A note on characterization of h-convex functions via Hermite-Hadamard type inequality

DOI: 10.15393/j3.art.2019.5790

UDC 517.51

M. Rostamian Delavar, S. S. Dragomir, M. De La Sen

A NOTE ON CHARACTERIZATION OF h-CONVEX FUNCTIONS VIA HERMITE-HADAMARD TYPE INEQUALITY

Abstract. A characterization of h-convex function via Hermite-Hadamard inequality related to the h-convex functions is investigated. In fact it is determined that under what conditions a function is h-convex, if it satisfies the h-convex version of Hermite-Hadamard inequality.

Key words: *h-convex function, Hermite-Hadamard inequality*

2010 Mathematical Subject Classification: *26A51, 26D15, 52A01*

1. Introduction. The following result is well-known in the literature:

Theorem 1. [6] A function $f : (a,b) \subset \mathbb{R} \to \mathbb{R}$ is convex if and only if

$$f\left(\frac{x+y}{2}\right) \leqslant \frac{1}{y-x}\int_a^b f(t)dt \leqslant \frac{f(x)+f(y)}{2} \tag{1}$$

holds for all $x, y \in (a,b)$ with $x \neq y$.

Inequality (1) is known as the Hermite-Hadamard integral inequality for convex functions. Note that the left-hand part and the right-hand part of (1) separately are equivalent to the convexity of f (see [5,6]).

In 2006, the concept of h-convex functions related to the nonnegative real functions has been introduced in [9] by S. Varošanec. This class includes a large class of nonnegative functions, such as nonnegative convex functions, Godunova-Levin functions [3], s-convex functions in the second sense [1], and P-functions [2]. In [4], A. Házy used the following definition of h-convex functions, which is a generalization of convexexity:

Definition 1. *Let $h : [0, 1] \to \mathbb{R}$ be a function, such that $h \not\equiv 0$. We say that $f : (a, b) \to \mathbb{R}$ is an h-convex function, if for all $x, y \in (a, b)$, $\lambda \in [0, 1]$, we have*

$$f\big(\lambda x + (1 - \lambda)y\big) \leqslant h(\lambda)f(x) + h(1 - \lambda)f(y). \tag{2}$$

We use this definition for the real functions defined on open intervals $(a, b) \subseteq \mathbb{R}$ in this paper. The h-convex version of the Hermite-Hadamard inequality was introduced in [8] by Sarikaya et al. as the following:

Theorem 2. *Let $f : I \to [0, \infty]$ be an integrable h-convex function. If $a, b \in I$, with $a < b$, then*

$$\frac{1}{2h(\frac{1}{2})} f\left(\frac{a + b}{2}\right) \leqslant \frac{1}{b - a} \int_a^b f(x)dx \leqslant \big[f(a) + f(b)\big]\left(\int_0^1 h(t)dt\right). \tag{3}$$

Motivated by the abovementioned works and results, we, in this paper, reply to the problem of conditions h-convexity of a function that satisfies (3). Since inequality (3) is double, we separate the problem to the right-hand and the left-hand versions, for the sake of convenience.

2. Main results. To achieve our main results about the characterization of an h-convex function via (3), we introduce a primary definition along with an example and then establish a basic lemma related to h-convex functions.

Definition 2. *A function $h : [0, 1] \to \mathbb{R}$ is said to be self-concave if*

$$h\big(zx + (1 - z)y\big) \geqslant h(z)h(x) + h(1 - z)h(y),$$

for all $z \in (0, 1)$ and $x, y \in [0, 1]$.

We can find some simple functions that are self concave.

Example. Consider the function $h(x) = x^n$ for $n \in \mathbb{N}$ and $x \in [0, 1]$. It is not hard to see that this function is self-concave. In fact, since the function h is nonnegative,

$$h\big(\lambda x + (1 - \lambda)y\big) = \big(\lambda x + (1 - \lambda)y\big)^n = \textstyle\sum_{i=0}^n \binom{n}{i} \big(\lambda x\big)^{n-i}\big((1 - \lambda)y\big)^i \geqslant$$
$$\geqslant \binom{n}{0} \big(\lambda x\big)^n + \binom{n}{n} \big((1 - \lambda)y\big)^n = h(\lambda)h(x) + h(1 - \lambda)h(y).$$

