Upper and lower bounds for the difference between the weighted arithmetic and harmonic operator means

EXTRACTA MATHEMATICAE
Vol. **34**, Num. 1 (2019), 41 – 60
doi:10.17398/2605-5686.34.1.41
Available online March 4, 2019

Upper and lower bounds for the difference between the weighted arithmetic and harmonic operator means

S.S. Dragomir [1,2]

[1] *Mathematics, College of Engineering & Science, Victoria University PO Box 14428, Melbourne City, MC 8001, Australia*
sever.dragomir@vu.edu.au, http://rgmia.org/dragomir

[2] *School of Computer Science & Applied Mathematics, University of the Witwatersrand Private Bag 3, Johannesburg 2050, South Africa*

Received October 11, 2018
Accepted February 4, 2019
Presented by Mostafa Mbekhta

Abstract: In this paper we establish some new upper and lower bounds for the difference between the weighted arithmetic and harmonic operator means under various assumption for the positive invertible operators A, B. Some applications when A, B are bounded above and below by positive constants are given as well.

Key words: Young's inequality, convex functions, arithmetic mean-Harmonic mean inequality, operator means, operator inequalities.

AMS *Subject Class.* (2010): 47A63, 47A30, 15A60,.26D15; 26D10.

1. Introduction

Throughout this paper A, B are positive invertible operators on a complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$. We use the following notations for operators

$$A \nabla_\nu B := (1-\nu) A + \nu B,$$

the *weighted operator arithmetic mean*,

$$A \sharp_\nu B := A^{1/2} \left(A^{-1/2} B A^{-1/2} \right)^\nu A^{1/2},$$

the *weighted operator geometric mean* and

$$A !_\nu B := \left((1-\nu) A^{-1} + \nu B^{-1} \right)^{-1}$$

the *weighted operator harmonic mean*, where $\nu \in [0,1]$.

When $\nu = \frac{1}{2}$, we write $A \nabla B$, $A \sharp B$ and $A!B$ for brevity, respectively.

ISSN: 0213-8743 (print), 2605-5686 (online)

The following fundamental inequality between the weighted arithmetic, geometric and harmonic operator means holds

$$A!_{\nu}B \le A\sharp_{\nu}B \le A\nabla_{\nu}B \tag{1.1}$$

for any $\nu \in [0,1]$.

For various recent inequalities between these means we recommend the recent papers [3]-[6], [8]-[12] and the references therein.

In the recent work [7] we obtained between others the following result:

THEOREM 1. *Let A, B be positive invertible operators and $M > m > 0$ such that*

$$MA \ge B \ge mA. \tag{1.2}$$

Then for any $\nu \in [0,1]$ we have

$$rk\,(m,M)\,A \le A\nabla_{\nu}B - A!_{\nu}B \le RK\,(m,M)\,A, \tag{1.3}$$

where $r = \min\{\nu, 1-\nu\}$, $R = \max\{\nu, 1-\nu\}$ and the bounds $K(m,M)$ and $k(m,M)$ are given by

$$K(m,M) \tag{1.4}$$
$$:= \begin{cases} (m-1)^2 (m+1)^{-1} & \text{if } M < 1, \\ \max\left\{ (m-1)^2 (m+1)^{-1}, (M-1)^2 (M+1)^{-1} \right\} & \text{if } m \le 1 \le M, \\ (M-1)^2 (M+1)^{-1} & \text{if } 1 < m, \end{cases}$$

and

$$k\,(m,M) := \begin{cases} (M-1)^2 (M+1)^{-1} & \text{if } M < 1, \\ 0 & \text{if } m \le 1 \le M, \\ (m-1)^2 (m+1)^{-1} & \text{if } 1 < m. \end{cases} \tag{1.5}$$

In particular,

$$\frac{1}{2}k\,(m,M)\,A \le A\nabla B - A!B \le \frac{1}{2}K\,(m,M)\,A. \tag{1.6}$$

Let A, B positive invertible operators and positive real numbers m, m', M, M' such that the condition $0 < mI \le A \le m'I < M'I \le B \le MI$ holds. Put $h := \frac{M}{m}$ and $h' := \frac{M'}{m'}$, then for any $\nu \in [0,1]$ we have [7]

$$\begin{aligned} r\left(h'-1\right)^2 \left(h'+1\right)^{-1} A &\le A\nabla_{\nu}B - A!_{\nu}B \\ &\le R\,(h-1)^2 (h+1)^{-1} A, \end{aligned} \tag{1.7}$$

where $r=\min\{\nu,1-\nu\}$, $R=\max\{\nu,1-\nu\}$ and, in particular,

$$\begin{aligned}\frac{1}{2}\left(h'-1\right)^2\left(h'+1\right)^{-1}A &\le A\nabla B-A!B\\ &\le \frac{1}{2}\left(h-1\right)^2\left(h+1\right)^{-1}A.\end{aligned} \tag{1.8}$$

Let A, B positive invertible operators and positive real numbers m, m', M, M' such that the condition $0<mI\le B\le m'I<M'I\le A\le MI$ holds. Then for any $\nu\in[0,1]$ we also have [7]

$$\begin{aligned}r\left(h'-1\right)^2\left(h'+1\right)^{-1}\left(h'\right)^{-1}A &\le A\nabla_\nu B-A!_\nu B\\ &\le R\left(h-1\right)^2\left(h+1\right)^{-1}h^{-1}A,\end{aligned} \tag{1.9}$$

and, in particular,

$$\begin{aligned}\frac{1}{2}\left(h'-1\right)^2\left(h'+1\right)^{-1}\left(h'\right)^{-1}A &\le A\nabla B-A!B\\ &\le \frac{1}{2}\left(h-1\right)^2\left(h+1\right)^{-1}h^{-1}A.\end{aligned} \tag{1.10}$$

Motivated by the above facts, in this paper we establish some new upper and lower bounds for the difference $A\nabla_\nu B-A!_\nu B$ for $\nu\in[0,1]$ under various assumption for the positive invertible operators A, B. Some applications when A, B are bounded above and below by positive constants are given as well. A graphic comparison for upper bounds is provided as well.

