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This is the Published version of the following publication

Dragomir, Sever S (2020) Further inequalities for the generalized k - g -fractional integrals of functions with bounded variation. Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics, 69 (1). pp. 49-72. ISSN 2618-6470

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FURTHER INEQUALITIES FOR THE GENERALIZED k - g -FRACTIONAL INTEGRALS OF FUNCTIONS WITH BOUNDED VARIATION

SILVESTRU SEVER DRAGOMIR

ABSTRACT. Let g be a strictly increasing function on (a, b) , having a continuous derivative g' on (a, b) . For the Lebesgue integrable function $f : (a, b) \rightarrow \mathbb{C}$, we define the k - g -left-sided fractional integral of f by

$$S_{k,g,a+}f(x) = \int_a^x k(g(x) - g(t)) g'(t) f(t) dt, \quad x \in (a, b)$$

and the k - g -right-sided fractional integral of f by

$$S_{k,g,b-}f(x) = \int_x^b k(g(t) - g(x)) g'(t) f(t) dt, \quad x \in [a, b),$$

where the kernel k is defined either on $(0, \infty)$ or on $[0, \infty)$ with complex values and integrable on any finite subinterval.

In this paper we establish some new inequalities for the k - g -fractional integrals of functions of bounded variation. Examples for the *generalized left- and right-sided Riemann-Liouville fractional integrals* of a function f with respect to another function g and a general exponential fractional integral are also provided.

1. INTRODUCTION

Assume that the kernel k is defined either on $(0, \infty)$ or on $[0, \infty)$ with complex values and integrable on any finite subinterval. We define the function $K : [0, \infty) \rightarrow \mathbb{C}$ by

$$K(t) := \begin{cases} \int_0^t k(s) ds & \text{if } 0 < t, \\ 0 & \text{if } t = 0. \end{cases}$$

Received by the editors: March 21, 2019; Accepted: July 19, 2019.

2010 *Mathematics Subject Classification*. Primary 26D15, 26D10; Secondary 26D07, 26A33.

Key words and phrases. Generalized Riemann-Liouville fractional integrals, Hadamard fractional integrals, Functions of bounded variation, Ostrowski type inequalities, Trapezoid inequalities.

As a simple example, if $k(t) = t^{\alpha-1}$ then for $\alpha \in (0, 1)$ the function k is defined on $(0, \infty)$ and $K(t) := \frac{1}{\alpha}t^\alpha$ for $t \in [0, \infty)$. If $\alpha \geq 1$, then k is defined on $[0, \infty)$ and $K(t) := \frac{1}{\alpha}t^\alpha$ for $t \in [0, \infty)$.

Let g be a strictly increasing function on (a, b) , having a continuous derivative g' on (a, b) . For the Lebesgue integrable function $f : (a, b) \rightarrow \mathbb{C}$, we define the *k-g-left-sided fractional integral* of f by

$$S_{k,g,a+}f(x) = \int_a^x k(g(x) - g(t))g'(t)f(t)dt, \quad x \in (a, b] \quad (1)$$

and the *k-g-right-sided fractional integral* of f by

$$S_{k,g,b-}f(x) = \int_x^b k(g(t) - g(x))g'(t)f(t)dt, \quad x \in [a, b). \quad (2)$$

If we take $k(t) = \frac{1}{\Gamma(\alpha)}t^{\alpha-1}$, where Γ is the *Gamma function*, then

$$\begin{aligned} S_{k,g,a+}f(x) &= \frac{1}{\Gamma(\alpha)} \int_a^x [g(x) - g(t)]^{\alpha-1} g'(t) f(t) dt \\ &=: I_{a+,g}^\alpha f(x), \quad a < x \leq b \end{aligned} \quad (3)$$

and

$$\begin{aligned} S_{k,g,b-}f(x) &= \frac{1}{\Gamma(\alpha)} \int_x^b [g(t) - g(x)]^{\alpha-1} g'(t) f(t) dt \\ &=: I_{b-,g}^\alpha f(x), \quad a \leq x < b, \end{aligned} \quad (4)$$

which are the *generalized left- and right-sided Riemann-Liouville fractional integrals* of a function f with respect to another function g on $[a, b]$ as defined in [23, p. 100].

For $g(t) = t$ in (4) we have the classical *Riemann-Liouville fractional integrals* while for the logarithmic function $g(t) = \ln t$ we have the *Hadamard fractional integrals* [23, p. 111]

$$H_{a+}^\alpha f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x \left[\ln \left(\frac{x}{t} \right) \right]^{\alpha-1} \frac{f(t) dt}{t}, \quad 0 \leq a < x \leq b \quad (5)$$

and

$$H_{b-}^\alpha f(x) := \frac{1}{\Gamma(\alpha)} \int_x^b \left[\ln \left(\frac{t}{x} \right) \right]^{\alpha-1} \frac{f(t) dt}{t}, \quad 0 \leq a < x < b. \quad (6)$$

One can consider the function $g(t) = -t^{-1}$ and define the "*Harmonic fractional integrals*" by

$$R_{a+}^\alpha f(x) := \frac{x^{1-\alpha}}{\Gamma(\alpha)} \int_a^x \frac{f(t) dt}{(x-t)^{1-\alpha} t^{\alpha+1}}, \quad 0 \leq a < x \leq b \quad (7)$$

and

$$R_{b-}^\alpha f(x) := \frac{x^{1-\alpha}}{\Gamma(\alpha)} \int_x^b \frac{f(t) dt}{(t-x)^{1-\alpha} t^{\alpha+1}}, \quad 0 \leq a < x < b. \quad (8)$$

Also, for $g(t) = \exp(\beta t)$, $\beta > 0$, we can consider the " β -Exponential fractional integrals"

$$E_{a+,\beta}^\alpha f(x) := \frac{\beta}{\Gamma(\alpha)} \int_a^x [\exp(\beta x) - \exp(\beta t)]^{\alpha-1} \exp(\beta t) f(t) dt, \quad (9)$$

for $a < x \leq b$ and

$$E_{b-,\beta}^\alpha f(x) := \frac{\beta}{\Gamma(\alpha)} \int_x^b [\exp(\beta t) - \exp(\beta x)]^{\alpha-1} \exp(\beta t) f(t) dt, \quad (10)$$

for $a \leq x < b$.

If we take $g(t) = t$ in (1) and (2), then we can consider the following k -fractional integrals

$$S_{k,a+} f(x) = \int_a^x k(x-t) f(t) dt, \quad x \in (a, b] \quad (11)$$

and

$$S_{k,b-} f(x) = \int_x^b k(t-x) f(t) dt, \quad x \in [a, b). \quad (12)$$

In [26], Raina studied a class of functions defined formally by

$$\mathcal{F}_{\rho,\lambda}^\sigma(x) := \sum_{k=0}^{\infty} \frac{\sigma(k)}{\Gamma(\rho k + \lambda)} x^k, \quad |x| < R, \quad \text{with } R > 0 \quad (13)$$

where $\rho, \lambda > 0$ where the coefficients $\sigma(k)$ generate a bounded sequence of positive real numbers. With the help of (13), Raina defined the following left-sided fractional integral operator

$$\mathcal{J}_{\rho,\lambda,a+;w}^\sigma f(x) := \int_a^x (x-t)^{\lambda-1} \mathcal{F}_{\rho,\lambda}^\sigma(w(x-t)^\rho) f(t) dt, \quad x > a \quad (14)$$

where $\rho, \lambda > 0$, $w \in \mathbb{R}$ and f is such that the integral on the right side exists.

In [1], the right-sided fractional operator was also introduced as

$$\mathcal{J}_{\rho,\lambda,b-;w}^\sigma f(x) := \int_x^b (t-x)^{\lambda-1} \mathcal{F}_{\rho,\lambda}^\sigma(w(t-x)^\rho) f(t) dt, \quad x < b \quad (15)$$

where $\rho, \lambda > 0$, $w \in \mathbb{R}$ and f is such that the integral on the right side exists. Several Ostrowski type inequalities were also established.

We observe that for $k(t) = t^{\lambda-1} \mathcal{F}_{\rho,\lambda}^\sigma(wt^\rho)$ we re-obtain the definitions of (14) and (15) from (11) and (12).