Now consider the function $h(x) = \tan(x)$, for $x \in (0, 1)$ and $z \in (0, 1)$. Expanding this function and using the self-concavity of x^n for $n \in \mathbb{N}$ and

$x \in [0,1]$, we get

$$\tan\big(\lambda x + (1-\lambda)y\big) = \big(\lambda x + (1-\lambda)y\big) + \frac{1}{3}\big(\lambda x + (1-\lambda)y\big)^3 +$$

$$+\frac{2}{15}\big(\lambda x+(1-\lambda)y\big)^5+\frac{17}{315}\big(\lambda x+(1-\lambda)y\big)^7+\frac{62}{2835}\big(\lambda x+(1-\lambda)y\big)^9+\ldots \geqslant$$

$$\geqslant \lambda x + \frac{1}{3}(\lambda x)^3 + \frac{2}{15}(\lambda x)^5 + \frac{17}{315}(\lambda x)^7 + \frac{62}{2835}(\lambda x)^9 + \cdots + \big((1-\lambda)y\big)$$

$$+\frac{1}{3}\big((1-\lambda)y\big)^3+\frac{2}{15}\big((1-\lambda)y\big)^5+\frac{17}{315}\big((1-\lambda)y\big)^7+\frac{62}{2835}\big((1-\lambda)y\big)^9+\cdots =$$

$$= \tan(\lambda x) + \tan((1-\lambda)y) > \tan(\lambda)\tan(x) + \tan(1-\lambda)\tan(y),$$

which implies the self-concavity of $h(x) = \tan(x)$ on $(0,1)$. Note that we have used the fact that $\tan(xy) > \tan(x)\tan(y)$ for all $x, y \in (0,1)$.

The following lemma plays an important role in obtaining our expected results.

Lemma 1. *Let $f:(a,b) \to \mathbb{R}$ be a continuous function and $h:[0,1] \to \mathbb{R}$ be a continuous self-concave function. Suppose that for any $x, y \in (a,b)$ with $x \neq y$ there is a $\lambda \in (0,1)$ such that $f\big(\lambda x + (1-\lambda)y\big) \leqslant h(\lambda)f(x) + h(1-\lambda)f(y)$. Then f is h-convex on (a,b).*

Proof. Without loss of generality, consider $x, y \in (a,b)$ with $x < y$. Define

$$M_{x,y} = \Big\{\lambda \in [0,1];\, f\big(\lambda x + (1-\lambda)y\big) \leqslant h(\lambda)f(x) + h(1-\lambda)f(y)\Big\}.$$

It is obvious that $M_{x,y}$ is nonempty. Since f and h are continuous on their domains, $M_{x,y}$ is closed in $[0,1]$. We prove that $M_{x,y} = [0,1]$. On the contrary, suppose that $M_{x,y}$ is a proper subset of $[0,1]$; then we can find $\alpha, \beta \in M_{x,y}$ such that $(\alpha, \beta) \subset [0,1] \setminus M_{x,y}$. Set

$$w = \alpha x + (1-\alpha)y \quad , \quad z = \beta x + (1-\beta)y. \tag{4}$$

From the assumption, there is a $\lambda \in (0,1)$ such that

$$f\big(\lambda w + (1-\lambda)z\big) \leqslant h(\lambda)f(w) + h(1-\lambda)f(z). \tag{5}$$

Also

$$\begin{cases} f(w) = f\big(\alpha x + (1-\alpha)y\big) \leqslant h(\alpha)f(x) + h(1-\alpha)f(y), \\ f(z) = f\big(\beta x + (1-\beta)y\big) \leqslant h(\beta)f(x) + h(1-\beta)f(y). \end{cases} \tag{6}$$

