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On some Chebyshev type inequalities for the complex integral

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Abstract. Assume that f and g are continuous on γ, $\gamma \subset \mathbb{C}$ is a piecewise smooth path parametrized by $z(t)$, $t \in [a,b]$ from $z(a) = u$ to $z(b) = w$ with $w \neq u$, and the *complex Chebyshev functional* is defined by

$$\mathcal{D}_\gamma (f,g) := \frac{1}{w-u}\int_\gamma f(z)\,g(z)\,dz - \frac{1}{w-u}\int_\gamma f(z)\,dz \frac{1}{w-u}\int_\gamma g(z)\,dz.$$

In this paper we establish some bounds for the magnitude of the functional $\mathcal{D}_\gamma (f,g)$ under Lipschitzian assumptions for the functions f and g, and provide a complex version for the well known Chebyshev inequality.
Keywords: Complex integral, Continuous functions, Holomorphic functions, Chebyshev inequality.
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Sobre algunas desigualdades tipo Chebyshev para la integral compleja

Resumen. Sean f y g funciones continuas sobre γ, siendo $\gamma \subset \mathbb{C}$ un camino suave por partes parametrizado por $z(t)$, $t \in [a,b]$ con $z(a) = u$ y $z(b) = w$, $w \neq u$, y el *funcional de Chebyshev complejo* definido por

$$\mathcal{D}_\gamma (f,g) := \frac{1}{w-u}\int_\gamma f(z)\,g(z)\,dz - \frac{1}{w-u}\int_\gamma f(z)\,dz \frac{1}{w-u}\int_\gamma g(z)\,dz.$$

En este artículo establecemos algunas cotas para la magnitud del funcional $\mathcal{D}_\gamma (f,g)$ bajo condiciones de lipschitzianidad para las funciones f y g, y damos una versión compleja para la conocida desigualdad de Chebyshev.
Palabras clave: Integral compleja, funciones continuas, funciones holomórficas, desigualdad de Chebyshev.

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1. Introduction

For two Lebesgue integrable functions $f,\, g : [a, b] \to \mathbb{C}$, in order to compare the integral mean of the product with the product of the integral means, we consider the *Chebyshev functional* defined by

$$C(f,g) := \frac{1}{b-a}\int_a^b f(t)\, g(t)\, dt - \frac{1}{b-a}\int_a^b f(t)\, dt \frac{1}{b-a}\int_a^b g(t)\, dt.$$

In 1934, G. Grüss [17] showed that

$$|C(f,g)| \le \frac{1}{4}(M-m)(N-n), \tag{1}$$

provided m, M, n, N are real numbers with the property that

$$-\infty < m \le f \le M < \infty, \quad -\infty < n \le g \le N < \infty \quad \text{a.e. on} \quad [a,b]. \tag{2}$$

The constant $\frac{1}{4}$ in (1) is sharp.

Another, however less known result, even though it was obtained by Chebyshev in 1882, [8], states that

$$|C(f,g)| \le \frac{1}{12}\|f'\|_\infty \|g'\|_\infty (b-a)^2, \tag{3}$$

provided that f', g' exist and are continuous on $[a, b]$ and $\|f'\|_\infty = \sup_{t\in[a,b]} |f'(t)|$. The constant $\frac{1}{12}$ cannot be improved in the general case.

The Chebyshev inequality (3) also holds if $f,\, g : [a, b] \to \mathbb{R}$ are assumed to be *absolutely continuous* and $f',\, g' \in L_\infty[a, b]$, while $\|f'\|_\infty = essup_{t\in[a,b]} |f'(t)|$.

For other inequality of Grüss' type see [1]-[16] and [18]-[28].

In order to extend Grüss' inequality to complex integral we need the following preparations.

Suppose γ is a smooth path parametrized by $z(t)$, $t \in [a, b]$ and f is a complex valued function which is continuous on γ. Put $z(a) = u$ and $z(b) = w$ with $u, w \in \mathbb{C}$. We define the integral of f on $\gamma_{u,w} = \gamma$ as

$$\int_\gamma f(z)\, dz = \int_{\gamma_{u,w}} f(z)\, dz := \int_a^b f(z(t))\, z'(t)\, dt.$$

We observe that the actual choice of parametrization of γ does not matter.