In [24], Kirane and Torebek introduced the following *exponential fractional integrals*

$$\mathcal{T}_{a+}^\alpha f(x) := \frac{1}{\alpha} \int_a^x \exp\left\{-\frac{1-\alpha}{\alpha}(x-t)\right\} f(t) dt, \quad x > a \quad (16)$$

and

$$\mathcal{T}_{b-}^\alpha f(x) := \frac{1}{\alpha} \int_x^b \exp\left\{-\frac{1-\alpha}{\alpha}(t-x)\right\} f(t) dt, \quad x < b \quad (17)$$

where $\alpha \in (0, 1)$.

We observe that for $k(t) = \frac{1}{\alpha} \exp(-\frac{1-\alpha}{\alpha}t)$, $t \in \mathbb{R}$ we re-obtain the definitions of (16) and (17) from (11) and (12).

Let g be a strictly increasing function on (a, b) , having a continuous derivative g' on (a, b) . We can define the more general exponential fractional integrals

$$\mathcal{T}_{g,a+}^{\alpha} f(x) := \frac{1}{\alpha} \int_a^x \exp\left\{-\frac{1-\alpha}{\alpha}(g(x) - g(t))\right\} g'(t) f(t) dt, \quad x > a \quad (18)$$

and

$$\mathcal{T}_{g,b-}^{\alpha} f(x) := \frac{1}{\alpha} \int_x^b \exp\left\{-\frac{1-\alpha}{\alpha}(g(t) - g(x))\right\} g'(t) f(t) dt, \quad x < b \quad (19)$$

where $\alpha \in (0, 1)$.

Let g be a strictly increasing function on (a, b) , having a continuous derivative g' on (a, b) . Assume that $\alpha > 0$. We can also define the *logarithmic fractional integrals*

$$\mathcal{L}_{g,a+}^{\alpha} f(x) := \int_a^x (g(x) - g(t))^{\alpha-1} \ln(g(x) - g(t)) g'(t) f(t) dt, \quad (20)$$

for $0 < a < x \leq b$ and

$$\mathcal{L}_{g,b-}^{\alpha} f(x) := \int_x^b (g(t) - g(x))^{\alpha-1} \ln(g(t) - g(x)) g'(t) f(t) dt, \quad (21)$$

for $0 < a \leq x < b$, where $\alpha > 0$. These are obtained from (11) and (12) for the kernel $k(t) = t^{\alpha-1} \ln t$, $t > 0$.

For $\alpha = 1$ we get

$$\mathcal{L}_{g,a+} f(x) := \int_a^x \ln(g(x) - g(t)) g'(t) f(t) dt, \quad 0 < a < x \leq b \quad (22)$$

and

$$\mathcal{L}_{g,b-} f(x) := \int_x^b \ln(g(t) - g(x)) g'(t) f(t) dt, \quad 0 < a \leq x < b. \quad (23)$$

For $g(t) = t$, we have the simple forms

$$\mathcal{L}_{a+}^{\alpha} f(x) := \int_a^x (x-t)^{\alpha-1} \ln(x-t) f(t) dt, \quad 0 < a < x \leq b, \quad (24)$$

$$\mathcal{L}_{b-}^{\alpha} f(x) := \int_x^b (t-x)^{\alpha-1} \ln(t-x) f(t) dt, \quad 0 < a \leq x < b, \quad (25)$$

$$\mathcal{L}_{a+} f(x) := \int_a^x \ln(x-t) f(t) dt, \quad 0 < a < x \leq b \quad (26)$$

and

$$\mathcal{L}_{b-} f(x) := \int_x^b \ln(t-x) f(t) dt, \quad 0 < a \leq x < b. \quad (27)$$

For several Ostrowski type inequalities for Riemann-Liouville fractional integrals see [2]-[17], [21]-[34] and the references therein.

For k and g as at the beginning of Introduction, we consider the mixed operator

$$\begin{aligned} S_{k,g,a+,b-}f(x) & \quad (28) \\ & := \frac{1}{2} [S_{k,g,a+}f(x) + S_{k,g,b-}f(x)] \\ & = \frac{1}{2} \left[\int_a^x k(g(x) - g(t)) g'(t) f(t) dt + \int_x^b k(g(t) - g(x)) g'(t) f(t) dt \right] \end{aligned}$$

for the Lebesgue integrable function $f : (a, b) \rightarrow \mathbb{C}$ and $x \in (a, b)$.

We also define the function $\mathbf{K} : [0, \infty) \rightarrow [0, \infty)$ by

$$\mathbf{K}(t) := \begin{cases} \int_0^t |k(s)| ds & \text{if } 0 < t, \\ 0 & \text{if } t = 0. \end{cases}$$

In the recent paper [19] we obtained the following result for functions of bounded variation:

Theorem 1. *Assume that the kernel k is defined either on $(0, \infty)$ or on $[0, \infty)$ with complex values and integrable on any finite subinterval. Let $f : [a, b] \rightarrow \mathbb{C}$ be a function of bounded variation on $[a, b]$ and g be a strictly increasing function on (a, b) , having a continuous derivative g' on (a, b) . Then we have the Ostrowski type inequality*

$$\begin{aligned} & \left| S_{k,g,a+,b-}f(x) - \frac{1}{2} [K(g(b) - g(x)) + K(g(x) - g(a))] f(x) \right| \\ & \leq \frac{1}{2} \left[\int_x^b |k(g(t) - g(x))| \bigvee_x^t(f) g'(t) dt + \int_a^x |k(g(x) - g(t))| \bigvee_t^x(f) g'(t) dt \right] \\ & \leq \frac{1}{2} \left[\mathbf{K}(g(b) - g(x)) \bigvee_x^b(f) + \mathbf{K}(g(x) - g(a)) \bigvee_a^x(f) \right] \\ & \leq \frac{1}{2} \begin{cases} \max \{ \mathbf{K}(g(b) - g(x)), \mathbf{K}(g(x) - g(a)) \} \bigvee_a^b(f); \\ [\mathbf{K}^p(g(b) - g(x)) + \mathbf{K}^p(g(x) - g(a))]^{1/p} \left((\bigvee_a^x(f))^q + (\bigvee_x^b(f))^q \right)^{1/q} \\ \text{with } p, q > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ [\mathbf{K}(g(b) - g(x)) + \mathbf{K}(g(x) - g(a))] \left[\frac{1}{2} \bigvee_a^b(f) + \frac{1}{2} \left| \bigvee_a^x(f) - \bigvee_x^b(f) \right| \right] \end{cases} \quad (29) \end{aligned}$$

and the trapezoid type inequality

$$\left| S_{k,g,a+,b-}f(x) - \frac{1}{2} [K(g(b) - g(x)) f(b) + K(g(x) - g(a)) f(a)] \right|$$

$$\begin{aligned}
&\leq \frac{1}{2} \left[\int_a^x |k(g(x) - g(t))| \bigvee_a^t(f) g'(t) dt + \int_x^b |k(g(t) - g(x))| \bigvee_t^b(f) g'(t) dt \right] \\
&\leq \frac{1}{2} \left[\mathbf{K}(g(b) - g(x)) \bigvee_x^b(f) + \mathbf{K}(g(x) - g(a)) \bigvee_a^x(f) \right] \\
&\leq \frac{1}{2} \left\{ \begin{array}{l} \max \{ \mathbf{K}(g(b) - g(x)), \mathbf{K}(g(x) - g(a)) \} \bigvee_a^b(f); \\ [\mathbf{K}^p(g(b) - g(x)) + \mathbf{K}^p(g(x) - g(a))]^{1/p} \\ \times \left((\bigvee_a^x(f))^q + (\bigvee_x^b(f))^q \right)^{1/q} \\ \text{with } p, q > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ [\mathbf{K}(g(b) - g(x)) + \mathbf{K}(g(x) - g(a))] \\ \times \left[\frac{1}{2} \bigvee_a^b(f) + \frac{1}{2} \left| \bigvee_a^x(f) - \bigvee_x^b(f) \right| \right] \end{array} \right. \quad (30)
\end{aligned}$$

for any $x \in (a, b)$, where $\bigvee_c^d(f)$ denoted the total variation on the interval $[c, d]$.

