# *Generalizations of the Hermite-Hadamard type inequalities for functions whose derivatives are s-convex*

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# Generalizations of the Hermite–Hadamard type inequalities for functions whose derivatives are $s$-convex

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Abstract. Some new results related to the right-hand side of the Hermite–Hadamard type inequality for the class of functions whose derivatives at certain powers are $s$-convex functions in the second sense are obtained.

## 1. Introduction

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping defined on the interval $I$ of real numbers and $a, b \in I$, with $a < b$. The following double inequality is well known in the literature as the *Hermite–Hadamard inequality* [9]:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

For recent results, refinements, counterparts, generalizations of the Hermite–Hadamard inequality see [4] – [11] and [13] – [17].

Dragomir and Agarwal [5] established the following result connected with the right-hand side of (1.1).

**Theorem 1.** *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I^\circ$, where $a, b \in I$ with $a < b$. If $|f'|$ is convex on $[a, b]$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{b-a}{8}\left(\left|f'(a)\right| + \left|f'(b)\right|\right). \tag{1.2}$$

Hudzik and Maligranda [12] considered among others the class of functions which are $s$-convex in the second sense. This class is defined in the following

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way: a function $f : \mathbb{R}^+ \to \mathbb{R}$, where $\mathbb{R}^+ = [0, \infty)$, is said to be *$s$-convex in the second sense* if

$$f(\alpha x + \beta y) \leq \alpha^s f(x) + \beta^s f(y)$$

for all $x, y \in [0, \infty)$, $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$ and for some fixed $s \in (0, 1]$. The class of $s$-convex functions in the second sense is usually denoted by $K_s^2$.

It can be easily seen that for $s = 1$, $s$-convexity reduces to the ordinary convexity of functions defined on $[0, \infty)$.

For recent results and generalizations concerning $s$-convex functions see [1] – [7] and [13].

Dragomir and Fitzpatrick [8] proved a variant of Hadamard's inequality which holds for $s$-convex functions in the second sense.

**Theorem 2.** *Suppose that $f : [0, \infty) \to [0, \infty)$ is an $s$-convex function in the second sense, where $s \in (0, 1)$, and let $a, b \in [0, \infty)$, $a < b$. If $f \in L^1[0, 1]$, then the following inequalities hold:*

$$2^{s-1} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)\, dx \leq \frac{f(a) + f(b)}{s+1}. \tag{1.3}$$

*The constant $k = \frac{1}{s+1}$ is the best possible in the second inequality in* (1.3). *The above inequalities are sharp.*

New inequalities of Hermite–Hadamard type for differentiable functions based on concavity and $s$-convexity established by Kirmaci et al. [13] are presented below.

**Theorem 3.** *Let $f : I \subset [0, \infty) \to \mathbb{R}$ be a differentiable mapping on $I^\circ$ such that $f' \in L[a, b]$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is $s$-convex on $[a, b]$ for some fixed $s \in (0, 1]$ and $q \geq 1$, then the following inequality holds:*

$$\begin{aligned} &\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)\, dx \right| \\ &\qquad \leq \frac{b-a}{2} \left(\frac{1}{2}\right)^{\frac{q-1}{q}} \left[\frac{s + \left(\frac{1}{2}\right)^s}{(s+1)(s+2)}\right]^{\frac{1}{q}} \left(\left|f'(a)\right|^q + \left|f'(b)\right|^q\right)^{\frac{1}{q}}. \end{aligned}$$

**Theorem 4.** *Let $f : I \subset [0, \infty) \to \mathbb{R}$ be a differentiable mapping on $I^\circ$ such that $f' \in L[a, b]$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is $s$-convex on $[a, b]$*

*for some fixed* $s \in (0,1]$ *and* $q > 1$, *then*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
&\quad\le \frac{b-a}{2}\left[\frac{q-1}{2(2q-1)}\right]^{\frac{q-1}{q}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}} \\
&\quad\quad\times\left[\left(\left|f'(a)\right|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\right. \\
&\quad\quad\left.+\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q+\left|f'(a)\right|^q\right)^{\frac{1}{q}}\right] \\
&\quad\le \frac{b-a}{2}\left[\left(\left|f'(a)\right|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\right. \\
&\quad\quad\left.+\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q+\left|f'(a)\right|^q\right)^{\frac{1}{q}}\right].
\end{aligned} \tag{1.4}$$