2. Min and max bounds

The following lemma is of interest in itself.

Lemma 1. *For any a, $b>0$ and $\nu\in[0,1]$ we have*

$$\begin{aligned}\nu\left(1-\nu\right)\frac{\left(b-a\right)^2}{\max\{b,a\}} &\le A_\nu\left(a,b\right)-H_\nu\left(a,b\right)\\ &\le \nu\left(1-\nu\right)\frac{\left(b-a\right)^2}{\min\{b,a\}},\end{aligned} \tag{2.1}$$

where $A_\nu(a,b)$ and $H_\nu(a,b)$ are the scalar weighted arithmetic mean and harmonic mean, respectively, namely

$$A_\nu\left(a,b\right):=\left(1-\nu\right)a+\nu b \text{ and } H_\nu\left(a,b\right):=\frac{ab}{\left(1-\nu\right)b+\nu a}.$$

In particular,

$$\frac{1}{4}\frac{(b-a)^2}{\max\{b,a\}} \le A(a,b) - H(a,b) \le \frac{1}{4}\frac{(b-a)^2}{\min\{b,a\}}, \tag{2.2}$$

where

$$A(a,b) := \frac{a+b}{2} \text{ and } H(a,b) := \frac{2ab}{b+a}.$$

Proof. Consider the function $\xi_\nu : (0,\infty) \to (0,\infty)$ defined by

$$\xi_\nu(x) = 1 - \nu + \nu x - \frac{x}{(1-\nu)x + \nu},$$

where $\nu \in [0,1]$.

Then

$$\begin{aligned} \xi_\nu(x) &= \frac{(1-\nu+\nu x)[(1-\nu)x+\nu] - x}{(1-\nu)x+\nu} \\ &= \frac{(1-\nu)^2 x + \nu(1-\nu)x^2 + \nu(1-\nu) + \nu^2 x - x}{(1-\nu)x+\nu} \\ &= \frac{\nu(1-\nu)x^2 - 2\nu(1-\nu)x + \nu(1-\nu)}{(1-\nu)x+\nu} \\ &= \frac{\nu(1-\nu)(x-1)^2}{(1-\nu)x+\nu}, \end{aligned} \tag{2.3}$$

for any $x > 0$ and $\nu \in [0,1]$.

If we take in the definition of ξ_ν, $x = \frac{b}{a} > 0$, then we have

$$\xi_\nu\left(\frac{b}{a}\right) = \frac{1}{a}\left[A_\nu(a,b) - H_\nu(a,b)\right].$$

From the equality (2.3) we also have

$$\xi_\nu\left(\frac{b}{a}\right) = \frac{\nu(1-\nu)(b-a)^2}{aA_\nu(b,a)}.$$

Therefore, we have the equality

$$A_\nu(a,b) - H_\nu(a,b) = \frac{\nu(1-\nu)(b-a)^2}{A_\nu(b,a)} \tag{2.4}$$

for any a, $b > 0$ and $\nu \in [0,1]$.

Since for any a, $b > 0$ and $\nu \in [0,1]$ we have

$$\min\{a,b\} \le A_\nu(b,a) \le \max\{a,b\}$$

then

$$\frac{\nu(1-\nu)(b-a)^2}{\max\{a,b\}} \le \frac{\nu(1-\nu)(b-a)^2}{A_\nu(b,a)} \le \frac{\nu(1-\nu)(b-a)^2}{\min\{a,b\}} \tag{2.5}$$

and by (2.4) we get the desired result (2.1). ∎

Remark 1. We show that there is no constant $K_1 > 1$ and $K_2 < 1$ such that

$$\begin{aligned} \nu(1-\nu)\frac{(b-a)^2}{\max\{b,a\}} &\le A_\nu(a,b) - H_\nu(a,b) \\ &\le \nu(1-\nu)\frac{(b-a)^2}{\min\{b,a\}}, \end{aligned} \tag{2.6}$$

for some $\nu \in (0,1)$ and any a, $b > 0$.

Assume that there exist K_1, $K_2 > 0$ such that

$$\begin{aligned} K_1\nu(1-\nu)\frac{(b-a)^2}{\max\{b,a\}} &\le A_\nu(a,b) - H_\nu(a,b) \\ &\le K_2\nu(1-\nu)\frac{(b-a)^2}{\min\{b,a\}}, \end{aligned} \tag{2.7}$$

for some $\nu \in (0,1)$ and any a, $b > 0$.

Let $\varepsilon > 0$ and write the inequality (2.7) for $a > 0$ and $b = a + \varepsilon$ to get, via (2.4) that

$$K_1\nu(1-\nu)\frac{\varepsilon^2}{a+\varepsilon} \le \frac{\nu(1-\nu)\varepsilon^2}{(1-\nu)\varepsilon + a} \le K_2\nu(1-\nu)\frac{\varepsilon^2}{a}. \tag{2.8}$$

If we divide by $\nu(1-\nu)\varepsilon^2 > 0$ in (2.8), then we get

$$K_1\frac{1}{a+\varepsilon} \le \frac{1}{(1-\nu)\varepsilon + a} \le K_2\frac{1}{a}, \tag{2.9}$$

for any $a > 0$ and $\varepsilon > 0$.

By letting $\varepsilon \to 0+$ in (2.9), we get $K_1 \le 1 \le K_2$ and the statement is proved.

