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Robust Constrained Model Predictive Control Based on Parameter-Dependent Lyapunov Functions

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Abstract: The problem of robust constrained model predictive control (MPC) of systems with polytopic uncertainty is considered in this paper. New sufficient conditions for the existence of parameter-dependent Lyapunov functions are proposed in terms of linear matrix inequalities (LMIs), which will reduce the conservativeness resulting from using a single Lyapunov function. At each sampling instant, the corresponding parameter-dependent Lyapunov function is an upper bound for a worst-case objective function, which can be minimized using the LMI convex optimization approach. Based on the solution of optimization at each sampling instant, the corresponding state feedback controller is designed, which can guarantee that the resulting closed-loop system is robustly asymptotically stable. In addition, the feedback controller will meet the specifications for systems with input or output constraints, for all admissible time-varying parameter uncertainties. Numerical examples are presented to demonstrate the effectiveness of the proposed techniques.

Keywords: MPC, robust control; polytopic uncertainty; stability; optimization; LMI

I. Introduction

MPC, also known as Receding horizon control (RHC), is an on-line technique in which the current control action is computed at each time step by solving a finite horizon open-loop optimal control problem that extends from the current time to the current time plus a specified horizon length. The current state of the plant is used as the initial state, and the first control in the sequence obtained by optimization is applied to the plant. A full review of the recent advances related to MPC can be found in a survey paper [1]. In practice, there are always modelling errors, and it is necessary to consider robust MPC in the presence of model uncertainty. Recently, many research results in the design of robust MPC have appeared, see for example, ~~[2], [3], [4], [5], [6]~~ and the reference therein. On the other hand, the main drawback associated to the above mentioned methods proposed in MPC is that a single Lyapunov matrix is used to guarantee the desired closed-loop multiobjective specifications. This must work for all matrices in the uncertain domain to ensure that the hard constraints on inputs and outputs are satisfied. This condition is generally conservative if used in time-invariant systems. Furthermore, the hard constraints on outputs of closed-loop systems can not be transformed into an LMI form using the method proposed in [2], [3], [5].

This paper considers the problem of robust MPC of the closed-loop systems that satisfies hard constraint on controls and states of the closed-loop system with polytopic type uncertainties. A new LMI characterization of minimal quadratic objective is derived with hard constraints on inputs and outputs, respectively. The idea of using the parameter-dependent Lyapunov function was introduced in [7] for linear continues-time uncertain systems, and in [8], [9] for linear discrete-time uncertain systems. The results obtained in this paper generalize the ones derived by the quadratic approach for MPC with respect to polytopic uncertainty. It is expressed as LMIs and exhibits a kind of separation property between the Lyapunov matrices and the uncertain dynamic matrices. In terms of the new LMI characterization, sufficient conditions are obtained for the existence of an upper bound on the quadratic objective of the closed-loop system with hard constraints on inputs and outputs. By minimizing this upper bound at each sampling time, the corresponding state feedback will guarantee that the resulting closed-loop system is robustly stable and the hard constraints on inputs and outputs at each sampling instant are satisfied. Finally, numerical example is presented to illustrate the effectiveness and the potential of the obtained theoretic results.

II. Problem statement

We consider the following class of uncertain discrete-time systems

$$\begin{aligned} x(k+1) &= A(k)x(k) + B(k)u(k) \\ y(k) &= C(k)x(k) \end{aligned} \tag{1}$$

where $x(k) \in R^n$ is the state, $u(k) \in R^l$ is the control input, $y(k) \in R^m$ is the plant output, $A(k), B(k), C(k)$ are uncertain matrices which are assumed to belong to a polytopic convex domain:

$$\begin{bmatrix} A(k) & B(k) \\ C(k) & 0 \end{bmatrix} \in \Omega \tag{2}$$

For polytopic uncertainty, Ω is the polytope $Co\{\begin{bmatrix} A_1 & B_1 \\ C_1 & 0 \end{bmatrix}, \cdots, \begin{bmatrix} A_p & B_p \\ C_p & 0 \end{bmatrix}\}$, where Co denotes the convex hull, $\begin{bmatrix} A_i & B_i \\ C_i & 0 \end{bmatrix}$ are vertices of the convex hull. Any $\begin{bmatrix} A(k) & B(k) \\ C(k) & 0 \end{bmatrix}$ within the convex set Ω is a linear combination of the vertices

$$\begin{bmatrix} A(k) & B(k) \\ C(k) & 0 \end{bmatrix} = \sum_{i=1}^{p} \xi_i(k) \begin{bmatrix} A_i & B_i \\ C_i & 0 \end{bmatrix} \tag{3}$$

where $\sum_{i=1}^{p} \xi_i(k) = 1, \xi_i(k) \geq 0$.

Remark 1: Note that relation (2) defines a polyhedral type of uncertainty domain. In fact, the important case of interval matrices can exactly be modelled by this kind of uncertainty with appropriate choice of the extreme matrices. Being convex and polyhedral, this kind of uncertainty is clearly more general than interval matrix domains. The design of MPC with polytopic uncertainty is considered in [2], [3], [5], [6], and the matrix C was assumed to be constant in [2], [3]. In (3), the parameters are time-varying in a certain range during operation. Then, the uncertain discrete-time system (1) is time-varying. So the robust stability conditions for system (1) will be more strict than the case that unknown parameter are constant [10].