**Theorem 5.** *Let* $f : I \subset [0,\infty) \to \mathbb{R}$ *be a differentiable mapping on* $I^\circ$ *such that* $f' \in L[a,b]$, *where* $a, b \in I$ *with* $a < b$. *If* $|f'|^q$ *is* $s$*-convex on* $[a,b]$ *for some fixed* $s \in (0,1]$ *and* $q > 1$, *then*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
&\quad\le \frac{b-a}{2}\left[\frac{q-1}{2(2q-1)}\right]^{\frac{q-1}{q}} 2^{\frac{s-1}{q}} \\
&\quad\quad\times\left(\left|f'\left(\frac{a+3b}{2}\right)\right|+\left|f'\left(\frac{3a+b}{2}\right)\right|\right) \\
&\quad\le \frac{b-a}{2}\left(\left|f'\left(\frac{a+3b}{2}\right)\right|+\left|f'\left(\frac{3a+b}{2}\right)\right|\right).
\end{aligned}$$

The main aim of this paper is to establish new inequalities of Hermite–Hadamard type for the class of functions whose derivatives at certain powers are $s$-convex functions in the second sense.

## 2. Hermite–Hadamard type inequalities for $s$-convex functions

In order to prove our main results we consider the following lemma.

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**Lemma 1.** *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be an absolutely continuous function on $I^{\circ}$, where $a, b \in I$ with $a < b$. Then the following equality holds:*

$$\begin{aligned}\frac{f(a)+rf(b)}{r+1}&-\frac{1}{b-a}\int_a^b f(x)\,dx\\&=\frac{b-a}{r+1}\int_0^1 [(r+1)t-1]\,f'(tb+(1-t)a)\,dt\end{aligned}\tag{2.1}$$

*for every fixed $r \in [0,1]$.*

*Proof.* We note that

$$\begin{aligned}\mathcal{I}&=\int_0^1 [(r+1)t-1]\,f'(tb+(1-t)a)\,dt\\&=[(r+1)t-1]\left.\frac{f(tb+(1-t)a)}{b-a}\right|_0^1-\frac{r+1}{b-a}\int_0^1 f(tb+(1-t)a)\,dt\\&=\frac{rf(b)+f(a)}{b-a}-\frac{r+1}{b-a}\int_0^1 f(tb+(1-t)a)\,dt.\end{aligned}$$

Setting $x = tb + (1-t)a$, and $dx = (b-a)dt$ gives

$$\mathcal{I}=\frac{f(a)+rf(b)}{b-a}-\frac{r+1}{(b-a)^2}\int_a^b f(x)\,dx.$$

Therefore,

$$\left(\frac{b-a}{r+1}\right)I=\frac{f(a)+rf(b)}{r+1}-\frac{1}{b-a}\int_a^b f(x)\,dx$$

which gives the desired representation (2.1). □

The next theorem gives a new refinement of the upper Hermite–Hadamard inequality for $s$-convex functions.

**Theorem 6.** *Let $f : I \subset [0,\infty) \to \mathbb{R}$ be an absolutely continuous function on $I^{\circ}$ and $a, b \in I$ with $a < b$. If $|f'|$ is $s$-convex on $[a,b]$ for some fixed $s \in (0,1]$, then*

$$\begin{aligned}&\left|\frac{f(a)+rf(b)}{r+1}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\\&\quad\le\frac{(b-a)}{(r+1)(s+1)(s+2)}\left[\left(r(s+1)+2\left(\frac{1}{r+1}\right)^{s+1}-1\right)|f'(b)|\right.\\&\qquad\left.+\left(s-r+2(r+1)\left(\frac{r}{r+1}\right)^{s+2}+1\right)|f'(a)|\right]\end{aligned}$$