We have the following operator double inequality:

THEOREM 2. *Let A, B be positive invertible operators and $M > m > 0$ such that the condition (1.2). Then for any $\nu \in [0,1]$ we have*

$$\begin{aligned}\nu(1-\nu)\,c(m,M)\,A &\le \frac{\nu(1-\nu)}{\max\{M,1\}}(B-A)A^{-1}(B-A)\\ &\le A\nabla_\nu B - A!_\nu B\\ &\le \frac{\nu(1-\nu)}{\min\{m,1\}}(B-A)A^{-1}(B-A)\\ &\le \nu(1-\nu)\,C(m,M)\,A,\end{aligned} \tag{2.10}$$

where

$$c(m,M) := \begin{cases}(M-1)^2 & \text{if } M<1,\\ 0 & \text{if } m\le 1\le M,\\ \frac{(m-1)^2}{M} & \text{if } 1<m,\end{cases}$$

and

$$C(m,M) := \begin{cases}\frac{(m-1)^2}{m} & \text{if } M<1,\\ \frac{1}{m}\max\left\{(m-1)^2,(M-1)^2\right\} & \text{if } m\le 1\le M,\\ (M-1)^2 & \text{if } 1<m.\end{cases}$$

In particular,

$$\begin{aligned}\frac{1}{4}c(m,M)\,A &\le \frac{1}{4\max\{M,1\}}(B-A)A^{-1}(B-A) \le A\nabla B - A!B\\ &\le \frac{1}{4\min\{m,1\}}(B-A)A^{-1}(B-A) \le \frac{1}{4}C(m,M)\,A.\end{aligned} \tag{2.11}$$

Proof. If we write the inequality (2.1) for $a=1$ and $b=x$, then we get

$$\begin{aligned}\nu(1-\nu)\frac{(x-1)^2}{\max\{x,1\}} &\le 1-\nu+\nu x - \left((1-\nu)+\nu x^{-1}\right)^{-1}\\ &\le \nu(1-\nu)\frac{(x-1)^2}{\min\{x,1\}}\end{aligned} \tag{2.12}$$

for any $x>0$ and for any $\nu \in [0,1]$.

If $x \in [m, M] \subset (0, \infty)$, then $\max\{x, 1\} \leq \max\{M, 1\}$ and $\min\{m, 1\} \leq \min\{x, 1\}$ and by (2.12) we get

$$\begin{aligned}\nu(1-\nu)\frac{\min_{x\in[m,M]}(x-1)^2}{\max\{M,1\}} &\leq \nu(1-\nu)\frac{(x-1)^2}{\max\{M,1\}} \\ &\leq 1-\nu+\nu x-\left((1-\nu)+\nu x^{-1}\right)^{-1} \\ &\leq \nu(1-\nu)\frac{(x-1)^2}{\min\{m,1\}} \\ &\leq \nu(1-\nu)\frac{\max_{x\in[m,M]}(x-1)^2}{\min\{m,1\}}\end{aligned} \tag{2.13}$$

for any $x \in [m, M]$ and for any $\nu \in [0, 1]$.

Observe that

$$\min_{x\in[m,M]}(x-1)^2 = \begin{cases}(M-1)^2 & \text{if } M<1,\\ 0 & \text{if } m\leq 1\leq M,\\ (m-1)^2 & \text{if } 1<m,\end{cases}$$

and

$$\max_{x\in[m,M]}(x-1)^2 = \begin{cases}(m-1)^2 & \text{if } M<1,\\ \max\left\{(m-1)^2,(M-1)^2\right\} & \text{if } m\leq 1\leq M,\\ (M-1)^2 & \text{if } 1<m.\end{cases}$$

Then

$$\begin{aligned}\frac{\min_{x\in[m,M]}(x-1)^2}{\max\{M,1\}} &= \begin{cases}(M-1)^2 & \text{if } M<1,\\ 0 & \text{if } m\leq 1\leq M,\\ \frac{(m-1)^2}{M} & \text{if } 1<m,\end{cases} \\ &= c(m,M)\end{aligned}$$

and

$$\begin{aligned}\frac{\max_{x\in[m,M]}(x-1)^2}{\min\{m,1\}} &= \begin{cases}\frac{(m-1)^2}{m} & \text{if } M<1,\\ \frac{1}{m}\max\left\{(m-1)^2,(M-1)^2\right\} & \text{if } m\leq 1\leq M,\\ (M-1)^2 & \text{if } 1<m,\end{cases} \\ &= C(m,M).\end{aligned}$$

Using the inequality (2.13) we have

$$\begin{aligned}\nu(1-\nu)\,c(m,M) &\leq \nu(1-\nu)\,\frac{(x-1)^2}{\max\{M,1\}} \\ &\leq 1-\nu+\nu x-\left((1-\nu)+\nu x^{-1}\right)^{-1} \\ &\leq \nu(1-\nu)\,\frac{(x-1)^2}{\min\{m,1\}} \\ &\leq \nu(1-\nu)\,C(m,M)\end{aligned} \tag{2.14}$$

for any $x \in [m,M]$ and for any $\nu \in [0,1]$.

If we use the continuous functional calculus for the positive invertible operator X with $mI \leq X \leq MI$, then we have from (2.14) that

$$\begin{aligned}\nu(1-\nu)\,c(m,M)\,I &\leq \frac{\nu(1-\nu)}{\max\{M,1\}}\,(X-I)^2 \\ &\leq (1-\nu)\,I+\nu X-\left((1-\nu)\,I+\nu X^{-1}\right)^{-1} \\ &\leq \frac{\nu(1-\nu)}{\min\{m,1\}}\,(X-I)^2 \\ &\leq \nu(1-\nu)\,C(m,M)\,I\end{aligned} \tag{2.15}$$

for any $\nu \in [0,1]$.

If we multiply (1.2) both sides by $A^{-1/2}$ we get $MI \geq A^{-1/2}BA^{-1/2} \geq mI$.