Consider the following problem, which minimizes the worst case quadratic objective function in an infinite horizon:

$$\min_{u(k+i|k)=F(k)x(k+i|k)} \ \max_{[A(k+i) \ \ B(k+i)] \in \Omega, i \geq 0} J_\infty(k), \tag{4}$$

$$J_\infty = \sum_{i=0}^{\infty}[x(k+i|k)^T Q_0 (k+i|k) + u(k+i|k)^T R_0 u(k+i|k) \tag{5}$$

where $Q_0 > 0$ and $R_0 > 0$ are known weighting matrices, $F(k)$ is the feedback matrix gain obtained at sampling instant k, which will be denoted as F for simplicity. The optimal problem (4) is subject to (1) and

$$|u_h(k+i|k)| \leq u_{h,max}, i \geq 0, h = 1, 2, \cdots, l \tag{6}$$

$$|y_h(k+i|k| \leq y_{h,max}, i \geq 1, h = 1, 2, \cdots, m \tag{7}$$

$x(k+i|k)$ and $y(k+i|k)$ are the state and output respectively, at time $k+i$, predicted based on the measurements at time k; $x(k|k)$ and $y(k|k)$ refer respectively to the state and output measured at time k; $u(k+i|k)$ is the control move at time $k+i$, computed by the optimization problem (4) at time k; $u(k|k)$ is the control move to be implemented at time k.

Remark 2: In the control theory, especially control engineering, control problems where variables need to satisfy constraints and where the control action also has to minimize a cost arise naturally and received much consideration. One effective control algorithm that addresses such problems is model predictive control (MPC), this method proposes a framework to deal with constrained control problems for which the classical off-line computation of control laws is rendered difficult or impossible. Due to the merit of this method, MPC has become a widely used technique to address advanced control problems in industrial applications, such as chemical process control [11].

III. Main Results

To begin this section, we first recall the following lemmas which will be used in the proof of our main results.

Lemma 1: ([12]) The following conditions are equivalent:

i There exists a symmetric matrix $P > 0$ such that

$$A^T P A - P < 0 \tag{8}$$

ii There exist a symmetric matrix P and a matrix G such that

$$\begin{bmatrix} P & A^T G \\ G^T A & G + G^T - P \end{bmatrix} > 0 \tag{9}$$

Lemma 2: The following statements are equivalent:

i There exists a symmetric positive definite matrix P such that

$$x(k|k)^T P x(k|k) < \gamma \tag{10}$$

ii There exist a symmetric positive definite matrix $\overline{P}$ and a matrix V such that

$$\begin{bmatrix} -\gamma & x(k|k)^T \\ x(k|k) & \overline{P} - V - V^T \end{bmatrix} < 0 \tag{11}$$

Proof: Firstly, we will prove that (10) is equivalent to the existence of matrix G and positive definite matrix $\tilde{P}$ such that

$$\begin{bmatrix} -\gamma & x(k|k)^T G \\ G^T x(k|k) & \tilde{P} - G^T - G \end{bmatrix} < 0 \tag{12}$$

Sufficiency: Since the matrix $\begin{bmatrix} 1 & x(k|k)^T \end{bmatrix}$ has full rank, (12) implies that

$$\begin{bmatrix} 1 & x(k|k)^T \end{bmatrix} \begin{bmatrix} -\gamma & x(k|k)^T G \\ G^T x(k|k) & \tilde{P} - G^T - G \end{bmatrix} \begin{bmatrix} 1 \\ x(k|k) \end{bmatrix} < 0 \tag{13}$$

letting $\tilde{P} = P$, which leads to (10).

Necessity: Assuming (10) is satisfied, and using Schur complement, (10) is equivalent to

$$\begin{bmatrix} -\gamma & x(k|k)^T P \\ P^T x(k|k) & -P \end{bmatrix} < 0 \tag{14}$$

Then choosing $G = G^T = P = \tilde{P}$, the above inequality can be written as

$$\begin{bmatrix} -\gamma & x(k|k)^T G \\ G^T x(k|k) & \tilde{P} - G - G^T \end{bmatrix} < 0 \tag{15}$$

Secondly, it will be shown that (12) is equivalent to the existence of matrix G and a symmetric positive definite matrix $\overline{P}$ such that (11) holds. Pre-and post-multiplying (12) by $\begin{bmatrix} 1 & 0 \\ 0 & G^{-T} \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & G^{-1} \end{bmatrix}$ on both sides, and letting $V = G^{-1}$ and $\overline{P} = G^{-T} \tilde{P} G^{-1}$, we obtain (11).

As quadratic objective J_∞ defined (5) is difficult to be obtained. An possible way is to find an upper bound on J_∞, then we can design the controller such that the upper bound is minimized respect to uncertainty [2]. In order to derive such an upper bound, it is assumed that a quadratic function

$V(x(k|k)) = x(k|k)^T P(\xi(k))x(k|k)$, is defined at sampling time k, where $P(\xi(k))$ is a parameter-dependent positive definite matrix. For any $[A(k+i)\ B(k+i)] \in \Omega, i > 0$, suppose that $V(x(k|k))$ satisfies the following robust stability constraint:

$$V(x(k+i+1|k)) - V(x(k+i|k)) \leq -[x(k+i|k)^T Q_0 x(k+i|k) + u(k+i|k)^T R_0 u(k+i|k)] \quad (16)$$

As it is assumed that the summation is up to ∞, i.e., $i \to \infty$, $x(i|k)$ should approach zero, that is, $x(\infty|k) = 0$. Summing (16) from $i = 0$ to ∞ leads to the following inequality

$$\max_{[A(k+i)\ B(k+i)] \in \Omega, i>0} J_\infty(k) \leq V(x(k|k)) \quad (17)$$

From the above inequality, it shows that $V(x(k|k))$ is just an upper bound on J_∞, thus in the following theorem, the controller is designed such that $V(x(k|k))$ is minimized at sampling time k.