*for every fixed $r \in [0,1]$.*

*Proof.* From Lemma 1 we have

$$
\begin{aligned}
&\left|\frac{f(a)+rf(b)}{r+1}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
&\quad\le \frac{b-a}{r+1}\int_0^1 |(r+1)t-1|\left|f'(tb+(1-t)a)\right|dt \\
&\quad= \frac{b-a}{r+1}\int_0^{\frac{1}{r+1}} (1-(r+1)t)\left|f'(tb+(1-t)a)\right|dt \\
&\qquad+\frac{b-a}{r+1}\int_{\frac{1}{r+1}}^1 ((r+1)t-1)\left|f'(tb+(1-t)a)\right|dt \\
&\quad\le \frac{b-a}{r+1}\int_0^{\frac{1}{r+1}} (1-(r+1)t)\left[t^s\left|f'(b)\right|+(1-t)^s\left|f'(a)\right|\right]dt \\
&\qquad+\frac{b-a}{r+1}\int_{\frac{1}{r+1}}^1 ((r+1)t-1)\left[t^s\left|f'(b)\right|+(1-t)^s\left|f'(a)\right|\right]dt \\
&\quad= \frac{b-a}{r+1}\left[\frac{\left(\frac{1}{r+1}\right)^{s+1}}{(s+1)(s+2)}\left|f'(b)\right|\right. \\
&\qquad\left.+\frac{s+2+(r+1)\left[\left(\frac{r}{r+1}\right)^{s+2}-1\right]}{(s+1)(s+2)}\left|f'(a)\right|\right] \\
&\qquad+\frac{b-a}{r+1}\left[\frac{r(s+1)+\left(\frac{1}{r+1}\right)^{s+1}-1}{(s+1)(s+2)}\left|f'(b)\right|\right. \\
&\qquad\left.+\frac{(r+1)\left(\frac{r}{r+1}\right)^{s+2}}{(s+1)(s+2)}\left|f'(a)\right|\right] \\
&\quad= \frac{(b-a)}{(r+1)(s+1)(s+2)}\left[\left(r(s+1)+2\left(\frac{1}{r+1}\right)^{s+1}-1\right)\left|f'(b)\right|\right. \\
&\qquad\left.+\left(s-r+2(r+1)\left(\frac{r}{r+1}\right)^{s+2}+1\right)\left|f'(a)\right|\right]
\end{aligned}
$$

which completes the proof. □

Therefore, we can deduce the following results.

**Corollary 1.** *Let $f : I \subset [0, \infty) \to \mathbb{R}$ be an absolutely continuous function on $I^\circ$ and $a, b \in I$ with $a < b$. Assume that $|f'|$ is $s$-convex on $[a, b]$ for some fixed $s \in (0, 1]$. Then the following inequalities hold:*

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \\ &\qquad \le \frac{(s+2^{-s})(b-a)}{2(s+1)(s+2)}\left[\left|f'(b)\right| + \left|f'(a)\right|\right] \end{aligned} \tag{2.2}$$

*and*

$$\begin{aligned} &\left|f(a) - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \\ &\qquad \le \frac{(b-a)}{(s+1)(s+2)}\left[\left|f'(b)\right| + (s+1)\left|f'(a)\right|\right]. \end{aligned}$$

*Proof.* This is obvious from Theorem 6 by taking $r = 1$ and $r = 0$. □

**Remark 1.** We note that the inequality (2.2) with $s = 1$ gives an improvement for the inequality (1.2).

A similar result is embodied in the following theorem.