Set $t = \lambda\alpha + (1-\lambda)\beta$. It is clear that $t \in (\alpha, \beta)$ and $t \notin M_{x,y}$. Therefore, from the self-concavity of h and relations (4)-(6), we have

$$f\big(tx + (1-t)y\big) > h(t)f(x) + h(1-t)f(y) =$$

$$= h\big(\lambda\alpha + (1-\lambda)\beta\big)f(x) + h\big(1 - (\lambda\alpha + (1-\lambda)\beta)\big)f(y) =$$

$$= h\big(\lambda\alpha + (1-\lambda)\beta\big)f(x) + h\big(\lambda(1-\alpha) + (1-\lambda)(1-\beta)\big)f(y) \geqslant$$

$$\geqslant \Big[h(\lambda)h(\alpha)+h(1-\lambda)h(\beta)\Big]f(x)+\Big[h(\lambda)h(1-\alpha)+h(1-\lambda)h(1-\beta)\Big]f(y) =$$

$$= h(\lambda)\Big[h(\alpha)f(x) + h(1-\alpha)f(y)\Big] + h(1-\lambda)\Big[h(\beta)f(x) + h(1-\beta)f(y)\Big] \geqslant$$

$$\geqslant h(\lambda)f(w) + h(1-\lambda)f(z) \geqslant f\big(\lambda w + (1-\lambda)z\big).$$

On the other hand,

$$\lambda w + (1-\lambda)z = \lambda\big(\alpha x + (1-\alpha)y\big) + (1-\lambda)\big(\beta x + (1-\beta)y\big) =$$

$$= \Big[\lambda\alpha + (1-\lambda)\beta\Big]x + \Big[\lambda(1-\alpha) + (1-\lambda)(1-\beta)\Big]y =$$

$$= \Big[\lambda\alpha + (1-\lambda)\beta\Big]x + \Big[1 - \big(\lambda\alpha + (1-\lambda)\beta\big)\Big]y = tx + (1-t)y.$$

So,

$$f\big(tx + (1-t)y\big) = f\big(\lambda w + (1-\lambda)z\big) < f\big(tx + (1-t)y\big),$$

which is a contradiction. It follows that $M_{x,y}$ is not a proper subset of $[0,1]$ and hence $M_{x,y} = [0,1]$. Since this happens for any $x, y \in (a,b)$ with $x < y$, we conclude that f is h-convex on (a,b). □

Theorem 3. *Let $f : (a,b) \to \mathbb{R}$ be a continuous function. Also suppose that $h : [0,1] \to \mathbb{R}$ is a continuous self-concave function, such that*

$$\frac{1}{y-x}\int_x^y f(t)dt \leqslant [f(x) + f(y)]\left(\int_0^1 h(t)dt\right),$$

for all $x, y \in (a,b)$ with $x \neq y$. Then f is h-convex on (a,b).

Proof. Suppose that f is not h-convex on (a,b). Then, by Lemma 1, there are $x, y \in (a,b)$ with $x < y$ such that

$$f(tx + (1-t)y) > h(t)f(x) + h(1-t)f(y) \quad \forall t \in (0,1).$$

For such x and y,

$$\frac{1}{y-x}\int_x^y f(t)dt = \int_0^1 f(tx+(1-t)y)dt > \int_0^1 [h(t)f(x)+h(1-t)f(y)]dt =$$

$$= \left(\int_0^1 h(t)dt\right)f(x) + \left(\int_0^1 h(1-t)dt\right)f(y) = \big[f(x)+f(y)\big]\left(\int_0^1 h(t)dt\right).$$

This is a contradiction. Hence, f is h-convex on (a,b). □

The following lemma, along with Lemma 1, are the base for characterization of a h-convex function via the left-hand side of (3).

Lemma 2. (Also see Theorem 1.1.4 in [5].) Suppose that $\varphi : [a,b] \to \mathbb{R}$ is a continuous function such that $\varphi(a) = \varphi(b) = 0$ and $\varphi(t) > 0$ for some $t \in (a,b)$. Then there exists an $x \in (a,b)$ such that

$$\varphi(x) = \max_{a \leqslant y \leqslant b} \varphi(y) \text{ and } \varphi(x) > \varphi(y) \text{ for all } a \leqslant y < x.$$

Proof. From Theorem 4.16 in [7], φ attains its maximum α in $[a,b]$. From the assumption, we have $\alpha \geqslant \varphi(t) > 0$. Set $M = \{y \in [a,b]; \varphi(y) = \alpha\}$. Since φ is continuous, M is a nonempty compact subset of $[a,b]$, such that $a, b \notin M$. If we put $x = \inf\{y; y \in M\}$, then

$$\varphi(x) = \alpha = \max_{a \leqslant y \leqslant b} \varphi(y),$$

and $f(y) < f(x)$ for all $a \leqslant y < x$. □

In what follows, we assume that the function $h : [0,1] \to \mathbb{R}$ satisfies the conditions

$$\begin{cases} h(\lambda) + h(1-\lambda) = 1 \text{ for all } \lambda \in (0,1), \\ h(0) = 0. \end{cases} \tag{7}$$