In [13] and [10], the method of parameter-dependent Lyapunov function has been adopted. Especially, in [10] the results have been improved further compared to the results in [13]. But, ~~the~~ it can not be proved that the resulting closed-loop system is robustly stable based on the algorithm proposed in [13] and [10], which was neglected in paper [13] and [10]. Firstly, at sampling k, the control feedback is designed such that the upper bound on $V(x(k|k))$ is minimized, which is little different to Theorem 1 in [13] and Theorem 2 in [10].

Theorem 1: Let $x(k) = x(k|k)$ be the state of the uncertain system (1) measured at sampling time k. If the convex optimization problem for

$$\min_{E, \overline{P}_j, V} \gamma \quad (18)$$

subject to

$$\begin{bmatrix} -\gamma & x(k|k)^T \\ x(k|k) & \overline{P}_j(k) - V - V^T \end{bmatrix} < 0 \quad (19)$$

$$\begin{bmatrix} -\overline{P}_i(k) & V^T A_j^T + E^T(k) B_j^T & E^T(k) R_0 & V^T Q_0 \\ A_j V + B_j E(k) & \overline{P}_j(k) - V - V^T & 0 & 0 \\ E(k) R_0 & 0 & -R_0 & 0 \\ V Q_0 & 0 & 0 & -Q_0 \end{bmatrix} < 0 \quad (20)$$

$$\forall j = 1, 2, \cdots, p \text{ and } i = 1, 2, \cdots, p$$

has a solution in the matrix variables $\overline{P}_j(k) > 0, j = 1, 2, \cdots, p$, $E(k)$, V and γ, then, the parameter-dependent Lyapunov function can be taken as

$$V(x(k+i|k)) = x(k+i|k)^T V^{-T} \overline{P}(i,k) V^{-1} x(k+i|k) \quad (21)$$

where $\overline{P}(i,k)=\sum_{j=1}^{p}\xi_j(i+k)\overline{P}_j(k)$ and a state feedback control gain matrix $F(k)=E(k)V^{-1}$ in the control law can be chosen as $u(k+i|k)=F(k)x(k+i|k), i\geq 0$ such that the upper bound of

$$\begin{aligned} V(x(k|k)) :=&\ x(k|k)^T P(\xi(k))x(k|k) \\ :=&\ x(k|k)^T V^{-T}\overline{P}(0,k)V^{-1}x(k|k) \\ :=&\ x(k|k)^T V^{-T}\textstyle\sum_{j=1}^{p}\xi_j(k)\overline{P}_j(k)V^{-1}x(k|k) \\ :=&\ x(k|k)^T V^{-T}\overline{P}(\xi(k))V^{-1}x(k|k) \end{aligned} \tag{22}$$

on the robust performance objective function is minimized at sampling time k.

Proof. The derivation of formula (20) is very similar to that in [10], which is omitted here.

Note that minimization of $V(k|k)=x(k|k)^T P(\xi(k))x(k|k)$ is equivalent to

$$\min_{\gamma,P(\xi(k))}\gamma \tag{23}$$

subject to

$$x(k|k)^T P(\xi(k))x(k|k)<\gamma \tag{24}$$

It follows from Lemma 2 that (24) is equivalent to the existence of matrices $\overline{P}(\xi(k))$ and V such that

$$\begin{bmatrix} -\gamma & x(k|k)^T \\ x(k|k) & \overline{P}(\xi(k))-V-V^T \end{bmatrix}<0 \tag{25}$$

For each j, multiply (19) corresponding $j=1,2,\cdots,p$ inequalities by $\xi_i(k)\geq 0$ and sum, then, it is shown that (25) is satisfied if (19) holds. Thus, the proof is completed.

Remark 3: The Lyapunov function, $V(x(k|k)) = x(k|k)^T V^{-T}\overline{P}(\xi(k))V^{-1}x(k|k)$, adopted in Theorem 1 is parameter-dependent. It is different from the result in Theorem 1 in [2], in which a single Lyapunov matrix $V^{-T}\overline{P}(\xi(k))V^{-1}=P$, i.e., Lyapunov function $V(x(k|k))=x(k|k)^T Px(k|k)$ is used, that is, the inequalities (18)-(20) are required to be satisfied with a fixed P for all $[A(k)\ B(k)]\in\Omega$, which is also called quadratic stability and has many successful applications in robust control theory and filtering design although it brings conservativeness in some sense. This concept has been extensively used in many papers, such as [14], [15] and the references therein. More recently, the problem of MPC with parameter-dependent Lyapunov function has been considered in [13] and [10], but the uncertainties do not exit in all system matrix, while it is very natural that all the systems are subjected to time-varying uncertainties.

Note that the feedback controller designed in Theorem 1 can guarantee that real polytopic uncertainty, at each sampling time k. However, it does not mean that the closed-loop system is stable. The stability of the closed-loop system will be presented in the following theorem.

Theorem 2: The feasible receding horizon state feedback control obtained by optimization at each sampling instant in Theorem 1, i.e., $(u(0|0), u(1|1), \cdots, u(k|k), k \rightarrow \infty)$, robustly asymptotically stabilizes the resulting closed-loop system (1).

Proof:: See the proof in the Appendix.

When there are constraint bounds on the input $u(k+i|k)$ and output $y(k+i|k)$, a similar idea to [2] can be used to transform the constraints into LMI forms, which leads to the following result.