**Theorem 7.** *Let $f : I \subset [0, \infty) \to \mathbb{R}$ be an absolutely continuous function on $I^\circ$ and $a, b \in I$ with $a < b$. If $|f'|^{p/(p-1)}$ is $s$-convex on $[a, b]$ for some fixed $s \in (0, 1]$ and $p > 1$, then the following inequality holds:*

$$\begin{aligned} &\left|\frac{f(a)+rf(b)}{r+1} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \\ &\quad \le \frac{(b-a)}{(s+1)^{1+\frac{1}{q}}\left[(r+1)(1+p)\right]^{\frac{1}{p}}}\left[\left(\left|f'(a)\right|^q + \left|f'\left(\frac{b+ra}{r+1}\right)\right|^q\right)^{\frac{1}{q}}\right. \\ &\quad \left. + r^{(p+1)/p}\left(\left|f'(b)\right|^q + \left|f'\left(\frac{b+ra}{r+1}\right)\right|^q\right)^{\frac{1}{q}}\right] \end{aligned} \tag{2.3}$$

*for every fixed $r \in [0, 1]$, where $q = p/(p-1)$.*

*Proof.* Suppose that $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. From Lemma 1, using Hölder's inequality, we have

$$\begin{aligned} &\left|\frac{f(a)+rf(b)}{r+1} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \\ &\quad \le \frac{b-a}{r+1}\int_0^{\frac{1}{r+1}} (1-(r+1)t)\left|f'(tb+(1-t)a)\right| dt \\ &\quad + \frac{b-a}{r+1}\int_{\frac{1}{r+1}}^1 ((r+1)t-1)\left|f'(tb+(1-t)a)\right| dt \end{aligned}$$

$$\leq \frac{b-a}{r+1}\left(\int_0^{\frac{1}{r+1}} (1-(r+1)\,t)^p\,dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{r+1}} \left|f'\left(tb+(1-t)\,a\right)\right|^q dt\right)^{\frac{1}{q}}$$
$$+\frac{b-a}{r+1}\left(\int_{\frac{1}{r+1}}^1 ((r+1)\,t-1)^p\,dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{r+1}}^1 \left|f'\left(tb+(1-t)\,a\right)\right|^q dt\right)^{\frac{1}{q}}.$$

Since $|f'|^{\,q}$ is convex, we have

$$\int_0^{\frac{1}{r+1}} \left|f'\left(tb+(1-t)\,a\right)\right|^q dt \leq \frac{|f'(a)|^q+\left|f'\left(\frac{b+ra}{r+1}\right)\right|^q}{s+1}$$

and

$$\int_{\frac{1}{r+1}}^1 \left|f'\left(tb+(1-t)\,a\right)\right|^q dt \leq \frac{|f'(b)|^q+\left|f'\left(\frac{b+ra}{r+1}\right)\right|^q}{s+1}.$$

Therefore, we get

$$\left|\frac{f(a)+rf(b)}{r+1}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq \frac{(b-a)}{(s+1)^{1+\frac{1}{q}}\left[(r+1)\,(1+p)\right]^{\frac{1}{p}}}\left[\left(\left|f'(a)\right|^q+\left|f'\left(\frac{b+ra}{r+1}\right)\right|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\,r^{(p+1)/p}\left(\left|f'(b)\right|^q+\left|f'\left(\frac{b+ra}{r+1}\right)\right|^q\right)^{\frac{1}{q}}\right],$$

which is required. □

**Corollary 2.** *Let $f: I\subset[0,\infty)\to\mathbb{R}$ be an absolutely continuous function on $I^\circ$ and $a,b\in I$ with $a<b$. Assume that $|f'|^{p/(p-1)}$ is $s$-convex on $[a,b]$ for some fixed $s\in(0,1]$ and $p>1$. Let $q=p/(p-1)$. Then the following inequalities hold:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq \frac{(b-a)}{(s+1)^{1+\frac{1}{q}}\left[2\,(1+p)\right]^{\frac{1}{p}}}\left[\left(\left|f'(a)\right|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\right. \tag{2.4}$$
$$\left.+\left(\left|f'(b)\right|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\right]$$

*and*

$$\left|f(a)-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\leq\frac{(b-a)}{(s+1)^{1+\frac{1}{q}}\,(1+p)^{\frac{1}{p}}}\left(\left|f'(a)\right|^q+\left|f'(b)\right|^q\right)^{\frac{1}{q}}.$$

*Proof.* This follows directly from Theorem 7 by taking $r=1$ and $r=0$. □

**Remark 2.** We observe that the inequality (2.4) is better than the inequality (1.4).

Our next result gives a new refinement for the upper Hermite–Hadamard inequality.