This definition immediately extends to paths that are piecewise smooth. Suppose γ is parametrized by $z(t)$, $t \in [a, b]$, which is differentiable on the intervals $[a, c]$ and $[c, b]$; then, assuming that f is continuous on γ, we define

$$\int_{\gamma_{u,w}} f(z)\, dz := \int_{\gamma_{u,v}} f(z)\, dz + \int_{\gamma_{v,w}} f(z)\, dz,$$

where $v := z(c)$. This can be extended for a finite number of intervals.

We also define the integral with respect to arc-length:

$$\int_{\gamma_{u,w}} f(z)\,|dz| := \int_a^b f(z(t))\,|z'(t)|\,dt,$$

and the length of the curve γ is then

$$\ell(\gamma) = \int_{\gamma_{u,w}} |dz| = \int_a^b |z'(t)|\,dt.$$

Let f and g be holomorphic in G, an open domain, and suppose $\gamma \subset G$ is a piecewise smooth path from $z(a) = u$ to $z(b) = w$. Then we have the *integration by parts formula*

$$\int_{\gamma_{u,w}} f(z)\,g'(z)\,dz = f(w)\,g(w) - f(u)\,g(u) - \int_{\gamma_{u,w}} f'(z)\,g(z)\,dz. \tag{4}$$

We recall also the *triangle inequality* for the complex integral, namely,

$$\left|\int_\gamma f(z)\,dz\right| \le \int_\gamma |f(z)|\,|dz| \le \|f\|_{\gamma,\infty}\,\ell(\gamma), \tag{5}$$

where $\|f\|_{\gamma,\infty} := \sup_{z\in\gamma} |f(z)|$.

We also define the p-norm with $p \ge 1$ by

$$\|f\|_{\gamma,p} := \left(\int_\gamma |f(z)|^p\,|dz|\right)^{1/p}.$$

For $p = 1$ we have

$$\|f\|_{\gamma,1} := \int_\gamma |f(z)|\,|dz|.$$

If $p,\ q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then, by Hölder's inequality, we have

$$\|f\|_{\gamma,1} \le [\ell(\gamma)]^{1/q}\,\|f\|_{\gamma,p}.$$

Suppose $\gamma \subset \mathbb{C}$ is a piecewise smooth path parametrized by $z(t)$, $t \in [a,b]$ from $z(a) = u$ to $z(b) = w$ with $w \ne u$. If f and g are continuous on γ, we consider the *complex Chebyshev functional* defined by

$$\mathcal{D}_\gamma(f,g) := \frac{1}{w-u}\int_\gamma f(z)\,g(z)\,dz - \frac{1}{w-u}\int_\gamma f(z)\,dz\,\frac{1}{w-u}\int_\gamma g(z)\,dz.$$

In this paper we establish some bounds for the magnitude of the functional $\mathcal{D}_\gamma(f,g)$ under various assumptions for the functions f and g, and provide a complex version for the Chebyshev inequality (3).

2. Chebyshev type results

We start with the following identity of interest:

Lemma 2.1. *Suppose $\gamma \subset \mathbb{C}$ is a piecewise smooth path parametrized by $z(t)$, $t \in [a,b]$ from $z(a) = u$ to $z(b) = w$ with $w \neq u$. If f and g are continuous on γ, then*

$$\begin{aligned} \mathcal{D}_\gamma(f,g) &= \frac{1}{2(w-u)^2} \int_\gamma \left(\int_\gamma (f(z) - f(w))(g(z) - g(w))\,dw \right) dz \qquad (6) \\ &= \frac{1}{2(w-u)^2} \int_\gamma \left(\int_\gamma (f(z) - f(w))(g(z) - g(w))\,dz \right) dw \\ &= \frac{1}{2(w-u)^2} \int_\gamma \int_\gamma (f(z) - f(w))(g(z) - g(w))\,dz dw. \end{aligned}$$

Proof. For any $z \in \gamma$ the integral $\int_\gamma (f(z) - f(w))(g(z) - g(w))\,dw$ exists and

$$\begin{aligned} I(z) &:= \int_\gamma (f(z) - f(w))(g(z) - g(w))\,dw \\ &= \int_\gamma (f(z)g(z) + f(w)g(w) - g(z)f(w) - f(z)g(w))\,dw \\ &= f(z)g(z)\int_\gamma dw + \int_\gamma f(w)g(w)\,dw - g(z)\int_\gamma f(w)\,dw - f(z)\int_\gamma g(w)\,dw \\ &= (w-u)f(z)g(z) + \int_\gamma f(w)g(w)\,dw - g(z)\int_\gamma f(w)\,dw - f(z)\int_\gamma g(w)\,dw. \end{aligned}$$