Theorem 3: Let $x(k) = x(k|k)$ be the state of the uncertain system (1) measured at sampling time k. If the convex optimization problem

$$\min_{E, \hat{P}_j, \overline{Q}_j, V} \gamma \tag{26}$$

subject to

$$\begin{bmatrix} -1 & x(k|k)^T \\ x(k|k) & \hat{P}_j - V - V^T \end{bmatrix} < 0 \tag{27}$$

subject to

$$\begin{bmatrix} -\hat{P}_i & V^T A_j^T + E^T B_j^T & E^T R_0 & V^T Q_0 \\ A_j V + B_j E & \hat{P}_j - V - V^T & 0 & 0 \\ E R_0 & 0 & -\gamma R_0 & 0 \\ V Q_0 & 0 & 0 & -\gamma Q_0 \end{bmatrix} < 0 \tag{28}$$
$$\forall j = 1, 2, \cdots, p \text{ and } i = 1, 2, \cdots, p$$

$$\begin{bmatrix} -Z & E \\ E^T & -\hat{P}_j \end{bmatrix} < 0, \text{with} \quad Z_{(hh)} < u_{h,max}^2, h = 1, 2, \cdots, r \tag{29}$$

$$\begin{bmatrix} -U_j & (A_j V + B_j E) \\ (A_j V + B_j E)^T & -\hat{P}_j \end{bmatrix} < 0 \tag{30}$$

$$\begin{bmatrix} -S & C_j H^T \\ H C_j^T & U_j - H - H^T \end{bmatrix} < 0, \text{with} \quad S_{(hh)} < y_{h,max}^2, h = 1, 2, \cdots, m \tag{31}$$

has a solution in the matrix variables $\hat{P}_j > 0, \overline{Q}_j > 0, j = 1, 2, \cdots, p$, E, H and V. Then, the parameter-dependent Lyapunov function can be taken as $V(x(k|k)) = x(k|k)^T V^{-T} \sum_{j=1}^{p} \xi_j \hat{P}_j V^{-1} x(k|k)$ with a state feedback control gain matrix $F = EV^{-1}$ in the control law $u(k+i|k) = Fx(k+i|k), i \geq 0$ such that i the feasible receding horizon state feedback control law $u(k|k) = Fx(k|k)$, $k \geq 0$, i.e., $(u(0|0), u(1|1), \cdots, u(k|k), k \to \infty)$, obtained by the optimization at each sampling instant robustly asymptotically stabilizes the resulting closed-loop system (1);

ii the upper bound of $V(x(k|k))$ on the robust performance objective function is minimized at sampling time k;

iii the component-wise peak bound on $u_h(k+i|k)$ satisfies

$$u_h(k+i|k)| \leq u_{h,max}, i \geq 0, h = 1, 2, \cdots, l; \tag{32}$$

iv the component-wise peak bound on $y_h(k+i|k)$ satisfies

$$y_h(k+i|k)| \leq y_{h,max}, i \geq 0, h = 1, 2, \cdots, m. \tag{33}$$

Proof: Assume that the inequalities of (28) are satisfied, it can be easily shown that (20) are also satisfied with $\hat{P}$ replaced by $\frac{1}{\gamma}\overline{P}$. Then using the same as that in Theorem 1, it can be proved that (16) are satisfied for $i \geq 0$. Thus, it follows from (16) and (17) that the following inequalities hold for all $i \geq 0$

$$V(x(k+i+1|k)) \leq V(x(k+i|k)) \leq V(x(k|k)) \leq \gamma \tag{34}$$

Let $\hat{P}_j = \frac{1}{\gamma}\overline{P}_j$ and $\tilde{P}(\xi(k)) = \frac{1}{\gamma}P(\xi(k))$. Then

$$\tilde{P}(\xi(k)) = \frac{1}{\gamma}V^{-T}\sum_{j=1}^{p}\xi_j(k)\overline{P}_j(k)V^{-1} = V^{-T}\sum_{j=1}^{p}\xi_j(k)\hat{P}_j(k)V^{-1}$$

The above inequality implies

$$\begin{aligned}&x(k+i+1|k)^T\tilde{P}(\xi(k))x(k+i+1|k) \leq x(k+i|k)^T\tilde{P}(\xi(k))x(k+i|k)^T \\ &\leq x(k|k)^T\tilde{P}(\xi(k))x(k|k) \leq 1\end{aligned} \tag{35}$$

So the peak bounds on each component of $u(k+i|k)$ at sampling time k can be expressed as

$$\begin{aligned}\max_{i\geq 0}|u_h(k+i|k)|^2 &= \max_{i\geq 0}|(EV^{-1}x(k+i|k))_h|^2 \\ &\leq \max_{i\geq 0}|(EP^{\frac{1}{2}}(\xi)\hat{P}^{-\frac{1}{2}}(\xi)V^{-1}x(k+i|k))))_h|^2 \\ &\leq (\hat{P}^{-\frac{1}{2}}(\xi)E^TE\hat{P}^{-\frac{1}{2}}(\xi))_{hh}\|x^T(k+i|k)V^{-T}\hat{P}(\xi(k))V^{-1}x(k+i|k)\| \\ &= (\hat{P}^{-\frac{1}{2}}(\xi)E^TE\hat{P}^{-\frac{1}{2}}(\xi))_{hh}x^T(k+i|k)\tilde{P}(\xi(k))x(k+i|k)\end{aligned}$$

From inequality (35), we have

$$\max_{i\geq 0}|u_h(k+i|k)|^2 \leq ((\hat{P}^{-\frac{1}{2}}(\xi)E^TE\hat{P}^{-\frac{1}{2}}(\xi))_{hh} \tag{36}$$

It follows from the above inequalities and matrix theory that (32) hold if the following inequalities are satisfied:

$$\begin{bmatrix} -Z & E \\ E^T & -\hat{P}_j \end{bmatrix} < 0, \text{with} \quad Z_{(hh)} < u_{h,max}^2, h = 1, 2, \cdots, l \tag{37}$$

The rest of the proof is similar to that of Theorem 1 with $\hat{P}_j = \frac{1}{\gamma}\overline{P}_j$, for $j = 1, 2, \cdots, p$.