**Theorem 8.** *Let $f : I \subset [0,\infty) \to \mathbb{R}$ be an absolutely continuous function on $I^\circ$ and $a, b \in I$ with $a < b$. If $|f'|^{p/(p-1)}$ is $s$-convex on $[a,b]$ for some fixed $s \in (0,1]$ and $p > 1$, then the following inequality holds:*

$$\begin{aligned} &\left| \frac{f(a) + rf(b)}{r+1} - \frac{1}{b-a} \int_a^b f(x)\,dx \right| \\ &\quad \leq \frac{(b-a)}{(s+1)^{1+\frac{1}{q}} (r+1)^{\frac{1}{p}+\frac{s}{q}} (p+1)^{1+p}} \\ &\quad\quad \times \left[ \left( [(r+1)^s + 1] \left| f'(a) \right|^q + r^s \left| f'(b) \right|^q \right)^{\frac{1}{q}} \right. \\ &\quad\quad\quad \left. + \left( r^s \left| f'(a) \right|^q + [(r+1)^s + 1] \left| f'(b) \right|^q \right)^{\frac{1}{q}} \right] \end{aligned} \tag{2.5}$$

*for every fixed $r \in [0,1]$, where $q = p/(p-1)$.*

*Proof.* Since $|f'|^{p/(p-1)}$ is $s$-convex on $[a,b]$, we have

$$\left| f'\left(\frac{a+rb}{r+1}\right) \right|^q \leq \left(\frac{1}{r+1}\right)^s \left| f'(a) \right|^q + \left(\frac{r}{r+1}\right)^s \left| f'(b) \right|^q.$$

This gives, by (2.3), that

$$\begin{aligned} &\left| \frac{f(a) + rf(b)}{r+1} - \frac{1}{b-a} \int_a^b f(x)\,dx \right| \\ &\leq \frac{(b-a)}{(s+1)^{1+\frac{1}{q}} [(r+1)(1+p)]^{\frac{1}{p}}} \left[ \left( \left| f'(a) \right|^q + \left| f'\left(\frac{b+ra}{r+1}\right) \right|^q \right)^{\frac{1}{q}} \right. \\ &\quad \left. + r^{(p+1)/p} \left( \left| f'(b) \right|^q + \left| f'\left(\frac{b+ra}{r+1}\right) \right|^q \right)^{\frac{1}{q}} \right] \\ &\leq \frac{(b-a)}{(s+1)^{1+\frac{1}{q}} (r+1)^{\frac{1}{p}+\frac{s}{q}} (p+1)^{1+p}} \left[ \left( [(r+1)^s+1] \left| f'(a) \right|^q + r^s \left| f'(b) \right|^q \right)^{\frac{1}{q}} \right. \\ &\quad \left. + \left( r^s \left| f'(a) \right|^q + [(r+1)^s + 1] \left| f'(b) \right|^q \right)^{\frac{1}{q}} \right], \end{aligned}$$

and the proof is completed. □

**Corollary 3.** *Let $f : I \subset [0,\infty) \to \mathbb{R}$ be an absolutely continuous function on $I^\circ$ and $a,b \in I$ with $a < b$. If $|f'|^{p/(p-1)}$ is $s$-convex on $[a,b]$ for some fixed $s \in (0,1]$ and $p > 1$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{(b-a)}{(s+1)^{1+\frac{1}{q}}\left[2(1+p)\right]^{\frac{1}{p}}}\Big[\left(\left(1+2^{-s}\right)\left|f'(a)\right|^q + 2^{-s}\left|f'(b)\right|^q\right)^{\frac{1}{q}} + \left(2^{-s}\left|f'(a)\right|^q + \left(1+2^{-s}\right)\left|f'(b)\right|^q\right)^{\frac{1}{q}}\Big], \tag{2.6}$$