Lemma 3. *Let $h : [0,1] \to \mathbb{R}$ be a continuous self-concave function. Suppose that $f : (a,b) \to \mathbb{R}$ is a continuous function and for any $x \in (a,b)$, $\varepsilon > 0$, there exist $y, z \in (a,b) \cap (x-\varepsilon, x+\varepsilon)$ with $y < x < z$ such that*

$$f(x) = f\big(\lambda y + (1-\lambda)z\big) \leqslant h(\lambda)f(y) + h(1-\lambda)f(z) \text{ for some } \lambda \in (0,1).$$

Then f is h-convex on (a,b).

Proof. If f is not h-convex, then by Lemma 1, there are $x_1, x_2 \in (a, b)$ with $x_1 \neq x_2$ (assume that $x_1 < x_2$) such that

$$f\big(\lambda x_1 + (1-\lambda)x_2\big) > h(\lambda)f(x_1) + h(1-\lambda)f(x_2) \text{ for all } \lambda \in (0,1). \quad (8)$$

Consider the function $g : [x_1, x_2] \to \mathbb{R}$ defined by

$$g(y) = g\big(\lambda x_1 + (1-\lambda)x_2\big) =$$

$$= f\big(\lambda x_1 + (1-\lambda)x_2\big) - f(x_1) - \frac{f(x_2) - f(x_1)}{x_2 - x_1}\Big(h(\lambda)x_1 + h(1-\lambda)x_2 - x_1\Big).$$

It is clear that g is continuous on $[x_1, x_2]$ and $g(x_1) = g(x_2) = 0$. Also, from (7) and (8), we get

$$g\big(\lambda x_1 + (1-\lambda)x_2\big) = f\big(\lambda x_1 + (1-\lambda)x_2\big) - f(x_1) - \quad (9)$$

$$-\frac{f(x_2) - f(x_1)}{x_2 - x_1}\Big((1-h(\lambda))x_2 - (1-h(\lambda))x_1\Big) =$$

$$= f\big(\lambda x_1 + (1-\lambda)x_2\big) - h(\lambda)f(x_1) - h(1-\lambda)f(x_2) > 0.$$

Lemma 2 and (9) imply that there is an $x \in (x_1, x_2)$ such that

$$g(x) = \max_{x_1 \leqslant y \leqslant x_2} g(y) \text{ and } g(x) > g(y) \text{ for } x_1 \leqslant y < x. \quad (10)$$

Hence, $x = tx_1 + (1-t)x_2$ for some $0 < t < 1$. Now choose $x_0, y_0 \in [x_1, x_2]$ such that $x_1 \leqslant x_0 < x < y_0 \leqslant x_2$. Therefore, from (10) for any $\lambda \in (0,1)$,

$$g(x) = [h(\lambda) + h(1-\lambda)]g(x) > h(\lambda)g(x_0) + h(1-\lambda)g(y_0). \quad (11)$$

$$f(x) - f(x_1) - \frac{f(x_2) - f(x_1)}{x_2 - x_1}\Big(h(\lambda)x_0 + h(1-\lambda)y_0 - x_1\Big) > \quad (12)$$

$$> h(\lambda)\Big[f(x_0) - f(x_1) - \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x_0 - x_1)\Big] +$$

$$+ h(1-\lambda)\Big[f(y_0) - f(x_1) - \frac{f(x_2) - f(x_1)}{x_2 - x_1}(y_0 - x_1)\Big].$$

From (7) we deduce, simplifying (12):

$$f(x) > h(\lambda)f(x_0) + h(1-\lambda)f(y_0) \text{ for all } \lambda \in (0,1). \quad (13)$$

Since x_0, y_0, and λ are arbitrary, (13) contradicts the assumption. Hence, f is an h-convex function on (a, b). □

Using Lemma 3, as an immediate consequence we have two following lemmas. For more details about this kind of results related to the convex functions, see [6].