In order to prove that (33) is satisfied, we only need to show that if (30) and (31) are satisfied, then the constraints on (33) hold for all $i \geq 0, j = 1, 2, \cdots, m$. With the feedback control law obtained at sampling time k, we have

$$\begin{aligned} \max_{i\geq 0} |y_h(k+i|k)|^2 &= \max_{i\geq 0} |C(A(k+i) + B(k+i)F)x(k+i|k))_h|^2 \\ &\leq \max_{i\geq 0} |(C(A(k+i) + B(k+i)F)V\hat{P}^{-\frac{1}{2}}(\xi)\hat{P}^{\frac{1}{2}}(\xi)V^{-1}x(k+i|k))))_h|^2 \\ &\leq |(C(A(k+i) + B(k+i)EV^{-1})V\hat{P}^{-\frac{1}{2}}(\xi))_j|^2 \times \\ &\quad \|x^T(k+i|k)V^{-T}\hat{P}(\xi(k))V^{-1}x(k+i|k)\| \end{aligned}$$

Note that $x^T(k+i|k)V^{-T}\hat{P}(\xi(k))V^{-1}x(k+i|k) = x^T(k+i|k)\tilde{P}(\xi(k))x(k+i|k)$ and (35) hold, then

$$\max_{i\geq 0} |y_h(k+i|k)|^2 \leq (\hat{P}^{-\frac{1}{2}}(\xi)(A(k+i)V + B(k+i)E)^T C^T C(A(k+i)V + B(k+i)E)\hat{P}^{-\frac{1}{2}}(\xi))_{hh}$$

Based on the above inequalities and the matrix theory, inequalities (7) hold if the following inequalities are satisfied for all $\xi(k+i) \in \Omega$:

$$\begin{bmatrix} -S & C(A_jV + B_jE) \\ (C(A_jV + B_jE))^T & -\hat{P}_j \end{bmatrix} < 0, \text{with} \quad S_{hh} < y_{h,max}^2, h = 1, 2, \cdots, m \tag{38}$$

It follows from (38) that

$$C(A_jV + B_jE)\hat{P}_j^{-1}(A_jV + B_hE)^T C^T < S$$

The above inequalities hold if there exist positive definite matrices U_j such that the following inequalities are satisfied for $j = 1, 2, \cdots, p$

$$U_j > (A_jV + B_jE)\hat{P}_j^{-1}(A_jV + B_jE)^T \tag{39}$$

$$CU_jC^T < S \tag{40}$$

Using Schur complement and Lemma 1, inequalities (39) and (40) are equivalent to the following inequalities

$$\begin{bmatrix} -U_j & (A_jV + B_jE) \\ (A_jV + B_jE)^T & -\hat{P}_j \end{bmatrix} < 0 \tag{41}$$

and

$$\begin{bmatrix} -S & C_jH^T \\ HC_j^T & U_j - H - H^T \end{bmatrix} < 0 \tag{42}$$

where H is an extra matrix variable. The rest of the proof is similar to that of Theorem 1, thus the proof is completed.

Remark 4: Note that in [2], it is assumed that the matrix C is known and constant, otherwise, the LMI condition of the constraints on outputs can not be obtained using the method proposed in [2]. In system (1), besides A and B, C can also be an uncertain matrix, and the constraints on outputs can be easily transformed into LMIs by the method proposed in this paper. Hence, the results obtained have covered those in [2] as a special case. It should be pointed out that the algorithm is easy to be implemented. The control input $u(k|k)$ is obtained at sampling time k by convex optimization in Theorem 1. If there are additional constraints on input and outputs, the control input $u(k|k)$ can be obtained at sampling time k by convex optimization in Theorem 3.

IV. Example

In this section, one example will be provided to illustrate the effectiveness of the techniques developed in this paper. As the method proposed in [2] is a special case of our methodology, the optimization problem should be feasible using the method proposed in this paper since it is solvable using the approach in [2]. However, the optimization may not have a solution by the result in [2], while it has a solution by our result.

Example 1: (input and output constraints) Consider the linear discrete-time parameter uncertain system (1) with

$$
\begin{aligned}
A_1 &= \begin{bmatrix} -0.90 & 0.80 \\ 0.35 & 0.45 \end{bmatrix}, A_2 = \begin{bmatrix} 0.90 & 0.85 \\ 0.40 & -0.85 \end{bmatrix}, A_3 = \begin{bmatrix} 0.96 & 0.13 \\ 0.28 & -0.90 \end{bmatrix} \\
B_1 &= \begin{bmatrix} 1 \\ -1 \end{bmatrix}, B_2 = \begin{bmatrix} 1 \\ -0.8 \end{bmatrix}, B_3 = \begin{bmatrix} 1 \\ -0.86 \end{bmatrix} \\
C_1 &= \begin{bmatrix} 1 & 0.3 \end{bmatrix}, C_2 = \begin{bmatrix} 0.8 & 0.2 \end{bmatrix}, C_3 = \begin{bmatrix} 1.2 & 0.4 \end{bmatrix},
\end{aligned} \tag{43}
$$

It is shown that the optimization is unfeasible with the method proposed in [2] without these constraints. However, taking output constraints with $y_{1,max} = 2$ and input constraints with $u_{1,max} = 0.8$, and uncertain parameters are assumed to be $\xi_1(k) = 0.5cos(k)cos(k), \xi_2(k) = 0.6sin(k)sin(k), \xi_3(k) = 0.5cos(k)cos(k) + 0.4sin(k)sin(k)$, it is feasible using the method proposed in this paper. The simulation results are given in Fig. 1:

