a_2|^{2q})}} + \frac{1}{\sqrt[q]{H(|ma_1|^{2q}, 2|a_2|^{2q})}} \right\}. \end{aligned} \quad (3.4)$$

Proof. Taking $\lambda = 1$, $f(t) = \frac{1}{t}$ and $\phi(t) = t$, in Theorem 2.11. \square

Next, we provide some new error estimates for the midpoint formula. Let Q be the partition of $a_1 = \ell_0 < \ell_1 < \dots < \ell_k = a_2$ of $[a_1, a_2]$. Let consider the following quadrature formula:

$$\int_{a_1}^{a_2} f(x)dx = M(f, Q) + E(f, Q),$$

where

$$M(f, Q) = \sum_{i=0}^{k-1} f\left(\frac{\ell_i + \ell_{i+1}}{2}\right) (\ell_{i+1} - \ell_i)$$

is the midpoint version and $E(f, Q)$ is denote their associated approximation error.

Proposition 3.5. Let $f : [a_1, a_2] \rightarrow \mathbb{R}$ be a differentiable function on (a_1, a_2) , where $a_1 < a_2$. If $|f'|^q$ is convex on $[a_1, a_2]$ for $q > 1$ and $p^{-1} + q^{-1} = 1$, then

$$\begin{aligned} & |E(f, Q)| \leq \frac{1}{\sqrt[q]{8} \sqrt[q]{2^{p+1}(p+1)}} \times \sum_{i=0}^{k-1} (\ell_{i+1} - \ell_i)^2 \\ & \times \left\{ \sqrt[q]{3|f'(\ell_i)|^q + |f'(\ell_{i+1})|^q} + \sqrt[q]{|f'(\ell_i)|^q + 3|f'(\ell_{i+1})|^q} \right\}. \end{aligned} \quad (3.5)$$

Proof. Applying Theorem 2.3 for $\lambda = m = 1$ and $\phi(t) = t$ on $[\ell_i, \ell_{i+1}]$ ($i = 0, \dots, k-1$) of Q , we have

$$\begin{aligned} & \left| f\left(\frac{\ell_i + \ell_{i+1}}{2}\right) - \frac{1}{(\ell_{i+1} - \ell_i)} \int_{\ell_i}^{\ell_{i+1}} f(x)dx \right| \leq \frac{(\ell_{i+1} - \ell_i)}{\sqrt[q]{8} \sqrt[q]{2^{p+1}(p+1)}} \\ & \times \left\{ \sqrt[q]{3|f'(\ell_i)|^q + |f'(\ell_{i+1})|^q} + \sqrt[q]{|f'(\ell_i)|^q + 3|f'(\ell_{i+1})|^q} \right\}. \end{aligned} \quad (3.6)$$

Hence from (3.6), we get

$$\begin{aligned}
 |E(f, Q)| &= \left| \int_{a_1}^{a_2} f(x) dx - M(f, Q) \right| \\
 &\leq \left| \sum_{i=0}^{k-1} \left\{ \int_{\ell_i}^{\ell_{i+1}} f(x) dx - f\left(\frac{\ell_i + \ell_{i+1}}{2}\right)(\ell_{i+1} - \ell_i) \right\} \right| \\
 &\leq \sum_{i=0}^{k-1} \left| \left\{ \int_{\ell_i}^{\ell_{i+1}} f(x) dx - f\left(\frac{\ell_i + \ell_{i+1}}{2}\right)(\ell_{i+1} - \ell_i) \right\} \right| \\
 &\leq \frac{1}{\sqrt[p]{8} \sqrt[p]{2^{p+1}(p+1)}} \times \sum_{i=0}^{k-1} (\ell_{i+1} - \ell_i)^2 \\
 &\times \left\{ \sqrt[q]{3|f'(\ell_i)|^q + |f'(\ell_{i+1})|^q} + \sqrt[q]{|f'(\ell_i)|^q + 3|f'(\ell_{i+1})|^q} \right\}.
 \end{aligned}$$

□

Proposition 3.6. Let $f : [a_1, a_2] \rightarrow \mathbb{R}$ be a differentiable function on (a_1, a_2) , where $a_1 < a_2$. If $|f'|^q$ is convex on $[a_1, a_2]$ for $q \geq 1$, then

$$\begin{aligned}
 |E(f, Q)| &\leq \frac{1}{8\sqrt[3]{3}} \times \sum_{i=0}^{k-1} (\ell_{i+1} - \ell_i)^2 \\
 &\times \left\{ \sqrt[q]{2|f'(\ell_i)|^q + |f'(\ell_{i+1})|^q} + \sqrt[q]{|f'(\ell_i)|^q + 2|f'(\ell_{i+1})|^q} \right\}.
 \end{aligned} \tag{3.7}$$

Proof. The proof is analogous as to that of Proposition 3.5 but use Theorem 2.11. □

Remark 3.7. Applying our Theorems 2.3 and 2.11, where $m = 1$, for appropriate choices of function $\phi(t)$, we can deduce some new bounds for midpoint formula using above ideas and techniques. The details are left to the interested reader.

4. Conclusion

The authors have proved an identity for a generalized integral operator via differentiable function. By applying the established identity, the generalized trapezium type integral inequalities have been discovered. Some applications of presented results have been analyzed. Interested reader can establish new inequalities by using different integral operators and they can be applied in convex analysis, optimization and different area of pure and applied mathematics.

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Conflict of interest

The authors declare that they have no competing interests.

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