By writing the inequality (2.15) for $X = A^{-1/2}BA^{-1/2}$ we obtain

$$\begin{aligned}&\nu(1-\nu)c(m,M)I \\ &\quad\leq \frac{\nu(1-\nu)}{\max\{M,1\}}\left(A^{-1/2}BA^{-1/2}-I\right)^2 \\ &\quad\leq (1-\nu)I+\nu A^{-1/2}BA^{-1/2}-A^{-1/2}\left((1-\nu)A^{-1}+\nu B^{-1}\right)^{-1}A^{-1/2} \\ &\quad\leq \frac{\nu(1-\nu)}{\min\{m,1\}}\left(A^{-1/2}BA^{-1/2}-I\right)^2 \\ &\quad\leq \nu(1-\nu)C(m,M)I\end{aligned} \tag{2.16}$$

for any $\nu \in [0,1]$.

If we multiply the inequality (2.16) both sides with $A^{1/2}$, then we get

$$\begin{aligned}\nu(1-\nu)\,c(m,M)\,A &\le \frac{\nu(1-\nu)}{\max\{M,1\}}A^{1/2}\left(A^{-1/2}BA^{-1/2}-I\right)^2A^{1/2}\\ &\le (1-\nu)A+\nu B-\left((1-\nu)A^{-1}+\nu B^{-1}\right)^{-1}\\ &\le \frac{\nu(1-\nu)}{\min\{m,1\}}A^{1/2}\left(A^{-1/2}BA^{-1/2}-I\right)^2A^{1/2}\\ &\le \nu(1-\nu)\,C(m,M)\,A,\end{aligned} \tag{2.17}$$

and since

$$\begin{aligned}&A^{1/2}\left(A^{-1/2}\;BA^{-1/2}-I\right)^2A^{1/2}\\ &\quad= A^{1/2}\left(A^{-1/2}(B-A)A^{-1/2}\right)^2A^{1/2}\\ &\quad= A^{1/2}A^{-1/2}(B-A)A^{-1/2}A^{-1/2}(B-A)A^{-1/2}A^{1/2}\\ &\quad= (B-A)A^{-1}(B-A),\end{aligned}$$

then by (2.17) we get the desired result (2.10). ∎

When the operators A and B are bounded above and below by constants we have the following result as well:

Corollary 1. *Let A, B be two positive operators and m, m', M, M' be positive real numbers. Put $h := \frac{M}{m}$ and $h' := \frac{M'}{m'}$.*
(i) *if $0 < mI \le A \le m'I < M'I \le B \le MI$, then*

$$\begin{aligned}\nu(1-\nu)\frac{(h'-1)^2}{h}A &\le \frac{\nu(1-\nu)}{h}(B-A)A^{-1}(B-A)\\ &\le A\nabla_\nu B - A!_\nu B\\ &\le \nu(1-\nu)(B-A)A^{-1}(B-A)\\ &\le \nu(1-\nu)(h-1)^2A,\end{aligned} \tag{2.18}$$

and, in particular,

$$\begin{aligned}\frac{(h'-1)^2}{4h}A &\le \frac{1}{4h}(B-A)A^{-1}(B-A) \le A\nabla B - A!B\\ &\le \frac{1}{4}(B-A)A^{-1}(B-A) \le \frac{1}{4}(h-1)^2A.\end{aligned} \tag{2.19}$$

(ii) *if* $0 < mI \le B \le m'I < M'I \le A \le MI$, *then*

$$\begin{aligned} \nu(1-\nu)\left(\frac{h'-1}{h'}\right)^2 A &\le \nu(1-\nu)(B-A)A^{-1}(B-A) \\ &\le A\nabla_\nu B - A!_\nu B \\ &\le \nu(1-\nu)h(B-A)A^{-1}(B-A) \\ &\le \nu(1-\nu)\frac{(h-1)^2}{h}A \end{aligned} \tag{2.20}$$

and, in particular,

$$\begin{aligned} \frac{1}{4}\left(\frac{h'-1}{h'}\right)^2 A &\le \frac{1}{4}(B-A)A^{-1}(B-A) \le A\nabla B - A!B \\ &\le \frac{1}{4}h(B-A)A^{-1}(B-A) \le \frac{(h-1)^2}{4h}A. \end{aligned} \tag{2.21}$$

Proof. We observe that h, $h' > 1$ and if either of the condition (i) or (ii) holds, then $h \ge h'$.

If (i) is valid, then we have

$$A < h'A = \frac{M'}{m'}A \le B \le \frac{M}{m}A = hA, \tag{2.22}$$

while, if (ii) is valid, then we have

$$\frac{1}{h}A \le B \le \frac{1}{h'}A < A. \tag{2.23}$$

If we use the inequality (2.10) and the assumption (i), then we get (2.18). If we use the inequality (2.10) and the assumption (ii), then we get (2.20). ∎

3. Bounds in term of Kantorovich's constant

We consider the *Kantorovich's constant* defined by

$$K(h) := \frac{(h+1)^2}{4h}, \qquad h > 0. \tag{3.1}$$

The function K is decreasing on $(0,1)$ and increasing on $[1,\infty)$, $K(h) \ge 1$ for any $h > 0$ and $K(h) = K(\frac{1}{h})$ for any $h > 0$.