Observe that

$$S_{k,g,x+}f(b) = \int_x^b k(g(b) - g(t)) g'(t) f(t) dt, \quad x \in [a, b] \quad (31)$$

and

$$S_{k,g,x-}f(a) = \int_a^x k(g(t) - g(a)) g'(t) f(t) dt, \quad x \in (a, b]. \quad (32)$$

We can define also the mixed operator

$$\begin{aligned}
&\check{S}_{k,g,a+,b-}f(x) \quad (33) \\
&:= \frac{1}{2} [S_{k,g,x+}f(b) + S_{k,g,x-}f(a)] \\
&= \frac{1}{2} \left[\int_x^b k(g(b) - g(t)) g'(t) f(t) dt + \int_a^x k(g(t) - g(a)) g'(t) f(t) dt \right]
\end{aligned}$$

for any $x \in (a, b)$.

In this paper we establish some inequalities for the k - g -fractional integrals of functions with bounded variation $f : [a, b] \rightarrow \mathbb{C}$ that provide error bounds in approximating the composite operators $S_{k,g,a+,b-}f$ and $\check{S}_{k,g,a+,b-}f$ in terms of the *double trapezoid rule*

$$\frac{1}{2} \left[\frac{f(x) + f(b)}{2} K(g(b) - g(x)) + \frac{f(a) + f(x)}{2} K(g(x) - g(a)) \right], \quad x \in (a, b).$$

Examples for the *generalized left- and right-sided Riemann-Liouville fractional integrals* of a function f with respect to another function g and a general exponential fractional integral are also provided.

2. FURTHER INEQUALITIES FOR FUNCTIONS OF BV

The following two parameters representation for the operators $S_{k,g,a+,b-}$ and $\check{S}_{k,g,a+,b-}$ hold [20]:

Lemma 2. *Assume that the kernel k is defined either on $(0, \infty)$ or on $[0, \infty)$ with complex values and integrable on any finite subinterval. Let $f : [a, b] \rightarrow \mathbb{C}$ be an integrable function on $[a, b]$ and g be a strictly increasing function on (a, b) , having a continuous derivative g' on (a, b) . Then*

$$\begin{aligned} S_{k,g,a+,b-}f(x) &= \frac{1}{2} [\gamma K(g(b) - g(x)) + \lambda K(g(x) - g(a))] \\ &\quad + \frac{1}{2} \int_a^x k(g(x) - g(t)) g'(t) [f(t) - \lambda] dt \\ &\quad + \frac{1}{2} \int_x^b k(g(t) - g(x)) g'(t) [f(t) - \gamma] dt \end{aligned} \quad (34)$$

and

$$\begin{aligned} \check{S}_{k,g,a+,b-}f(x) &= \frac{1}{2} [\lambda K(g(b) - g(x)) + \gamma K(g(x) - g(a))] \\ &\quad + \frac{1}{2} \int_a^x k(g(t) - g(a)) g'(t) [f(t) - \gamma] dt \\ &\quad + \frac{1}{2} \int_x^b k(g(b) - g(t)) g'(t) [f(t) - \lambda] dt \end{aligned} \quad (35)$$

for $x \in (a, b)$ and for any $\lambda, \gamma \in \mathbb{C}$.

Proof. We have, by taking the derivative over t and using the chain rule, that

$$[K(g(x) - g(t))] = K'(g(x) - g(t)) (g(x) - g(t))' = -k(g(x) - g(t)) g'(t)$$

for $t \in (a, x)$ and

$$[K(g(t) - g(x))] = K'(g(t) - g(x)) (g(t) - g(x))' = k(g(t) - g(x)) g'(t)$$

for $t \in (x, b)$.

Therefore, for any $\lambda, \gamma \in \mathbb{C}$ we have

$$\begin{aligned} &\int_a^x k(g(x) - g(t)) g'(t) [f(t) - \lambda] dt \\ &= \int_a^x k(g(x) - g(t)) g'(t) f(t) dt - \lambda \int_a^x k(g(x) - g(t)) g'(t) dt \\ &= S_{k,g,a+}f(x) + \lambda \int_a^x [K(g(x) - g(t))] dt \\ &= S_{k,g,a+}f(x) + \lambda [K(g(x) - g(t))]_a^x = S_{k,g,a+}f(x) - \lambda K(g(x) - g(a)) \end{aligned} \quad (36)$$

and

$$\begin{aligned}
& \int_x^b k(g(t) - g(x)) g'(t) [f(t) - \gamma] dt \tag{37} \\
&= \int_x^b k(g(t) - g(x)) g'(t) f(t) dt - \gamma \int_x^b k(g(t) - g(x)) g'(t) dt \\
&= S_{k,g,b-} f(x) - \gamma \int_x^b [K(g(t) - g(x))]_x' dt \\
&= S_{k,g,b-} f(x) - \gamma [K(g(t) - g(x))]_x^b = S_{k,g,b-} f(x) - \gamma K(g(b) - g(x))
\end{aligned}$$

for $x \in (a, b)$.

If we add the equalities (36) and (37) and divide by 2 then we get the desired result (34).

Moreover, by taking the derivative over t and using the chain rule, we have that

$$[K(g(b) - g(t))]_t' = K'(g(b) - g(t)) (g(b) - g(t))' = -k(g(b) - g(t)) g'(t)$$

for $t \in (x, b)$ and

$$[K(g(t) - g(a))]_t' = K'(g(t) - g(a)) (g(t) - g(a))' = k(g(t) - g(a)) g'(t)$$

for $t \in (a, x)$.

For any $\lambda, \gamma \in \mathbb{C}$ we have

$$\begin{aligned}
& \int_x^b k(g(b) - g(t)) g'(t) [f(t) - \lambda] dt \tag{38} \\
&= \int_x^b k(g(b) - g(t)) g'(t) f(t) dt - \lambda \int_x^b k(g(b) - g(t)) g'(t) dt \\
&= S_{k,g,x+} f(b) + \lambda \int_x^b [K(g(b) - g(t))]_t' dt \\
&= S_{k,g,x+} f(b) - \lambda K(g(b) - g(x))
\end{aligned}$$

and

$$\begin{aligned}
& \int_a^x k(g(t) - g(a)) g'(t) [f(t) - \gamma] dt \tag{39} \\
&= \int_a^x k(g(t) - g(a)) g'(t) f(t) dt - \gamma \int_a^x k(g(t) - g(a)) g'(t) dt \\
&= \int_a^x k(g(t) - g(a)) g'(t) f(t) dt - \gamma \int_a^x [K(g(t) - g(a))]_t' dt \\
&= \int_a^x k(g(t) - g(a)) g'(t) f(t) dt - \gamma K(g(x) - g(a))
\end{aligned}$$

for $x \in (a, b)$.

If we add the equalities (38) and (39) and divide by 2 then we get the desired result (35). \square

If g is a function which maps an interval I of the real line to the real numbers, and is both continuous and injective then we can define the g -mean of two numbers $a, b \in I$ as

$$M_g(a, b) := g^{-1} \left(\frac{g(a) + g(b)}{2} \right).$$

If $I = \mathbb{R}$ and $g(t) = t$ is the *identity function*, then $M_g(a, b) = A(a, b) := \frac{a+b}{2}$, the *arithmetic mean*. If $I = (0, \infty)$ and $g(t) = \ln t$, then $M_g(a, b) = G(a, b) := \sqrt{ab}$, the *geometric mean*. If $I = (0, \infty)$ and $g(t) = \frac{1}{t}$, then $M_g(a, b) = H(a, b) := \frac{2ab}{a+b}$, the *harmonic mean*. If $I = (0, \infty)$ and $g(t) = t^p$, $p \neq 0$, then $M_g(a, b) = M_p(a, b) := \left(\frac{a^p + b^p}{2} \right)^{1/p}$, the *power mean with exponent p* . Finally, if $I = \mathbb{R}$ and $g(t) = \exp t$, then

$$M_g(a, b) = LME(a, b) := \ln \left(\frac{\exp a + \exp b}{2} \right),$$

the *LogMeanExp function*.