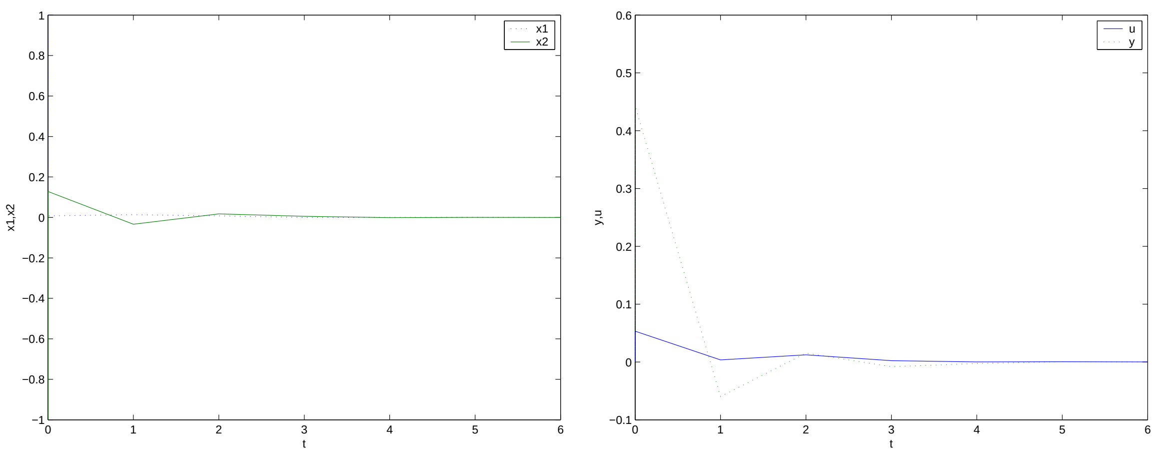

Fig. 1: States (x_1, x_2), output y and input u (method in this paper)

From the simulation with LMI-toolbox in Matlab, it is very effective to design the controller based on the method proposed in this paper. However, it should be mentioned that the complexity of computation will grow quickly with the increase of vertices of the convex set. Due to development of computer techniques, the difficulties can be overcomed.

V. Conclusion

The problem of robust constrained model predictive control based on parameter-dependent Lyapunov functions with polytopic type uncertainty has been addressed in this paper. The results are based on a new extended LMI characterization of the quadratic objective, hard constraints on inputs and outputs. Sufficient conditions in LMI do not involve the product of the Lyapunov matrices and the system dynamic matrices. The state feedback control guarantees the closed-loop system is robustly stable and the hard constraints on inputs and outputs are satisfied. The approach developed here provides a way to reduce the conservativeness of the existing conditions by decoupling the control parameterization from the Lyapunov matrix.

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APPENDIX

Proof of Theorem 2: In order to show that the closed-loop system (1) is robustly asymptotically stable, we need to prove $x(k|k) \to 0$, as $k \to 0$. Thus, in the following, we will prove the parameter-dependent Lyapunov function

$$V(x(k|k)) = x(k|k)^T P(\xi(k))x(k|k) = x(k|k)^T V^{-T} \sum_{j=1}^{p} \xi_j(k)\overline{P}_j(k)V^{-1}x(k|k) \tag{44}$$

$k = 1, 2, \cdots, \infty$, obtained at each sampling instant is a strictly decreasing function.

First, we will show: $V(x(k+1|k+1)$ with the convex minimal Lypunov matrix solution $\overline{P}(\xi(k))$ obtained at sampling k must be less than $V(x(k|k)$ with the convex minimal Lypunov matrix solution $\overline{P}(\xi(k))$ obtained at sampling time k, that is,

$$x(k+1|k+1)^T P(\xi(k))x(k+1|k+1) < x(k|k)^T P(\xi(k))x(k|k) \tag{45}$$

At sampling time k, taking the parameter-dependent Lyapunov function as

$$V(x(k|k)) = x(k|k)^T V^{-T} \sum_{j=1}^{p} \xi_j \overline{P}_j(k) V^{-1} x(k|k) = x(k|k)^T P(\xi(k)) x(k|k) \tag{46}$$

which is defined in Theorem 1. From Theorem 1, an upper bound of $V(x(k|k))$ is minimized at sampling time k with optimal solution ($\gamma(k)$, $\overline{P}_j(k)$, $j = 1, 2, \cdots, p$, $E(k)$, V). Then (18)-(20) in Theorem 1 are satisfied with this optimal solution. When $i = j$, multiplying inequality (20) in Theorem 1 with ξ_l, for $l = 1, 2, \cdots, p$ for each j, ~~which~~ leads to

$$\begin{bmatrix} -\overline{P}(\xi(k)) & V^T A(k)^T + E^T(k) B(k)^T & E^T(k) R_0 & V^T Q_0 \\ A(k)V + B(k)E & \overline{P}(\xi(k)) - V - V^T & 0 & 0 \\ E(k) R_0 & 0 & -R_0 & 0 \\ V Q_0 & 0 & 0 & -Q_0 \end{bmatrix} < 0 \tag{47}$$