Observe that for any $h > 0$

$$K(h) - 1 = \frac{(h-1)^2}{4h} = K\left(\frac{1}{h}\right) - 1.$$

Observe that

$$K\left(\frac{b}{a}\right) - 1 = \frac{(b-a)^2}{4ab} \qquad \text{for } a, b > 0.$$

Since, obviously

$$ab = \min\{a, b\} \max\{a, b\} \qquad \text{for } a, b > 0,$$

then we have the following version of Lemma 1:

Lemma 2. *For any a, $b > 0$ and $\nu \in [0, 1]$ we have*

$$\begin{aligned} 4\nu(1-\nu)\min\{a,b\}\left[K\left(\frac{b}{a}\right) - 1\right] &\le A_\nu(a,b) - H_\nu(a,b) \\ &\le 4\nu(1-\nu)\max\{a,b\}\left[K\left(\frac{b}{a}\right) - 1\right]. \end{aligned} \tag{3.2}$$

For positive invertible operators A, B we define

$$A\nabla_\infty B := \frac{1}{2}(A+B) + \frac{1}{2}A^{1/2}\left|A^{-1/2}(B-A)A^{-1/2}\right|A^{1/2},$$

$$A\nabla_{-\infty} B := \frac{1}{2}(A+B) - \frac{1}{2}A^{1/2}\left|A^{-1/2}(B-A)A^{-1/2}\right|A^{1/2}.$$

If we consider the continuous functions f_∞, $f_{-\infty} : [0, \infty) \to [0, \infty)$ defined by

$$f_\infty(x) = \max\{x, 1\} = \frac{1}{2}(x+1) + \frac{1}{2}|x-1|,$$

$$f_{-\infty}(x) = \max\{x, 1\} = \frac{1}{2}(x+1) - \frac{1}{2}|x-1|,$$

then, obviously, we have

$$A\nabla_{\pm\infty} B = A^{1/2} f_{\pm\infty}\left(A^{-1/2}BA^{-1}\right)A^{1/2}. \tag{3.3}$$

If A and B are commutative, then

$$A\nabla_{\pm\infty} B = \frac{1}{2}(A+B) \pm \frac{1}{2}|B-A| = B\nabla_{\pm\infty} A.$$

Theorem 3. *Let A, B be positive invertible operators and $M > m > 0$ such that the condition* (1.2) *holds. Then we have*

$$\begin{aligned} 4\nu\left(1-\nu\right) g\left(m,M\right) A\nabla_{-\infty}B &\le A\nabla_{\nu}B - A!_{\nu}B \\ &\le 4\nu\left(1-\nu\right) G\left(m,M\right) A\nabla_{\infty}B, \end{aligned} \tag{3.4}$$

where

$$g\left(m,M\right) := \begin{cases} K\left(M\right)-1 & \text{if } M<1, \\ 0 & \text{if } m\le 1\le M, \\ K\left(m\right)-1 & \text{if } 1<m, \end{cases}$$

$$G\left(m,M\right) := \begin{cases} K\left(m\right)-1 & \text{if } M<1, \\ \max\left\{K\left(m\right),K\left(M\right)\right\}-1 & \text{if } m\le 1\le M, \\ K\left(M\right)-1 & \text{if } 1<m. \end{cases}$$

In particular,

$$g\left(m,M\right) A\nabla_{-\infty}B \le A\nabla B - A!B \le G\left(m,M\right) A\nabla_{\infty}B. \tag{3.5}$$

Proof. From (3.2) we have for $a=1$ and $b=x$ that

$$\begin{aligned} 4\nu\left(1-\nu\right)\min\left\{1,x\right\}\left[K\left(x\right)-1\right] &\le 1-\nu+\nu x-\left(\left(1-\nu\right)+\nu x^{-1}\right)^{-1} \\ &\le 4\nu\left(1-\nu\right)\max\left\{1,x\right\}\left[K\left(x\right)-1\right] \end{aligned} \tag{3.6}$$

for any $x>0$.

From (3.6) we then have

$$\begin{aligned} 4\nu\left(1-\nu\right) f_{-\infty}(x)\min_{x\in[m,M]}\left[K\left(x\right)-1\right] &\le 1-\nu+\nu x-\left(\left(1-\nu\right)+\nu x^{-1}\right)^{-1} \\ &\le 4\nu\left(1-\nu\right) f_{\infty}\left(x\right)\max_{x\in[m,M]}\left[K\left(x\right)-1\right] \end{aligned} \tag{3.7}$$

for any $x\in[m,M]$.

Observe that

$$\begin{aligned} \max_{x\in[m,M]}\left[K\left(x\right)-1\right] &= \begin{cases} K\left(m\right)-1 & \text{if } M<1, \\ \max\left\{K\left(m\right),K\left(M\right)\right\}-1 & \text{if } m\le 1\le M, \\ K\left(M\right)-1 & \text{if } 1<m, \end{cases} \\ &= G\left(m,M\right) \end{aligned}$$

and

$$\min_{x\in[m,M]}[K(x)-1]=\begin{cases}K(M)-1 & \text{if } M<1,\\ 0 & \text{if } m\le 1\le M,\\ K(m)-1 & \text{if } 1<m.\end{cases}$$
$$= g(m,M).$$

Therefore by (3.7) we get

$$\begin{aligned}4\nu(1-\nu)f_{-\infty}(x)\,g(m,M) &\le 1-\nu+\nu x-\left((1-\nu)+\nu x^{-1}\right)^{-1}\\ &\le 4\nu(1-\nu)f_{\infty}(x)\,G(m,M)\end{aligned}\tag{3.8}$$

for any $x\in[m,M]$ and $\nu\in[0,1]$.

If we use the continuous functional calculus for the positive invertible operator X with $mI\le X\le MI$, then we have from (3.8) that

$$\begin{aligned}4\nu(1-\nu)f_{-\infty}(X)\,g(m,M) &\le (1-\nu)I+\nu X-\left((1-\nu)+\nu X^{-1}\right)^{-1}\\ &\le 4\nu(1-\nu)f_{\infty}(X)\,G(m,M)\end{aligned}\tag{3.9}$$

for any $x\in[m,M]$ and $\nu\in[0,1]$.