Using the g -mean of two numbers we can introduce

$$\begin{aligned} P_{k,g,a+,b-}f &:= S_{k,g,a+,b-}f(M_g(a, b)) \\ &= \frac{1}{2} \int_a^{M_g(a,b)} k \left(\frac{g(a) + g(b)}{2} - g(t) \right) g'(t) f(t) dt \\ &\quad + \frac{1}{2} \int_{M_g(a,b)}^b k \left(g(t) - \frac{g(a) + g(b)}{2} \right) g'(t) f(t) dt. \end{aligned} \quad (40)$$

Using the representation (34) we have

$$\begin{aligned} P_{k,g,a+,b-}f &= K \left(\frac{g(b) - g(a)}{2} \right) \frac{\gamma + \lambda}{2} \\ &\quad + \frac{1}{2} \int_a^{M_g(a,b)} k \left(\frac{g(a) + g(b)}{2} - g(t) \right) g'(t) [f(t) - \lambda] dt \\ &\quad + \frac{1}{2} \int_{M_g(a,b)}^b k \left(g(t) - \frac{g(a) + g(b)}{2} \right) g'(t) [f(t) - \gamma] dt \end{aligned} \quad (41)$$

for any $\lambda, \gamma \in \mathbb{C}$.

Also, if

$$\begin{aligned} \check{P}_{k,g,a+,b-}f &:= \check{S}_{k,g,a+,b-}f(M_g(a, b)) \\ &= \frac{1}{2} \int_{M_g(a,b)}^b k (g(b) - g(t)) g'(t) f(t) dt \\ &\quad + \frac{1}{2} \int_a^{M_g(a,b)} k (g(t) - g(a)) g'(t) f(t) dt. \end{aligned} \quad (42)$$

then by (35) we get

$$\begin{aligned} \check{P}_{k,g,a+,b-}f &= K \left(\frac{g(b) - g(a)}{2} \right) \frac{\gamma + \lambda}{2} \\ &+ \frac{1}{2} \int_a^{M_g(a,b)} k(g(t) - g(a)) g'(t) [f(t) - \gamma] dt \\ &+ \frac{1}{2} \int_{M_g(a,b)}^b k(g(b) - g(t)) g'(t) [f(t) - \lambda] dt \end{aligned} \quad (43)$$

for any $\lambda, \gamma \in \mathbb{C}$.

Theorem 3. *Assume that the kernel k is defined either on $(0, \infty)$ or on $[0, \infty)$ with complex values and integrable on any finite subinterval. Let $f : [a, b] \rightarrow \mathbb{C}$ be a function of bounded variation on $[a, b]$ and g be a strictly increasing function on (a, b) , having a continuous derivative g' on (a, b) . Then we have the double trapezoid inequalities*

$$\begin{aligned} &|S_{k,g,a+,b-}f(x) \\ &\quad - \frac{1}{2} \left[\frac{f(x) + f(b)}{2} K(g(b) - g(x)) + \frac{f(a) + f(x)}{2} K(g(x) - g(a)) \right]| \\ &\quad \leq \frac{1}{4} \left[\mathbf{K}(g(x) - g(a)) \bigvee_a^x(f) + \mathbf{K}(g(b) - g(x)) \bigvee_x^b(f) \right] \\ &\leq \frac{1}{4} \left\{ \begin{array}{l} \max \{ \mathbf{K}(g(b) - g(x)), \mathbf{K}(g(x) - g(a)) \} \bigvee_a^b(f); \\ [\mathbf{K}^p(g(b) - g(x)) + \mathbf{K}^p(g(x) - g(a))]^{1/p} \left((\bigvee_a^x(f))^q + (\bigvee_x^b(f))^q \right)^{1/q} \\ \text{with } p, q > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ [\mathbf{K}(g(b) - g(x)) + \mathbf{K}(g(x) - g(a))] \left[\frac{1}{2} \bigvee_a^b(f) + \frac{1}{2} \left| \bigvee_a^x(f) - \bigvee_x^b(f) \right| \right] \end{array} \right. \end{aligned} \quad (44)$$

and

$$\begin{aligned} &|\check{S}_{k,g,a+,b-}f(x) \\ &\quad - \frac{1}{2} \left[\frac{f(x) + f(b)}{2} K(g(b) - g(x)) + \frac{f(a) + f(x)}{2} K(g(x) - g(a)) \right]| \\ &\quad \leq \frac{1}{4} \left[\mathbf{K}(g(x) - g(a)) \bigvee_a^x(f) + \mathbf{K}(g(b) - g(x)) \bigvee_x^b(f) \right] \end{aligned}$$

$$\leq \frac{1}{4} \begin{cases} \max \{ \mathbf{K}(g(b) - g(x)), \mathbf{K}(g(x) - g(a)) \} \mathbf{V}_a^b(f); \\ [\mathbf{K}^p(g(b) - g(x)) + \mathbf{K}^p(g(x) - g(a))]^{1/p} \left((\mathbf{V}_a^x(f))^q + (\mathbf{V}_x^b(f))^q \right)^{1/q} \\ \text{with } p, q > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ [\mathbf{K}(g(b) - g(x)) + \mathbf{K}(g(x) - g(a))] \left[\frac{1}{2} \mathbf{V}_a^b(f) + \frac{1}{2} \left| \mathbf{V}_a^x(f) - \mathbf{V}_x^b(f) \right| \right] \end{cases} \quad (45)$$

for $x \in (a, b)$.

Proof. Using the identity (34) for $\lambda = \frac{f(a)+f(x)}{2}$ and $\gamma = \frac{f(x)+f(b)}{2}$ we have

$$\begin{aligned} & S_{k,g,a+,b-}f(x) \\ &= \frac{1}{2} \left[\frac{f(x) + f(b)}{2} K(g(b) - g(x)) + \frac{f(a) + f(x)}{2} K(g(x) - g(a)) \right] \\ &+ \frac{1}{2} \int_a^x k(g(x) - g(t)) g'(t) \left[f(t) - \frac{f(a) + f(x)}{2} \right] dt \\ &+ \frac{1}{2} \int_x^b k(g(t) - g(x)) g'(t) \left[f(t) - \frac{f(x) + f(b)}{2} \right] dt \end{aligned} \quad (46)$$

for $x \in (a, b)$.

Since f is of bounded variation, then

$$\begin{aligned} \left| f(t) - \frac{f(a) + f(x)}{2} \right| &= \left| \frac{f(t) - f(a) + f(t) - f(x)}{2} \right| \\ &\leq \frac{1}{2} [|f(t) - f(a)| + |f(x) - f(t)|] \leq \frac{1}{2} \bigvee_a^x(f) \end{aligned}$$

and

$$\begin{aligned} \left| f(t) - \frac{f(x) + f(b)}{2} \right| &= \left| \frac{f(t) - f(x) + f(t) - f(b)}{2} \right| \\ &\leq \frac{1}{2} [|f(t) - f(x)| + |f(b) - f(t)|] \leq \frac{1}{2} \bigvee_x^b(f) \end{aligned}$$

for $x \in (a, b)$.

Using the equality (46) we have

$$\begin{aligned} & |S_{k,g,a+,b-}f(x) \\ & - \frac{1}{2} \left[\frac{f(x) + f(b)}{2} K(g(b) - g(x)) + \frac{f(a) + f(x)}{2} K(g(x) - g(a)) \right]| \\ & \leq \frac{1}{2} \left| \int_a^x k(g(x) - g(t)) g'(t) \left[f(t) - \frac{f(a) + f(x)}{2} \right] dt \right| \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2} \left| \int_x^b k(g(t) - g(x)) g'(t) \left[f(t) - \frac{f(x) + f(b)}{2} \right] dt \right| \\
& \leq \frac{1}{2} \int_a^x |k(g(x) - g(t))| \left| f(t) - \frac{f(a) + f(x)}{2} \right| g'(t) dt \\
& \quad + \frac{1}{2} \int_x^b |k(g(t) - g(x))| \left| f(t) - \frac{f(x) + f(b)}{2} \right| g'(t) dt \\
& \leq \frac{1}{4} \left[\bigvee_a^x(f) \int_a^x |k(g(x) - g(t))| g'(t) dt + \bigvee_x^b(f) \int_x^b |k(g(t) - g(x))| g'(t) dt \right] \\
& \qquad \qquad \qquad =: B(x) \quad (47)
\end{aligned}$$

for $x \in (a, b)$.