Note that the following inequality always holds:

$$V^T \overline{P}(\xi(k))^{-1} V - V - V^T + \overline{P}(\xi(k)) = (V^T \overline{P}(\xi(k))^{-1} - I) \overline{P}(\xi(k)) (V^T \overline{P}(\xi(k))^{-1} - I)^T \geq 0 \tag{48}$$

which means that

$$\overline{P}(\xi(k)) - V - V^T \geq -V^T \overline{P}(\xi(k))^{-1} V$$

then, it follows from (47) that

$$\begin{bmatrix} -\overline{P}(\xi(k)) & V^T A(k)^T + E^T(k) B(k)^T & E^T(k) R_0 & V^T Q_0 \\ A(k)V + B(k)E & -V^T \overline{P}(\xi(k))^{-1} V & 0 & 0 \\ E(k) R_0 & 0 & -R_0 & 0 \\ V Q_0 & 0 & 0 & -Q_0 \end{bmatrix} < 0 \tag{49}$$

Note that $E(k) = F(k)V$, then, by Schur complement, (49) is equivalent to the following inequality:

$$(A(k) + B(k)F(k))^T P(\xi(k))(A(k) + B(k)F(k)) - P(\xi(k)) < F^T R_0 F + Q_0 \tag{50}$$

Pre- and post-multiplying the above inequality by $x(k|k)^T$ and $x(k|k)$ on both sides, respectively, we have

$$\begin{aligned} & x(k|k)^T((A(k) + B(k)F(k))^T P(\xi(k))(A(k) + B(k)F(k)))x(k|k) \\ & \quad - x(k|k)^T P(\xi(k)) x(k|k) + x(k|k)^T (F^T R_0 F + Q_0) x(k|k) < 0 \end{aligned}$$

for $x(k|k) \neq 0$. Note that $x(k+1|k+1) = [A(k) + B(k)F]x(k|k)$ for $[A(k)\ B(k)] \in \Omega$, where $x(k+1|k+1)$ means the state measured at $k+1$, then it follows from the above inequality that the following inequality holds for $x(k|k) \neq 0$

$$x(k+1|k+1)^T P(\xi(k))x(k+1|k+1) - x(k|k)^T P(\xi(k))x(k|k) < x(k|k)^T (F^T R_0 F + Q_0)x(k|k)$$

which leads to

$$x(k+1|k+1)^T P(\xi(k))x(k+1|k+1) < x(k|k)^T P(\xi(k))x(k|k) \tag{51}$$

Next, we will prove that the optimal solution $(\gamma(k), \overline{P}_j(k), j = 1, 2, \cdots, p, V, E(k))$ obtained by optimization at sampling time k is also a feasible solution to that at the sampling time $k+1$, that is, we should prove $(\gamma(k), \overline{P}_j(k), j = 1, 2, \cdots, p, V, E(k))$ satisfies the following inequalities:

$$\begin{bmatrix} -\gamma & x(k+1|k+1)^T \\ x(k+1|k+1) & \overline{P}_j(k) - V - V^T \end{bmatrix} < 0 \tag{52}$$

$$\begin{bmatrix} -\overline{P}_i(k) & V^T A_j^T + E^T(k) B_j^T & E^T(k) R_0 & V^T Q_0 \\ A_j V + B_j E(k) & \overline{P}_j(k) - V - V^T & 0 & 0 \\ E(k) R_0 & 0 & -R_0 & 0 \\ V Q_0 & 0 & 0 & -Q_0 \end{bmatrix} < 0 \tag{53}$$

$$\forall j = 1, 2, \cdots, p \text{ and } i = 1, 2, \cdots, p.$$

Obviously, $(\gamma(k), \overline{P}_j(k), j = 1, 2, \cdots, p, V, E(k))$ satisfies (53) since it is the same as the one at sampling time k. Then we only need to show that $(\gamma(k), \overline{P}_j(k), j = 1, 2, \cdots, p, V, E(k))$ satisfies (52).

It follows from (53) by Schur complement that

$$\begin{bmatrix} -\overline{P}_i(k) + E^T(k) R_0 E + V^T Q_0 V & V^T A_j^T + E^T(k) B_j^T \\ A_j V + B_j E(k) & \overline{P}_j(k) - V - V^T \end{bmatrix} < 0 \tag{54}$$

which implies

$$\begin{bmatrix} -\overline{P}_i(k) & V^T A_j^T + E^T(k) B_j^T \\ A_j V + B_j E(k) & \overline{P}_j - V - V^T \end{bmatrix} < 0 \tag{55}$$

For each i, multiply the above corresponding $j = 1, 2, \cdots, p$ inequalities by $\sigma_i(k) \geq 0$ and sum. Multiply the resulting $i = 1, 2, \cdots, p$ inequalities by $\xi(k) \geq 0$ and sum. Taking into account $\sum_{j=1}^{p} \sigma_j(k) = 1$ and $\sum_{i=1}^{p} \xi_i(k) = 1$, results in

$$\begin{bmatrix} -\tilde{P}(\sigma(k)) & V^T A^T(k) + E^T B^T(k) \\ A(k) V + B(k) E & \overline{P}(\xi(k)) - V - V^T \end{bmatrix} < 0 \tag{56}$$

where $\tilde{P}(\sigma(k)) = \sum_{i=1}^{p} \sigma_i(k)\overline{P}_i(k)$. Pre- and post-multiplying (56) by $\begin{bmatrix} V^{-T} & 0 \\ 0 & I \end{bmatrix}$ and $\begin{bmatrix} V^{-1} & 0 \\ 0 & I \end{bmatrix}$ on both sides, and from $F(k) = E(k)V^{-1}$, we obtain

$$\begin{bmatrix} -V^{-T}\tilde{P}(\sigma(k))V^{-1} & (A(k)+B(k)F)^T \\ (A(k)+B(k)F) & \overline{P}(\xi(k)) - V - V^T \end{bmatrix} < 0 \tag{57}$$