Corollary 1. *Let $h : [0, 1] \to \mathbb{R}$ be a continuous self-concave function. Suppose that $f : (a, b) \to \mathbb{R}$ is a continuous function and for any $x \in (a, b)$, $\varepsilon > 0$, there exists a $\delta \in (0, \varepsilon)$ such that*

$$f(x) \leqslant h(1/2)\big[f(x-\delta) + f(x+\delta)\big].$$

Then f is h-convex on (a, b).

Proof. In Lemma 3, take $y = x - \delta$, $z = x + \delta$ and $\lambda = 1/2$. □

Lemma 4. *Let $h : [0, 1] \to \mathbb{R}$ be a continuous self-concave function. Suppose that $f : (a, b) \to \mathbb{R}$ is a continuous function and for any $x \in (a, b)$, $\varepsilon > 0$, there exists $\delta \in (0, \varepsilon)$ such that*

$$f(x) \leqslant \frac{h(1/2)}{\delta} \int_{x-\delta}^{x+\delta} f(u)du.$$

Then f is h-convex on (a, b).

Proof. Suppose that f is not h-convex on (a, b). From Corollary 1, there are $x \in (a, b)$ and $\varepsilon > 0$ such that $a < x - \varepsilon < x + \varepsilon < b$ and

$$f(x) > h(1/2)\big[f(x-\delta) + f(x+\delta)\big] \text{ for any } 0 < \delta < \varepsilon.$$

Integrating with respect to δ in the above inequality, we get

$$\frac{1}{h(1/2)} \int_0^{\delta} f(x)dt > \int_0^{\delta} f(x-t)dt + \int_0^{\delta} f(x+t)dt =$$

$$= -\int_x^{x-\delta} f(u)du + \int_x^{x+\delta} f(u)du = \int_{x-\delta}^{x+\delta} f(u)du.$$

So,

$$f(x) \cdot \delta \leqslant h(1/2) \int_{x-\delta}^{x+\delta} f(u)du.$$

This contradicts the assumption and, hence, f is h-convex on (a, b). □

Now, using Lemma 4, we can obtain a characterization-type theorem for h-convex functions via the left-hand side of (3).

Theorem 4. *Let $h : [0, 1] \to \mathbb{R}$ be a continuous self-concave function. Suppose that $f : (a, b) \to \mathbb{R}$ is a continuous function and for all $y, z \in (a, b)$ with $y \neq z$ we have*

$$\frac{1}{2h(1/2)} f\left(\frac{y+z}{2}\right) \leqslant \frac{1}{z-y} \int_y^z f(u)du; \tag{14}$$

then f is h-convex on (a, b).

Proof. Suppose that f is not h-convex on (a,b). From Lemma 4, there exist $x \in (a, b)$ and $\varepsilon > 0$ such that for all $\delta \in (0, \varepsilon)$

$$f(x) > \frac{h(1/2)}{\delta} \int_{x-\delta}^{x+\delta} f(u)du.$$

Now, if we choose $\delta < \varepsilon$ and $y, z \in (a, b)$ with $y < z$ such that

$$\begin{cases} x = \frac{1}{2}y + \frac{1}{2}z, \\ x - y = z - x = \delta, \end{cases}$$

then we have

$$f\left(\frac{y+z}{2}\right) > \frac{2h(1/2)}{z-y} \int_y^z f(u)du.$$

This contradicts (14). Thus, f is h-convex on (a, b). □

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Received December 18, 2018.
In revised form, March 11, 2019.
Accepted March 12, 2019.
Published online April 1, 2019.

M. Rostamian Delavar
Department of Mathematics, Faculty of Basic Sciences, University of Bojnord, Bojnord, Iran
E-mail: m.rostamian@ub.ac.ir

S. S. Dragomir
Mathematics, College of Engineering & Science, Victoria University,
PO Box 14428, Melbourne City, MC 8001, Australia
E-mail: sever.dragomir@vu.edu.au

M. De La Sen
Institute of Research and Development of Processes, University of Basque Country, Campus of Leioa (Bizkaia) - Aptdo. 644 - Bilbao, Bilbao, 48080, Spain
E-mail: manuel.delasen@ehu.eus