We have, by taking the derivative over t and using the chain rule, that

$$[\mathbf{K}(g(x) - g(t))] = \mathbf{K}'(g(x) - g(t)) (g(x) - g(t))' = -|k(g(x) - g(t))| g'(t)$$

for $t \in (a, x)$ and

$$[\mathbf{K}(g(t) - g(x))] = \mathbf{K}'(g(t) - g(x)) (g(t) - g(x))' = |k(g(t) - g(x))| g'(t)$$

for $t \in (x, b)$.

Then

$$\int_a^x |k(g(x) - g(t))| g'(t) dt = - \int_a^x [\mathbf{K}(g(x) - g(t))] dt = \mathbf{K}(g(x) - g(a))$$

and

$$\int_x^b |k(g(t) - g(x))| g'(t) dt = \int_x^b [\mathbf{K}(g(t) - g(x))] dt = \mathbf{K}(g(b) - g(x)).$$

Therefore

$$\begin{aligned}
B(x) &= \frac{1}{4} \left[\bigvee_a^x(f) \int_a^x |k(g(x) - g(t))| g'(t) dt + \bigvee_x^b(f) \int_x^b |k(g(t) - g(x))| g'(t) dt \right] \\
&= \frac{1}{4} \left[\mathbf{K}(g(x) - g(a)) \bigvee_a^x(f) + \mathbf{K}(g(b) - g(x)) \bigvee_x^b(f) \right].
\end{aligned}$$

The last part of (44) is obvious by making use of the elementary Hölder type inequalities for positive real numbers $c, d, m, n \geq 0$

$$mc + nd \leq \begin{cases} \max\{m, n\} (c + d); \\ (m^p + n^p)^{1/p} (c^q + d^q)^{1/q} \text{ with } p, q > 1, \frac{1}{p} + \frac{1}{q} = 1. \end{cases}$$

Using the identity (35) for $\lambda = \frac{f(x)+f(b)}{2}$ and $\gamma = \frac{f(x)+f(a)}{2}$ we also have

$$\left| \check{S}_{k,g,a+,b-} f(x) \right|$$

$$\begin{aligned}
& -\frac{1}{2} \left[\frac{f(x) + f(b)}{2} K(g(b) - g(x)) + \frac{f(x) + f(a)}{2} K(g(x) - g(a)) \right] \Big| \\
& \leq \frac{1}{2} \int_a^x |k(g(t) - g(a))| \left| f(t) - \frac{f(x) + f(a)}{2} \right| g'(t) dt \\
& \quad + \frac{1}{2} \int_x^b |k(g(b) - g(t))| \left| f(t) - \frac{f(x) + f(b)}{2} \right| g'(t) dt \\
& \leq \frac{1}{4} \mathcal{V}_a^x(f) \int_a^x |k(g(t) - g(a))| g'(t) dt + \frac{1}{4} \mathcal{V}_x^b(f) \int_x^b |k(g(b) - g(t))| g'(t) dt \\
& \qquad \qquad \qquad =: C(x).
\end{aligned}$$

We also have, by taking the derivative over t and using the chain rule, that

$$[\mathbf{K}(g(b) - g(t))] = \mathbf{K}'(g(b) - g(t)) (g(b) - g(t))' = -|k(g(b) - g(t))| g'(t)$$

for $t \in (x, b)$ and

$$[\mathbf{K}(g(t) - g(a))] = \mathbf{K}'(g(t) - g(a)) (g(t) - g(a))' = |k(g(t) - g(a))| g'(t)$$

for $t \in (a, x)$.

Therefore

$$\int_a^x |k(g(t) - g(a))| g'(t) dt = \mathbf{K}(g(x) - g(a))$$

and

$$\int_x^b |k(g(b) - g(t))| g'(t) dt = \mathbf{K}(g(b) - g(x))$$

giving that

$$C(x) = \frac{1}{4} \mathcal{V}_a^x(f) \mathbf{K}(g(x) - g(a)) + \frac{1}{4} \mathcal{V}_x^b(f) \mathbf{K}(g(b) - g(x))$$

for $x \in (a, b)$, and the inequality (45) is thus proved. \square

Corollary 4. *With the assumptions of Theorem 3 we have*

$$\begin{aligned}
& \left| P_{k,g,a+,b-f} - \frac{1}{2} K \left(\frac{g(b) - g(a)}{2} \right) \left[f(M_g(a,b)) + \frac{f(a) + f(b)}{2} \right] \right| \quad (48) \\
& \leq \frac{1}{4} \mathbf{K} \left(\frac{g(b) - g(a)}{2} \right) \mathcal{V}_a^b(f)
\end{aligned}$$

and

$$\begin{aligned}
& \left| \check{P}_{k,g,a+,b-f} - \frac{1}{2} K \left(\frac{g(b) - g(a)}{2} \right) \left[f(M_g(a,b)) + \frac{f(a) + f(b)}{2} \right] \right| \quad (49) \\
& \leq \frac{1}{4} \mathbf{K} \left(\frac{g(b) - g(a)}{2} \right) \mathcal{V}_a^b(f).
\end{aligned}$$

If we take $x = \frac{a+b}{2}$ in (44) and (45), then we get

$$\begin{aligned}
& \left| S_{k,g,a+,b-f} \left(\frac{a+b}{2} \right) - \frac{f \left(\frac{a+b}{2} \right) + f(b)}{4} K \left(g(b) - g \left(\frac{a+b}{2} \right) \right) \right. \\
& \quad \left. - \frac{f(a) + f \left(\frac{a+b}{2} \right)}{4} K \left(g \left(\frac{a+b}{2} \right) - g(a) \right) \right| \\
& \leq \frac{1}{4} \left[\mathbf{K} \left(g \left(\frac{a+b}{2} \right) - g(a) \right) \overset{\frac{a+b}{2}}{\underset{a}{\mathbb{V}}} (f) + \mathbf{K} \left(g(b) - g \left(\frac{a+b}{2} \right) \right) \overset{b}{\underset{\frac{a+b}{2}}{\mathbb{V}}} (f) \right] \\
& \leq \frac{1}{4} \begin{cases} \max \{ \mathbf{K} (g(b) - g \left(\frac{a+b}{2} \right)), \mathbf{K} (g \left(\frac{a+b}{2} \right) - g(a)) \} \mathbb{V}_a^b (f); \\ [\mathbf{K}^p (g(b) - g \left(\frac{a+b}{2} \right)) + \mathbf{K}^p (g \left(\frac{a+b}{2} \right) - g(a))]^{1/p} \\ \left(\left(\mathbb{V}_a^{\frac{a+b}{2}} (f) \right)^q + \left(\mathbb{V}_{\frac{a+b}{2}}^b (f) \right)^q \right)^{1/q} \\ \text{with } p, q > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ [\mathbf{K} (g(b) - g \left(\frac{a+b}{2} \right)) + \mathbf{K} (g \left(\frac{a+b}{2} \right) - g(a))] \\ \left[\frac{1}{2} \mathbb{V}_a^b (f) + \frac{1}{2} \left| \mathbb{V}_a^{\frac{a+b}{2}} (f) - \mathbb{V}_{\frac{a+b}{2}}^b (f) \right| \right] \end{cases} \quad (50)
\end{aligned}$$

and

$$\begin{aligned}
& \left| \check{S}_{k,g,a+,b-f} \left(\frac{a+b}{2} \right) - \frac{f \left(\frac{a+b}{2} \right) + f(b)}{4} K \left(g(b) - g \left(\frac{a+b}{2} \right) \right) \right. \\
& \quad \left. - \frac{f(a) + f \left(\frac{a+b}{2} \right)}{4} K \left(g \left(\frac{a+b}{2} \right) - g(a) \right) \right| \\
& \leq \frac{1}{4} \left[\mathbf{K} \left(g \left(\frac{a+b}{2} \right) - g(a) \right) \overset{\frac{a+b}{2}}{\underset{a}{\mathbb{V}}} (f) + \mathbf{K} \left(g(b) - g \left(\frac{a+b}{2} \right) \right) \overset{b}{\underset{x}{\mathbb{V}}} (f) \right]
\end{aligned}$$

$$\leq \frac{1}{4} \begin{cases} \max \left\{ \mathbf{K} \left(g(b) - g\left(\frac{a+b}{2}\right) \right), \mathbf{K} \left(g\left(\frac{a+b}{2}\right) - g(a) \right) \right\} V_a^b(f); \\ \left[\mathbf{K}^p \left(g(b) - g\left(\frac{a+b}{2}\right) \right) + \mathbf{K}^p \left(g\left(\frac{a+b}{2}\right) - g(a) \right) \right]^{1/p} \\ \left(\left(V_a^{\frac{a+b}{2}}(f) \right)^q + \left(V_{\frac{a+b}{2}}^b(f) \right)^q \right)^{1/q} \\ \text{with } p, q > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \left[\mathbf{K} \left(g(b) - g\left(\frac{a+b}{2}\right) \right) + \mathbf{K} \left(g\left(\frac{a+b}{2}\right) - g(a) \right) \right] \\ \left[\frac{1}{2} V_a^b(f) + \frac{1}{2} \left| V_a^{\frac{a+b}{2}}(f) - V_{\frac{a+b}{2}}^b(f) \right| \right] \end{cases} \quad (51)$$

for $x \in (a, b)$.