*where $q = p/(p-1)$.*

*Proof.* Since $|f'|^{p/(p-1)}$ is $s$-convex on $[a,b]$,

$$\left|f'\left(\frac{a+b}{2}\right)\right|^q \leq \frac{|f'(a)|^q + |f'(b)|^q}{2^s},$$

which gives, in view of (2.4),

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
&\leq \frac{(b-a)}{(s+1)^{1+\frac{1}{q}}\left[2(1+p)\right]^{\frac{1}{p}}}\left[\left(\left|f'(a)\right|^q + \left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}} + \left(\left|f'(b)\right|^q + \left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\right] \\
&\leq \frac{(b-a)}{(s+1)^{1+\frac{1}{q}}\left[2(1+p)\right]^{\frac{1}{p}}}\Big[\left(\left(1+2^{-s}\right)\left|f'(a)\right|^q + 2^{-s}\left|f'(b)\right|^q\right)^{\frac{1}{q}} + \left(2^{-s}\left|f'(a)\right|^q + \left(1+2^{-s}\right)\left|f'(b)\right|^q\right)^{\frac{1}{q}}\Big].
\end{aligned}$$

This completes the proof. $\square$

**Corollary 4.** *Let $f : I \subset [0,\infty) \to \mathbb{R}$ be an absolutely continuous function on $I^\circ$ and $a,b \in I$ with $a < b$. If $|f'|^{p/(p-1)}$ is $s$-convex on $[a,b]$ for some fixed $s \in (0,1]$ and $p > 1$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{\left(1+2^{1-s}\right)^{\frac{1}{q}}(b-a)}{(s+1)^{1+\frac{1}{q}}\left[2(1+p)\right]^{\frac{1}{p}}}\left[\left|f'(a)\right| + \left|f'(b)\right|\right],$$

*where $q = p/(p-1)$.*

*Proof.* Let $a_1 = (1+2^{-s})\,|f'(a)|^q$, $b_1 = 2^{-s}\,|f'(b)|^q$, $a_2 = 2^{-s}\,|f'(a)|^q$ and $b_2 = (1+2^{-s})\,|f'(b)|^q$.

Here, $0 < \frac{1}{q} < 1$. Using the fact that

$$\sum_{i=1}^{n} (a_i + b_i)^k \le \sum_{i=1}^{n} a_i^k + \sum_{i=1}^{n} b_i^k,$$

for $0 < k < 1$, $a_1, a_2, ..., a_n \ge 0$ and $b_1, b_2, ..., b_n \ge 0$, by the inequality (2.6) we obtain

$$\begin{aligned}
&\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)\,dx \right| \\
&\quad \le \frac{(b-a)}{(s+1)^{1+\frac{1}{q}} [2(1+p)]^{\frac{1}{p}}} \left[ \left( \left(1+2^{-s}\right) \left|f'(a)\right|^q + 2^{-s} \left|f'(b)\right|^q \right)^{\frac{1}{q}} \right. \\
&\qquad \left. + \left( 2^{-s} \left|f'(a)\right|^q + \left(1+2^{-s}\right) \left|f'(b)\right|^q \right)^{\frac{1}{q}} \right] \\
&\quad \le \frac{\left(1+2^{1-s}\right)^{\frac{1}{q}} (b-a)}{(s+1)^{1+\frac{1}{q}} [2(1+p)]^{\frac{1}{p}}} \left[ \left|f'(a)\right| + \left|f'(b)\right| \right],
\end{aligned}$$

which is required. □

**Remark 3.** 1. Using the technique in Corollary 4, one can obtain in a similar manner another result by considering the inequality (2.5). However, the details are left to the interested reader.

2. All of the above inequalities obviously hold for convex functions. Simply choose $s = 1$ in each of those results to get the desired results.

3. Interchanging $a$ and $b$ in Lemma 1, we obtain the equality

$$\begin{aligned}
&\frac{1}{b-a} \int_a^b f(x)\,dx - \frac{rf(a) + f(b)}{r+1} \\
&\qquad = \frac{b-a}{r+1} \int_0^1 [(r+1)t - 1]\, f'((1-t)b + ta)\,dt.
\end{aligned} \tag{2.7}$$

For this reason, if we interchange $a$ and $b$ in all above results, then, using the equality (2.7), we can write new results.