By writing the inequality (3.9) for $X=A^{-1/2}BA^{-1/2}$ we obtain

$$\begin{aligned}&4\nu(1-\nu)f_{-\infty}\left(A^{-1/2}BA^{-1/2}\right)g(m,M)\\ &\quad\le(1-\nu)I+\nu A^{-1/2}BA^{-1/2}-A^{-1/2}\left((1-\nu)A^{-1}+\nu B^{-1}\right)^{-1}A^{-1/2}\\ &\quad\le 4\nu(1-\nu)f_{\infty}\left(A^{-1/2}BA^{-1/2}\right)G(m,M)\end{aligned}\tag{3.10}$$

for any $\nu\in[0,1]$.

If we multiply (3.10) both sides by $A^{1/2}$ we get

$$\begin{aligned}&4\nu(1-\nu)A^{1/2}f_{-\infty}\left(A^{-1/2}BA^{-1/2}\right)A^{1/2}g(m,M)\\ &\quad\le(1-\nu)A+\nu B-\left((1-\nu)A^{-1}+\nu B^{-1}\right)^{-1}\\ &\quad\le 4\nu(1-\nu)A^{1/2}f_{\infty}\left(A^{-1/2}BA^{-1/2}\right)A^{1/2}G(m,M)\end{aligned}$$

for any $\nu\in[0,1]$, which, by (3.3) produces the desired result (3.4). ∎

We have:

Corollary 2. *Let A, B be two positive operators and m, m', M, M' be positive real numbers. Put $h := \frac{M}{m}$ and $h' := \frac{M'}{m'}$. If either of the conditions* (i) *or* (ii) *from Corollary* 1 *holds, then*

$$4\nu(1-\nu)\left[K\left(h'\right)-1\right]A\nabla_{-\infty}B \leq A\nabla_{\nu}B - A!_{\nu}B \leq 4\nu(1-\nu)\left[K(h)-1\right]A\nabla_{\infty}B. \tag{3.11}$$

In particular,

$$\left[K\left(h'\right)-1\right]A\nabla_{-\infty}B \leq A\nabla B - A!B \leq \left[K(h)-1\right]A\nabla_{\infty}B. \tag{3.12}$$

Proof. If (i) is valid, then we have

$$A < h'A = \frac{M'}{m'}A \leq B \leq \frac{M}{m}A = hA.$$

By using the inequality (3.4) we get (3.11).

If (ii) is valid, then we have

$$\frac{1}{h}A \leq B \leq \frac{1}{h'}A < A.$$

By using the inequality (3.4) we get

$$4\nu(1-\nu)\left[K\left(\frac{1}{h'}\right)-1\right]A\nabla_{-\infty}B \leq A\nabla_{\nu}B - A!_{\nu}B \leq 4\nu(1-\nu)\left[K\left(\frac{1}{h}\right)-1\right]A\nabla_{\infty}B,$$

and since $K\left(\frac{1}{h'}\right) = K(h')$ and $K\left(\frac{1}{h}\right) = K(h)$, the inequality (3.11) is also obtained. ∎

4. Further bounds

The following result also holds:

Theorem 4. *Let A, B be positive invertible operators and $M > m > 0$ such that the condition* (1.2) *holds. Then we have*

$$p_{\nu}(m,M)A \leq A\nabla_{\nu}B - A!_{\nu}B \leq P_{\nu}(m,M)A \tag{4.1}$$

for any $\nu \in [0,1]$, *where*

$$p_\nu(m,M) := \begin{cases} \frac{\nu(1-\nu)(M-1)^2}{(1-\nu)M+\nu} & \text{if } M<1, \\ 0 & \text{if } m \le 1 \le M, \\ \frac{\nu(1-\nu)(m-1)^2}{(1-\nu)m+\nu} & \text{if } 1<m, \end{cases}$$

$$P_\nu(m,M) := \begin{cases} \frac{\nu(1-\nu)(m-1)^2}{(1-\nu)m+\nu} & \text{if } M<1, \\ \max\left\{\frac{\nu(1-\nu)(m-1)^2}{(1-\nu)m+\nu}, \frac{\nu(1-\nu)(M-1)^2}{(1-\nu)M+\nu}\right\} & \text{if } m \le 1 \le M, \\ \frac{\nu(1-\nu)(M-1)^2}{(1-\nu)M+\nu} & \text{if } 1<m. \end{cases}$$

Proof. Consider the function $\xi_\nu : (0,\infty) \to (0,\infty)$ defined by

$$\xi_\nu(x) = 1-\nu+\nu x - \frac{x}{(1-\nu)x+\nu},$$

where $\nu \in [0,1]$.

Taking the derivative, we have

$$\begin{aligned} \xi_\nu'(x) &= \nu - \frac{(1-\nu)x+\nu-x(1-\nu)}{[(1-\nu)x+\nu]^2} = \nu\frac{[(1-\nu)x+\nu]^2-1}{[(1-\nu)x+\nu]^2} \\ &= \frac{\nu(1-\nu)(x-1)[(1-\nu)x+\nu+1]}{[(1-\nu)x+\nu]^2} \end{aligned}$$

for any $x \ge 0$ and $\nu \in [0,1]$.

This shows that the function is decreasing on $[0,1]$ and increasing on $(1,\infty)$. We have $\xi_\nu(0) = 1-\nu$, $\xi_\nu(1) = 0$ and $\lim_{x\to\infty} \xi_\nu(x) = \infty$.