We use the classical Lebesgue p -norms defined as

$$\|h\|_{[c,d],\infty} := \operatorname{esssup}_{s \in [c,d]} |h(s)|$$

and

$$\|h\|_{[c,d],p} := \left(\int_c^d |h(s)|^p ds \right)^{1/p}, \quad p \geq 1.$$

Using Hölder's integral inequality we have for $t > 0$ that

$$K(t) = \int_0^t |k(s)| ds \leq \begin{cases} t \|k\|_{[0,t],\infty} & \text{if } k \in L_\infty[0, t] \\ t^{1/p} \|k\|_{[0,t],q} & \text{if } k \in L_q[0, t], \quad p, q > 1, \frac{1}{p} + \frac{1}{q} = 1. \end{cases}$$

Therefore by the first inequality in (44) and (45) we get for $p, q > 1, \frac{1}{p} + \frac{1}{q} = 1$

$$\begin{aligned} & |S_{k,g,a+,b-}f(x) \\ & - \frac{1}{2} \left[\frac{f(x) + f(b)}{2} K(g(b) - g(x)) + \frac{f(a) + f(x)}{2} K(g(x) - g(a)) \right] \Big| \\ & \leq \frac{1}{4} \bigvee_a^x(f) \begin{cases} (g(x) - g(a)) \|k\|_{[0,g(x)-g(a)],\infty} \\ (g(x) - g(a))^{1/p} \|k\|_{[0,g(x)-g(a)],q} \end{cases} \\ & \quad + \frac{1}{4} \bigvee_x^b(f) \begin{cases} (g(b) - g(x)) \|k\|_{[0,g(b)-g(x)],\infty} \\ (g(b) - g(x))^{1/p} \|k\|_{[0,g(b)-g(x)],q} \end{cases} \quad (52) \end{aligned}$$

and

$$\begin{aligned} & \left| \check{S}_{k,g,a+,b-}f(x) \right. \\ & \left. - \frac{1}{2} \left[\frac{f(x) + f(b)}{2} K(g(b) - g(x)) + \frac{f(a) + f(x)}{2} K(g(x) - g(a)) \right] \right| \end{aligned}$$

$$\begin{aligned} &\leq \frac{1}{4} \bigvee_a^x (f) \left\{ \begin{array}{l} (g(x) - g(a)) \|k\|_{[0, g(x) - g(a)], \infty} \\ (g(x) - g(a))^{1/p} \|k\|_{[0, g(x) - g(a)], q} \end{array} \right. \\ &\quad + \frac{1}{4} \bigvee_x^b (f) \left\{ \begin{array}{l} (g(b) - g(x)) \|k\|_{[0, g(b) - g(x)], \infty} \\ (g(b) - g(x))^{1/p} \|k\|_{[0, g(b) - g(x)], q} \end{array} \right. \end{aligned} \quad (53)$$

for $x \in (a, b)$.

From (48) and (49) we also have for $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ that

$$\begin{aligned} &\left| P_{k, g, a+, b-} f - \frac{1}{2} K \left(\frac{g(b) - g(a)}{2} \right) \left[f(M_g(a, b)) + \frac{f(a) + f(b)}{2} \right] \right| \\ &\leq \frac{1}{4} \bigvee_a^b (f) \left\{ \begin{array}{l} \left(\frac{g(b) - g(a)}{2} \right) \|k\|_{[0, \frac{g(b) - g(a)}{2}], \infty} \\ \left(\frac{g(b) - g(a)}{2} \right)^{1/p} \|k\|_{[0, \frac{g(b) - g(a)}{2}], q} \end{array} \right. \end{aligned} \quad (54)$$

and

$$\begin{aligned} &\left| \check{P}_{k, g, a+, b-} f - \frac{1}{2} K \left(\frac{g(b) - g(a)}{2} \right) \left[f(M_g(a, b)) + \frac{f(a) + f(b)}{2} \right] \right| \\ &\leq \frac{1}{4} \bigvee_a^b (f) \left\{ \begin{array}{l} \left(\frac{g(b) - g(a)}{2} \right) \|k\|_{[0, \frac{g(b) - g(a)}{2}], \infty} \\ \left(\frac{g(b) - g(a)}{2} \right)^{1/p} \|k\|_{[0, \frac{g(b) - g(a)}{2}], q} \end{array} \right. \end{aligned} \quad (55)$$

3. APPLICATIONS FOR GENERALIZED RIEMANN-LIOUVILLE FRACTIONAL INTEGRALS

If we take $k(t) = \frac{1}{\Gamma(\alpha)} t^{\alpha-1}$, where Γ is the *Gamma function*, then

$$S_{k, g, a+} f(x) = I_{a+, g}^\alpha f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x [g(x) - g(t)]^{\alpha-1} g'(t) f(t) dt$$

for $a < x \leq b$ and

$$S_{k, g, b-} f(x) = I_{b-, g}^\alpha f(x) := \frac{1}{\Gamma(\alpha)} \int_x^b [g(t) - g(x)]^{\alpha-1} g'(t) f(t) dt$$

for $a \leq x < b$, which are the *generalized left- and right-sided Riemann-Liouville fractional integrals* of a function f with respect to another function g on $[a, b]$ as defined in [23, p. 100].

We consider the mixed operators

$$I_{g, a+, b-}^\alpha f(x) := \frac{1}{2} [I_{a+, g}^\alpha f(x) + I_{b-, g}^\alpha f(x)] \quad (56)$$

and

$$\check{I}_{g,a+,b-}^\alpha f(x) := \frac{1}{2} [I_{x+,g}^\alpha f(b) + I_{x-,g}^\alpha f(a)] \quad (57)$$

for $x \in (a, b)$.

We observe that for $\alpha > 0$ we have

$$K(t) = \frac{1}{\Gamma(\alpha)} \int_0^t s^{\alpha-1} ds = \frac{t^\alpha}{\alpha\Gamma(\alpha)} = \frac{t^\alpha}{\Gamma(\alpha+1)}, \quad t \geq 0.$$

If we use the inequalities (44) and (45) we get

$$\begin{aligned} & \left| I_{g,a+,b-}^\alpha f(x) - \frac{1}{2\Gamma(\alpha+1)} \left[\frac{f(x)+f(b)}{2} (g(b)-g(x))^\alpha + \frac{f(a)+f(x)}{2} (g(x)-g(a))^\alpha \right] \right| \\ & \leq \frac{1}{4\Gamma(\alpha+1)} \left[(g(x)-g(a))^\alpha \bigvee_a^x(f) + (g(b)-g(x))^\alpha \bigvee_x^b(f) \right] \\ & \leq \frac{1}{4\Gamma(\alpha+1)} \\ & \times \begin{cases} \left[\frac{g(b)-g(a)}{2} + \left| g(x) - \frac{g(b)+g(a)}{2} \right| \right]^\alpha \bigvee_a^b(f); \\ [(g(b)-g(x))^{p\alpha} + (g(x)-g(a))^{p\alpha}]^{1/p} \left((\bigvee_a^x(f))^q + (\bigvee_x^b(f))^q \right)^{1/q} \\ \text{with } p, q > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ [(g(b)-g(x))^\alpha + (g(x)-g(a))^\alpha] \left[\frac{1}{2} \bigvee_a^b(f) + \frac{1}{2} \left| \bigvee_a^x(f) - \bigvee_x^b(f) \right| \right] \end{cases} \quad (58) \end{aligned}$$