## 3. Applications to special means

We consider the means for arbitrary real numbers $\alpha$, $\beta$ $(\alpha \neq \beta)$ as follows.

1) Arithmetic mean

$$A(\alpha, \beta) = \frac{\alpha + \beta}{2}, \quad \alpha, \beta \in \mathbb{R}.$$

2) Generalized log-mean

$$L_s(\alpha,\beta)=\left[\frac{\beta^{s+1}-\alpha^{s+1}}{(s+1)(\beta-\alpha)}\right]^{\frac{1}{s}},\ s\in\mathbb{R}\backslash\{-1,0\},\ \alpha,\beta\in\mathbb{R}.$$

Now, using results of Section 2, we give some applications to special means of real numbers. In [13], the following example is given.

Let $s\in(0,1)$ and $a,b,c\in\mathbb{R}$ . We define the function $f:[0,\infty)\to\mathbb{R}$,

$$f(t)=\begin{cases} a & \text{if } t=0,\\ bt^s+c & \text{if } t>0.\end{cases}$$

If $b\geq 0$ and $0\leq c\leq a$, then $f\in K_s^2$. Hence, for $a=c=0$, $b=1$, we have $f:[0,1]\to[0,1]$, $f(t)=t^s$, $f\in K_s^2$.

**Proposition 1.** *Let $a,b\in[0,1]$, $a<b$ and $0<s<1$. Then we have*

$$\left|L_s^s(a,b)-A(a^s,b^s)\right|\leq s(b-a)\frac{s+2^{-s}}{2(s+1)(s+2)}\left(|a|^{s-1}+|b|^{s-1}\right)$$

*and*

$$\left|L_s^s(a,b)-|a|^s\right|\leq\frac{s(b-a)}{(s+1)(s+2)}\left((s+1)|a|^{s-1}+|b|^{s-1}\right).$$

*Proof.* The assertions follow from Corollary 1 applied to the $s$-convex mapping $f:[0,1]\to[0,1]$, $f(t)=t^s$. □

**Proposition 2.** *Let $a,b\in[0,1]$, $a<b$ and $0<s<1$. Then for all $q>1$, we have*

$$\begin{aligned}&\left|L_s^s(a,b)-A(a^s,b^s)\right|\\&\quad\leq\frac{s(b-a)}{(s+1)^{1+\frac{1}{q}}[2(p+1)]^{\frac{1}{p}}}\left[\left(|a|^{q(s-1)}+\left|\frac{a+b}{2}\right|^{q(s-1)}\right)^{1/q}\right.\\&\qquad\left.+\left(\left|\frac{a+b}{2}\right|^{q(s-1)}+|b|^{q(s-1)}\right)^{1/q}\right]\end{aligned}$$

*and*

$$\left|L_s^s(a,b)-|a|^s\right|\leq\frac{s(b-a)}{(s+1)^{1+\frac{1}{q}}[2(p+1)]^{\frac{1}{p}}}\left((s+1)|a|^{q(s-1)}+|b|^{q(s-1)}\right)^{1/q}.$$

*Proof.* The assertions follow from Corollary 2 applied to the $s$-convex mapping $f:[0,1]\to[0,1]$, $f(t)=t^s$. □

**Proposition 3.** *Let $a, b \in [0,1]$, $a < b$ and $0 < s < 1$. Then for all $q > 1$, we have*

$$\left|L_s^s(a,b) - A(a^s,b^s)\right| \le \frac{s(b-a)\left(1+2^{1-s}\right)^{1/q}}{(s+1)^{1+\frac{1}{q}}\left[2(p+1)\right]^{\frac{1}{p}}}\left(|a|^{s-1}+|b|^{s-1}\right).$$

*Proof.* The assertion follows from Corollary 4 applied to the $s$-convex mapping $f : [0,1] \to [0,1]$, $f(t) = t^s$. □

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