Since, by (2.3)

$$\xi_\nu(x) = \frac{\nu(1-\nu)(x-1)^2}{(1-\nu)x+\nu}, \ x \ge 0,$$

then for $[m,M] \subset [0,\infty)$ we have

$$\begin{aligned} \min_{x\in[m,M]} \xi_\nu(x) &= \begin{cases} \frac{\nu(1-\nu)(M-1)^2}{(1-\nu)M+\nu} & \text{if } M<1, \\ 0 & \text{if } m \le 1 \le M, \\ \frac{\nu(1-\nu)(m-1)^2}{(1-\nu)m+\nu} & \text{if } 1<m, \end{cases} \\ &= p_\nu(m,M) \end{aligned}$$

and

$$\max_{x\in[m,M]}\xi_\nu(x)=\begin{cases}\frac{\nu(1-\nu)(m-1)^2}{(1-\nu)m+\nu} & \text{if } M<1,\\ \max\left\{\frac{\nu(1-\nu)(m-1)^2}{(1-\nu)m+\nu},\frac{\nu(1-\nu)(M-1)^2}{(1-\nu)M+\nu}\right\} & \text{if } m\le 1\le M,\\ \frac{\nu(1-\nu)(M-1)^2}{(1-\nu)M+\nu} & \text{if } 1<m,\end{cases}$$
$$= P_\nu\left(m,M\right).$$

Therefore

$$p_\nu\left(m,M\right)\le 1-\nu+\nu x-\left((1-\nu)+\nu x^{-1}\right)^{-1}\le P_\nu\left(m,M\right) \tag{4.2}$$

for any $x\in[m,M]$ and $\nu\in[0,1]$.

If we use the continuous functional calculus for the positive invertible operator X with $mI\le X\le MI$, then we have from (4.2) that

$$\begin{aligned} p\left(m,M\right)I &\le \left(1-\nu\right)I+\nu X-\left(\left(1-\nu\right)I+\nu X^{-1}\right)^{-1}\\ &\le P_\nu\left(m,M\right)I \end{aligned} \tag{4.3}$$

for any $\nu\in[0,1]$.

If we multiply (1.2) both sides by $A^{-1/2}$ we get

$$MI\ge A^{-1/2}BA^{-1/2}\ge mI.$$

By writing the inequality (4.3) for $X=A^{-1/2}BA^{-1/2}$ we obtain

$$\begin{aligned} p(m,M)I &\le (1-\nu)I+\nu A^{-1/2}BA^{-1/2}\\ &\quad - A^{-1/2}\left((1-\nu)A^{-1}+\nu B^{-1}\right)^{-1}A^{-1/2}\\ &\le P_\nu\left(m,M\right)I \end{aligned} \tag{4.4}$$

for any $\nu\in[0,1]$.

If we multiply (4.4) both sides by $A^{1/2}$ we get

$$\begin{aligned} p\left(m,M\right)A &\le \left(1-\nu\right)A+\nu B-\left(\left(1-\nu\right)A^{-1}+\nu B^{-1}\right)^{-1}\\ &\le P_\nu\left(m,M\right)A \end{aligned}$$

for any $\nu\in[0,1]$. ∎

Remark 2. If we consider

$$p\left(m,M\right):=\begin{cases}\frac{(M-1)^2}{2(M+1)} & \text{if } M<1,\\ 0 & \text{if } m\le 1\le M,\\ \frac{(m-1)^2}{2(m+1)} & \text{if } 1<m,\end{cases}$$

$$P\left(m,M\right):=\begin{cases}\frac{(m-1)^2}{2(m+1)} & \text{if } M<1,\\ \max\left\{\frac{(m-1)^2}{2(m+1)},\frac{(M-1)^2}{2(M+1)}\right\} & \text{if } m\le 1\le M,\\ \frac{(M-1)^2}{2(M+1)} & \text{if } 1<m,\end{cases}$$

then by (4.1) we have

$$p\left(m,M\right)A\le A\nabla B-A!B\le P\left(m,M\right)A, \tag{4.5}$$

provided that A, B are positive invertible operators and $M>m>0$ are such that the condition (1.2) holds.

Corollary 3. *Let A, B be two positive operators and m, m', M, M' be positive real numbers. Put $h:=\frac{M}{m}$ and $h':=\frac{M'}{m'}$.*

(i) *if $0<mI\le A\le m'I<M'I\le B\le MI$, then for any $\nu\in[0,1]$*

$$\begin{aligned}\frac{\nu\left(1-\nu\right)\left(h'-1\right)^2}{\left(1-\nu\right)h'+\nu}A&\le A\nabla_\nu B-A!_\nu B\\ &\le\frac{\nu\left(1-\nu\right)\left(h-1\right)^2}{\left(1-\nu\right)h+\nu}A\end{aligned} \tag{4.6}$$

and, in particular,

$$\frac{\left(h'-1\right)^2}{2\left(h'+1\right)}A\le A\nabla B-A!B\le\frac{\left(h-1\right)^2}{2\left(h+1\right)}A. \tag{4.7}$$

(ii) *if $0<mI\le B\le m'I<M'I\le A\le MI$, then for any $\nu\in[0,1]$*

$$\begin{aligned}\frac{\nu\left(1-\nu\right)\left(h'-1\right)^2}{h'\left(1-\nu+\nu h'\right)}A&\le A\nabla_\nu B-A!_\nu B\\ &\le\frac{\nu\left(1-\nu\right)\left(h-1\right)^2}{h\left(1-\nu+\nu h\right)}A\end{aligned} \tag{4.8}$$

and, in particular,

$$\frac{(h'-1)^2}{2h'(1+h')}A \leq A\nabla B - A!B \leq \frac{(h-1)^2}{2h(1+h)}A. \tag{4.9}$$

Proof. We observe that h, $h' > 1$ and if either of the condition (i) or (ii) holds, then $h \geq h'$.