and

$$\begin{aligned} & \left| \check{I}_{g,a+,b-}^\alpha f(x) - \frac{1}{2\Gamma(\alpha+1)} \left[\frac{f(x)+f(b)}{2} (g(b)-g(x))^\alpha + \frac{f(a)+f(x)}{2} (g(x)-g(a))^\alpha \right] \right| \\ & \leq \frac{1}{4\Gamma(\alpha+1)} \left[(g(x)-g(a))^\alpha \bigvee_a^x(f) + (g(b)-g(x))^\alpha \bigvee_x^b(f) \right] \\ & \leq \frac{1}{4\Gamma(\alpha+1)} \end{aligned}$$

$$\times \begin{cases} \left[\frac{g(b)-g(a)}{2} + \left| g(x) - \frac{g(b)+g(a)}{2} \right| \right]^\alpha V_a^b(f); \\ [(g(b) - g(x))^{p\alpha} + (g(x) - g(a))^{p\alpha}]^{1/p} \left((V_a^x(f))^q + (V_x^b(f))^q \right)^{1/q} \\ \text{with } p, q > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ [(g(b) - g(x))^\alpha + (g(x) - g(a))^\alpha] \left[\frac{1}{2} V_a^b(f) + \frac{1}{2} |V_a^x(f) - V_x^b(f)| \right] \end{cases} \quad (59)$$

for $x \in (a, b)$.

From (48) and (49) we get

$$\left| I_{g,a+,b-}^\alpha f(M_g(a,b)) - \frac{(g(b) - g(a))^\alpha}{2^{\alpha+1}\Gamma(\alpha+1)} \left[f(M_g(a,b)) + \frac{f(a) + f(b)}{2} \right] \right| \leq \frac{1}{2^{\alpha+2}\Gamma(\alpha+1)} (g(b) - g(a))^\alpha \bigvee_a^b(f) \quad (60)$$

and

$$\left| \tilde{I}_{g,a+,b-}^\alpha f(M_g(a,b)) - \frac{(g(b) - g(a))^\alpha}{2^{\alpha+1}\Gamma(\alpha+1)} \left[f(M_g(a,b)) + \frac{f(a) + f(b)}{2} \right] \right| \leq \frac{1}{2^{\alpha+2}\Gamma(\alpha+1)} (g(b) - g(a))^\alpha \bigvee_a^b(f). \quad (61)$$

4. EXAMPLE FOR AN EXPONENTIAL KERNEL

For $\alpha, \beta \in \mathbb{R}$ we consider the kernel $k(t) := \exp[(\alpha + \beta i)t]$, $t \in \mathbb{R}$. We have

$$K(t) = \frac{\exp[(\alpha + \beta i)t] - 1}{(\alpha + \beta i)}, \text{ if } t \in \mathbb{R}$$

for $\alpha, \beta \neq 0$.

Also, we have

$$|k(s)| := |\exp[(\alpha + \beta i)s]| = \exp(\alpha s) \text{ for } s \in \mathbb{R}$$

and

$$\mathbf{K}(t) = \int_0^t \exp(\alpha s) ds = \frac{\exp(\alpha t) - 1}{\alpha} \text{ if } 0 < t,$$

for $\alpha \neq 0$.

Let $f : [a, b] \rightarrow \mathbb{C}$ be a function of bounded variation on $[a, b]$ and g be a strictly increasing function on (a, b) , having a continuous derivative g' on (a, b) .

We consider the operator

$$\mathcal{H}_{g,a+,b-}^{\alpha+\beta i} f(x) := \frac{1}{2} \int_a^x \exp[(\alpha + \beta i)(g(x) - g(t))] g'(t) f(t) dt \quad (62)$$

$$+ \frac{1}{2} \int_x^b \exp[(\alpha + \beta i)(g(t) - g(x))] g'(t) f(t) dt$$

for $x \in (a, b)$.

If $g = \ln h$ where $h : [a, b] \rightarrow (0, \infty)$ is a strictly increasing function on (a, b) , having a continuous derivative h' on (a, b) , then we can consider the following operator as well

$$\begin{aligned} & \kappa_{h,a+,b-}^{\alpha+\beta i} f(x) \\ & := \mathcal{H}_{\ln h, a+, b-}^{\alpha+\beta i} f(x) \\ & = \frac{1}{2} \left[\int_a^x \left(\frac{h(x)}{h(t)} \right)^{\alpha+\beta i} \frac{h'(t)}{h(t)} f(t) dt + \int_x^b \left(\frac{h(t)}{h(x)} \right)^{\alpha+\beta i} \frac{h'(t)}{h(t)} f(t) dt \right], \end{aligned} \quad (63)$$

for $x \in (a, b)$.

Using the inequality (44) we have for $x \in (a, b)$

$$\begin{aligned} & \left| \mathcal{H}_{g,a+,b-}^{\alpha+\beta i} f(x) - \frac{1}{2} \frac{f(x) + f(b) \exp[(\alpha + \beta i)(g(b) - g(x))] - 1}{2(\alpha + \beta i)} \right. \\ & \quad \left. - \frac{f(a) + f(x) \exp[(\alpha + \beta i)(g(x) - g(a))] - 1}{2(\alpha + \beta i)} \right| \\ & \leq \frac{1}{4} \left[\frac{\exp(\alpha(g(x) - g(a))) - 1}{\alpha} \mathcal{V}_a^x(f) + \frac{\exp(\alpha(g(b) - g(x))) - 1}{\alpha} \mathcal{V}_x^b(f) \right] \\ & \leq \frac{1}{4} \left\{ \begin{aligned} & \max \left\{ \frac{\exp(\alpha(g(x) - g(a))) - 1}{\alpha}, \frac{\exp(\alpha(g(b) - g(x))) - 1}{\alpha} \right\} \mathcal{V}_a^b(f); \\ & \left[\left(\frac{\exp(\alpha(g(x) - g(a))) - 1}{\alpha} \right)^p + \left(\frac{\exp(\alpha(g(b) - g(x))) - 1}{\alpha} \right)^p \right]^{1/p} \\ & \times \left((\mathcal{V}_a^x(f))^q + (\mathcal{V}_x^b(f))^q \right)^{1/q} \\ & \text{with } p, q > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ & \left[\frac{\exp(\alpha(g(x) - g(a))) + \exp(\alpha(g(b) - g(x))) - 2}{\alpha} \right] \\ & \times \left[\frac{1}{2} \mathcal{V}_a^b(f) + \frac{1}{2} \left| \mathcal{V}_a^x(f) - \mathcal{V}_x^b(f) \right| \right] \end{aligned} \right. \quad (64) \end{aligned}$$

and if we take $g = \ln h$ where $h : [a, b] \rightarrow (0, \infty)$ is a strictly increasing function on (a, b) , having a continuous derivative h' on (a, b) , then we get

$$\left| \kappa_{h,a+,b-}^{\alpha+\beta i} f(x) - \frac{1}{2} \left[\frac{f(x) + f(b) \left(\frac{h(b)}{h(x)} \right)^{\alpha+\beta i} - 1}{2(\alpha + \beta i)} \right. \right.$$

$$\begin{aligned}
& \left| -\frac{f(a) + f(x)}{2} \frac{\left(\frac{h(x)}{h(a)}\right)^{\alpha+\beta i} - 1}{(\alpha + \beta i)} \right| \\
& \leq \frac{1}{4} \left[\frac{\left(\frac{h(x)}{h(a)}\right)^\alpha - 1}{\alpha} \mathcal{V}_a^x(f) + \frac{\left(\frac{h(b)}{h(x)}\right)^\alpha - 1}{\alpha} \mathcal{V}_x^b(f) \right] \\
& \leq \frac{1}{4} \left\{ \begin{array}{l} \max \left\{ \frac{\left(\frac{h(x)}{h(a)}\right)^\alpha - 1}{\alpha}, \frac{\left(\frac{h(b)}{h(x)}\right)^\alpha - 1}{\alpha} \right\} \mathcal{V}_a^b(f); \\ \left[\left(\frac{\left(\frac{h(x)}{h(a)}\right)^\alpha - 1}{\alpha} \right)^p + \left(\frac{\left(\frac{h(b)}{h(x)}\right)^\alpha - 1}{\alpha} \right)^p \right]^{1/p} \left((\mathcal{V}_a^x(f))^q + (\mathcal{V}_x^b(f))^q \right)^{1/q} \\ \text{with } p, q > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \left[\frac{\left(\frac{h(x)}{h(a)}\right)^\alpha + \left(\frac{h(b)}{h(x)}\right)^\alpha - 2}{\alpha} \right] \left[\frac{1}{2} \mathcal{V}_a^b(f) + \frac{1}{2} \left| \mathcal{V}_a^x(f) - \mathcal{V}_x^b(f) \right| \right]. \end{array} \right. \quad (65)
\end{aligned}$$