By Schur complement, (57) implies

$$(A(k)+B(k)F(k))^T(V+V^T-\overline{P}(\xi(k)))^{-1}(A(k)+B(k)F) < V^{-T}\tilde{P}(\sigma(k))V^{-1} \tag{58}$$

It follows from (48) that the following inequality always holds:

$$V^{-T}\tilde{P}(\sigma(k))V^{-1} \leq (V+V^T-\tilde{P}(\sigma(k)))^{-1} \tag{59}$$

Comparing (59) with (58), we have

$$(A(k)+B(k)F)^T(V+V^T-\overline{P}(\xi(k)))^{-1}(A(k)+B(k)F) < (V+V^T-\tilde{P}(\sigma(k)))^{-1}$$

Pre- and post-multiplying both sides of the above inequality with $x(k|k)^T$ and $x(k|k)$, respectively, and noting that $x(k+1|k+1) = (A(k)+B(k)F)x(k|k)$, we have

$$x(k+1|k+1)^T(V+V^T-\overline{P}(\xi(k)))^{-1}x(k+1|k+1) < x(k|k)(V+V^T-\tilde{P}(\sigma(k)))^{-1}x(k|k) \tag{60}$$

Since $(\gamma(k), \overline{P}_j(k), j = 1, 2, \cdots, p, E(k), V)$ is the optimal solution obtained at sampling k, it satisfies

$$\begin{bmatrix} -\gamma(k) & x(k|k)^T \\ x(k|k) & \overline{P}_j(k) - V^T - V \end{bmatrix} < 0, j = 1, 2, \cdots, p$$

By the convex combination with $\sigma_i(k)$, it follows from the above inequalities that

$$\begin{bmatrix} -\gamma(k) & x(k|k)^T \\ x(k|k) & \tilde{P}(\sigma(k)) - V^T - V \end{bmatrix} < 0$$

By Schur implement, the above inequality is equivalent to

$$x(k|k)^T(V+V^T-\tilde{P}(\sigma(k))^{-1}x(k|k) < \gamma \tag{61}$$

Comparing (61) with (60), we have

$$x(k+1|k+1)^T(\overline{V}+V^T-P(\xi(k)))^{-1}x(k+1|k+1) < \gamma(k) \tag{62}$$

By Schur complement, (62) is equivalent to

$$\begin{bmatrix} -\gamma(k) & x(k+1|k+1)^T \\ x(k+1|k+1) & \sum_{j=1}^{p}\xi_j\overline{P}_j(k) - V^T - V \end{bmatrix} < 0 \tag{63}$$

From Theorems 1 and 2 in [16], (63) is equivalent to

$$\begin{bmatrix} -\gamma(k) & x(k+1|k+1)^T \\ x(k+1|k+1) & \overline{P}_j(k) - V^T - V \end{bmatrix} < 0 \tag{64}$$

Then optimal solution ($\gamma(k), \overline{P}_j(k)$, $j = 1, 2, \cdots, p$, $E(k)$, V), i.e., ($P(\xi(k))$, $E(k)$, V) at sampling time k is also feasible solution at sampling time $k+1$, i.e, (64) and inequality (20) in Theorem 1 holds with the feasible solution obtained at sampling time k.

Finally, we will prove that $x(k|k) \to 0$ as $k \to \infty$. $P(\xi(k+1))$ denotes the convex optimal solution of minimization in Lyapunov matrix at sampling time $k+1$ by $P(\xi(k+1)$. Since the optimal solution ($\overline{P}_j$, $j = 1, 2, \cdots, p$, E, V), i.e., ($P(\xi(k))$, $j = 1, 2, \cdots, p$, E, V) at sampling time k is also a feasible solution at sampling time $k+1$, then $V(x(k+1|k+1)$ with the convex minimal Lypunov matrix solution $\overline{P}(\xi(k+1))$ obtained at sampling $k+1$ must be less than $V(x(k+1|k+1)$ with the feasible Lypunov matrix solution $\overline{P}(\xi(k))$ obtained at sampling time instant k, that is,

$$x(k+1|k+1)^T P(\xi(k+1))x(k+1|k+1) < x(k+1|k+1)^T P(\xi(k))x(k+1|k+1) \tag{65}$$

Comparing inequality (65) with inequality (51), the following inequality holds for $x(k|k) \neq 0$:

$$x(k+1|k+1)^T P(\xi(k+1))x(k+1|k+1) < x(k|k)^T P(\xi(k))x(k|k)$$

Which, in turn, implies that the parameter-dependent Lypunov function $x(k|k)^T P(\xi(k))x(k|k)$ is a strictly decreasing function. Thus,

$$x(k|k)^T P(\xi(k))x(k|k) \to 0, \text{when} \quad k \to \infty \tag{66}$$

As ($P_j, j = 1, 2, \cdots, p$, E, V) satisfies (20) in Theorem 1, then they satisfy (50), that is,

$$A(k) + B(k)F)^T P(\xi(k)))(A(k) + B(k)F) + F^T R_0 F + Q_0 < P(\xi(k))$$

It follows from the above inequality that

$$P(\xi(k)) > Q_0 \tag{67}$$

Comparing (68) and (67), it follows that

$$x(k|k)^T P(\xi(k))x(k|k) > x(k|k)^T Q_0 x(k|k) \tag{68}$$

Then

$$x(k|k)^T Q_0 x(k|k) \to 0, \text{when} \quad k \to \infty \tag{69}$$

Since $Q_0 > 0$ is a constant matrix, we have

$$x(k|k) \rightarrow 0, \text{when} \quad k \rightarrow \infty \tag{70}$$

Therefore, the feasible receding horizon state feedback control law obtained in Theorem 1 robustly asymptotically stabilizes the closed-loop system 1. The proof is completed.