If (i) is valid, then we have

$$A < h'A = \frac{M'}{m'}A \leq B \leq \frac{M}{m}A = hA,$$

while, if (ii) is valid, then we have

$$\frac{1}{h}A \leq B \leq \frac{1}{h'}A < A.$$

If we use the inequality (4.1) and the assumption (i), then we get (4.6). If we use the inequality (4.1) and the assumption (ii), then we get (4.8). ∎

5. A comparison

We observe that an upper bound for the difference $A\nabla_\nu B - A!_\nu B$ as provided in (1.3) is

$$B_1(\nu, m, M)A := \max\{\nu, 1-\nu\} \times \begin{cases} \frac{(m-1)^2}{m+1}A & \text{if } M < 1, \\ \max\left\{\frac{(m-1)^2}{m+1}, \frac{(M-1)^2}{M+1}\right\}A & \text{if } m \leq 1 \leq M, \\ \frac{(M-1)^2}{M+1}A & \text{if } 1 < m \end{cases}$$

while the one from (2.10) is

$$B_2(\nu, m, M)A := \nu(1-\nu) \times \begin{cases} \frac{(m-1)^2}{m}A & \text{if } M < 1, \\ \frac{1}{m}\max\left\{(m-1)^2, (M-1)^2\right\}A & \text{if } m \leq 1 \leq M, \\ (M-1)^2 A & \text{if } 1 < m, \end{cases}$$

where A, B are positive invertible operators and $M > m > 0$ such that the condition (1.2) holds.

We consider for $x = m \in (0,1)$ and $y = \nu \in [0,1]$ the difference

$$D_1(x,y) = \max\{y, 1-y\} \frac{(x-1)^2}{x+1} - y(1-y) \frac{(x-1)^2}{x}$$

that has the 3D plot on the box $[0.3, 0.6] \times [0,1]$ depicted in Figure 1 showing that it takes both positive and negative values, meaning that neither of the bounds $B_1(\nu, m, M) A$ and $B_2(\nu, m, M) A$ is better in the case $0 < m < M < 1$.

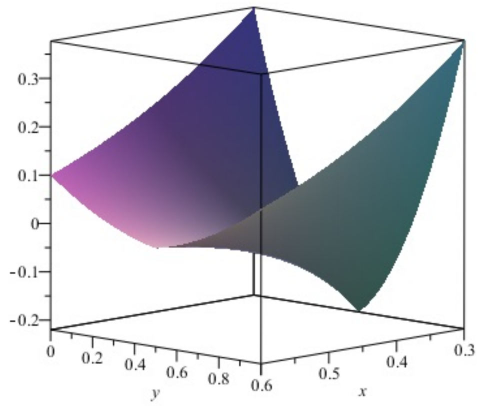

FIGURE 1: Plot of difference $D_1(x,y)$

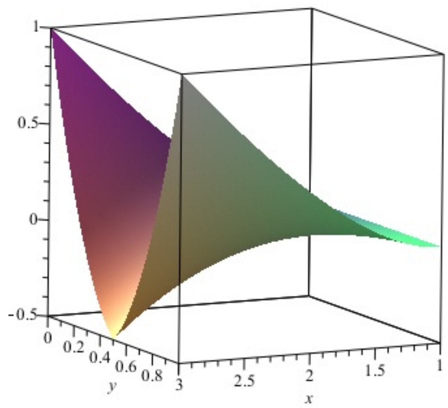

FIGURE 2: Plot of difference $D_2(x,y)$

We consider for $x = M \in (1, \infty)$ and $y = \nu \in [0, 1]$ the difference

$$D_2(x, y) = \max\{y, 1-y\} \frac{(x-1)^2}{x+1} - y(1-y)(x-1)^2$$

that has the 3D plot on the box $[1, 3] \times [0, 1]$ depicted in Figure 2 showing that it takes both positive and negative values, meaning that neither of the bounds $B_1(\nu, m, M) A$ and $B_2(\nu, m, M) A$ is better in the case $1 < m < M < \infty$.

Similar conclusions may be derived for lower bounds, however the details are left to the interested reader.

References

[1] S.S. Dragomir, Bounds for the normalised Jensen functional, *Bull. Austral. Math. Soc.* **74** (3) (2006), 417 – 478.

[2] S.S. Dragomir, A note on Young's inequality, preprint *RGMIA Res. Rep. Coll.* **18** (2015), Art. 126, `http://rgmia.org/papers/v18/v18a126.pdf`.

[3] S.S. Dragomir, Some new reverses of Young's operator inequality, preprint *RGMIA Res. Rep. Coll.* **18** (2015), Art. 130, `http://rgmia.org/papers/v18/v18a130.pdf`.

[4] S.S. Dragomir, On new refinements and reverses of Young's operator inequality, preprint *RGMIA Res. Rep. Coll.* **18** (2015), Art. 135, `http://rgmia.org/papers/v18/v18a135.pdf`.

[5] S.S. Dragomir, Some inequalities for operator weighted geometric mean, preprint *RGMIA Res. Rep. Coll.* **18** (2015), Art. 139, `http://rgmia.org/papers/v18/v18a139.pdf`.

[6] S.S. Dragomir, Some reverses and a refinement of Hölder operator inequality, preprint *RGMIA Res. Rep. Coll.* **18** (2015), Art. 147, `http://rgmia.org/papers/v18/v18a147.pdf`.

[7] S.S. Dragomir, Some inequalities for weighted harmonic and arithmetic operator means, preprint *RGMIA Res. Rep. Coll.* **19** (2016), Art. 5, `http://rgmia.org/papers/v19/v19a05.pdf`.

[8] S. Furuichi, Refined Young inequalities with Specht's ratio, *J. Egyptian Math. Soc.* **20** (2012), 46 – 49.

[9] S. Furuichi, On refined Young inequalities and reverse inequalities, *J. Math. Inequal.* **5** (2011), 21 – 31.

[10] W. Liao, J. Wu, J. Zhao, New versions of reverse Young and Heinz mean inequalities with the Kantorovich constant, *Taiwanese J. Math.* **19** (2) (2015), 467 – 479.

[11] M. Tominaga, Specht's ratio in the Young inequality, *Sci. Math. Japon.* **55** (2002), 583 – 588.

[12] G. Zuo, G. Shi, M. Fujii, Refined Young inequality with Kantorovich constant, *J. Math. Inequal.* **5** (2011), 551 – 556.