If we take if we take $x_h := h^{-1} \left(\sqrt{h(a)h(b)} \right) = h^{-1} (G(h(a), h(b))) \in (a, b)$, where G is the geometric mean, then from (65) we get

$$\begin{aligned}
& \left| \bar{\kappa}_{h,a+,b-}^{\alpha+\beta i} f - \frac{\left(\frac{h(b)}{h(a)}\right)^{\frac{\alpha+\beta i}{2}} - 1}{2(\alpha + \beta i)} \left[f(h^{-1}(G(h(a), h(b)))) + \frac{f(a) + f(b)}{2} \right] \right| \\
& \leq \frac{1}{4} \frac{\left(\frac{h(b)}{h(a)}\right)^{\frac{\alpha}{2}} - 1}{\alpha} \mathcal{V}_a^b(f), \quad (66)
\end{aligned}$$

where $\bar{\kappa}_{h,a+,b-}^{\alpha+\beta i} f = \kappa_{h,a+,b-}^{\alpha+\beta i} f(x_h)$.

Let $f : [a, b] \rightarrow \mathbb{C}$ be an integrable function on $[a, b]$ and g be a strictly increasing function on (a, b) , having a continuous derivative g' on (a, b) . Also define

$$\begin{aligned}
& \check{\mathcal{H}}_{g,a+,b-}^\alpha f(x) \\
& := \frac{1}{2} \int_x^b \exp[\alpha(g(b) - g(t))] g'(t) f(t) dt \\
& + \frac{1}{2} \int_a^x \exp[\alpha(g(t) - g(a))] g'(t) f(t) dt
\end{aligned} \quad (67)$$

for any $x \in (a, b)$.

If $g = \ln h$ where $h : [a, b] \rightarrow (0, \infty)$ is a strictly increasing function on (a, b) , having a continuous derivative h' on (a, b) , then we can consider the following

operator as well

$$\begin{aligned} & \check{\kappa}_{h,a+,b-}^{\alpha} f(x) \\ & := \check{\mathcal{H}}_{\ln h,a+,b-}^{\alpha} f(x) \\ & = \frac{1}{2} \left[\int_x^b \left(\frac{h(b)}{h(t)} \right)^{\alpha} \frac{h'(t)}{h(t)} f(t) dt + \int_a^x \left(\frac{h(t)}{h(a)} \right)^{\alpha} \frac{h'(t)}{h(t)} f(t) dt \right], \end{aligned} \quad (68)$$

for any $x \in (a, b)$.

Using the inequality (45) we have for $x \in (a, b)$ that

$$\begin{aligned} & \left| \check{\mathcal{H}}_{g,a+,b-}^{\alpha+\beta i} f(x) - \frac{1}{2} \left[\frac{f(x) + f(b)}{2} \frac{\exp[(\alpha + \beta i)(g(b) - g(x))] - 1}{(\alpha + \beta i)} \right. \right. \\ & \quad \left. \left. - \frac{f(a) + f(x)}{2} \frac{\exp[(\alpha + \beta i)(g(x) - g(a))] - 1}{(\alpha + \beta i)} \right] \right| \\ & \leq \frac{1}{4} \left[\frac{\exp(\alpha(g(x) - g(a))) - 1}{\alpha} \bigvee_a^x(f) + \frac{\exp(\alpha(g(b) - g(x))) - 1}{\alpha} \bigvee_x^b(f) \right] \\ & \leq \frac{1}{4} \left\{ \begin{aligned} & \max \left\{ \frac{\exp(\alpha(g(x) - g(a))) - 1}{\alpha}, \frac{\exp(\alpha(g(b) - g(x))) - 1}{\alpha} \right\} \bigvee_a^b(f); \\ & \left[\left(\frac{\exp(\alpha(g(x) - g(a))) - 1}{\alpha} \right)^p + \left(\frac{\exp(\alpha(g(b) - g(x))) - 1}{\alpha} \right)^p \right]^{1/p} \\ & \times \left(\left(\bigvee_a^x(f) \right)^q + \left(\bigvee_x^b(f) \right)^q \right)^{1/q} \\ & \text{with } p, q > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ & \left[\frac{\exp(\alpha(g(x) - g(a))) + \exp(\alpha(g(b) - g(x))) - 2}{\alpha} \right] \\ & \times \left[\frac{1}{2} \bigvee_a^b(f) + \frac{1}{2} \left| \bigvee_a^x(f) - \bigvee_x^b(f) \right| \right] \end{aligned} \right. \quad (69) \end{aligned}$$

and if we take $g = \ln h$ where $h : [a, b] \rightarrow (0, \infty)$ is a strictly increasing function on (a, b) , having a continuous derivative h' on (a, b) , then we get

$$\begin{aligned} & \left| \check{\kappa}_{h,a+,b-}^{\alpha+\beta i} f(x) - \frac{1}{2} \left[\frac{f(x) + f(b)}{2} \frac{\left(\frac{h(b)}{h(x)} \right)^{\alpha+\beta i} - 1}{(\alpha + \beta i)} \right. \right. \\ & \quad \left. \left. - \frac{f(a) + f(x)}{2} \frac{\left(\frac{h(x)}{h(a)} \right)^{\alpha+\beta i} - 1}{(\alpha + \beta i)} \right] \right| \\ & \leq \frac{1}{4} \left[\frac{\left(\frac{h(x)}{h(a)} \right)^{\alpha} - 1}{\alpha} \bigvee_a^x(f) + \frac{\left(\frac{h(b)}{h(x)} \right)^{\alpha} - 1}{\alpha} \bigvee_x^b(f) \right] \end{aligned}$$

$$\leq \frac{1}{4} \left\{ \begin{array}{l} \max \left\{ \frac{\left(\frac{h(x)}{h(a)}\right)^\alpha - 1}{\alpha}, \frac{\left(\frac{h(b)}{h(x)}\right)^\alpha - 1}{\alpha} \right\} V_a^b(f); \\ \left[\left(\frac{\left(\frac{h(x)}{h(a)}\right)^\alpha - 1}{\alpha} \right)^p + \left(\frac{\left(\frac{h(b)}{h(x)}\right)^\alpha - 1}{\alpha} \right)^p \right]^{1/p} \left((V_a^x(f))^q + (V_x^b(f))^q \right)^{1/q} \\ \text{with } p, q > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \left[\frac{\left(\frac{h(x)}{h(a)}\right)^\alpha + \left(\frac{h(b)}{h(x)}\right)^\alpha - 2}{\alpha} \right] \left[\frac{1}{2} V_a^b(f) + \frac{1}{2} |V_a^x(f) - V_x^b(f)| \right]. \end{array} \right. \quad (70)$$

If we take if we take $x_h = h^{-1}(G(h(a), h(b))) \in (a, b)$, where G is the geometric mean, then from (65) we get

$$\left| \bar{\ell}_{h,a+,b-}^{\alpha+\beta i} f - \frac{\left(\frac{h(b)}{h(a)}\right)^{\frac{\alpha+\beta i}{2}} - 1}{2(\alpha+\beta i)} \left[f(h^{-1}(G(h(a), h(b)))) + \frac{f(a) + f(b)}{2} \right] \right| \leq \frac{1}{4} \frac{\left(\frac{h(b)}{h(a)}\right)^{\frac{\alpha}{2}} - 1}{\alpha} \bigvee_a^b(f), \quad (71)$$

where $\bar{\ell}_{h,a+,b-}^{\alpha+\beta i} f = \check{\kappa}_{h,a+,b-}^{\alpha+\beta i} f(x_h)$.

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