The function $I(z)$ is also continuous on γ, then the integral $\int_\gamma I(z)\,dz$ exists and

$$\begin{aligned} \int_\gamma I(z)\,dz &= \int_\gamma \left[(w-u)f(z)g(z) + \int_\gamma f(w)g(w)\,dw \right. \\ &\qquad \left. -g(z)\int_\gamma f(w)\,dw - f(z)\int_\gamma g(w)\,dw \right] dz \\ &= (w-u)\int_\gamma f(z)g(z)\,dz + (w-u)\int_\gamma f(w)g(w)\,dw \\ &\quad - \int_\gamma f(w)\,dw \int_\gamma g(z)\,dz - \int_\gamma g(w)\,dw \int_\gamma f(z)\,dz \\ &= 2(w-u)\int_\gamma f(z)g(z)\,dz - 2\int_\gamma f(z)\,dz \int_\gamma g(z)\,dz \\ &= 2(w-u)^2 \mathcal{D}_\gamma(f,g), \end{aligned}$$

which proves the first equality in (6).

The rest follows in a similar manner and we omit the details. ☑

Suppose $\gamma \subset \mathbb{C}$ is a piecewise smooth path from $z(a) = u$ to $z(b) = w$ and $h : \gamma \to \mathbb{C}$ a continuous function on γ. Define the quantity:

$$\begin{aligned}\mathcal{P}_\gamma\left(h,\overline{h}\right) &= \frac{1}{\ell(\gamma)}\int_\gamma |h(z)|^2\,|dz| - \left|\frac{1}{\ell(\gamma)}\int_\gamma h(z)\,|dz|\right|^2 \\ &= \frac{1}{\ell(\gamma)}\int_\gamma \left|h(v) - \frac{1}{\ell(\gamma)}\int_\gamma h(z)\,|dz|\right|^2 |dv| \geq 0.\end{aligned} \tag{7}$$

We say that the function $f : G \subset \mathbb{C} \to \mathbb{C}$ is L-h-Lipschitzian on the subset G if

$$|f(z) - f(w)| \leq L\,|h(z) - h(w)|$$

for any $z, w \in G$. If $h(z) = z$, we recapture the usual concept of L-Lipschitzian functions on G.

Theorem 2.2. *Suppose $\gamma \subset \mathbb{C}$ is a piecewise smooth path parametrized by $z(t)$, $t \in [a,b]$ from $z(a) = u$ to $z(b) = w$ with $w \neq u$, $h : \gamma \to \mathbb{C}$ is continuous, f and g are L_1, L_2-h-Lipschitzian functions on γ; then*

$$|\mathcal{D}_\gamma(f,g)| \leq L_1 L_2 \frac{\ell^2(\gamma)}{|w-u|^2}\mathcal{P}_\gamma\left(h,\overline{h}\right). \tag{8}$$

Proof. Taking the modulus in the first equality in (6), we get

$$\begin{aligned}|\mathcal{D}_\gamma(f,g)| &= \frac{1}{2|w-u|^2}\left|\int_\gamma\left(\int_\gamma (f(z)-f(w))(g(z)-g(w))\,dw\right)dz\right| \\ &\leq \frac{1}{2|w-u|^2}\int_\gamma\left|\int_\gamma (f(z)-f(w))(g(z)-g(w))\,dw\right||dz| \\ &\leq \frac{1}{2|w-u|^2}\int_\gamma\left(\int_\gamma |(f(z)-f(w))(g(z)-g(w))|\,|dw|\right)|dz| \\ &\leq \frac{L_1L_2}{2|w-u|^2}\int_\gamma\left(\int_\gamma |h(z)-h(w)|^2\,|dw|\right)|dz| =: A.\end{aligned} \tag{9}$$

Now, observe that

$$\begin{aligned}
&\int_{\gamma}\left(\int_{\gamma}|h(z)-h(w)|^{2}|dw|\right)|dz| \qquad (10)\\
&=\int_{\gamma}\left(\int_{\gamma}\left(|h(z)|^{2}-2Re\left(h(z)\overline{h(w)}\right)+|h(w)|^{2}\right)|dw|\right)|dz|\\
&=\int_{\gamma}\left(\ell(\gamma)|h(z)|^{2}-2Re\left(h(z)\int_{\gamma}\overline{h(w)}|dw|\right)+\int_{\gamma}|h(w)|^{2}|dw|\right)|dz|\\
&=\ell(\gamma)\int_{\gamma}|h(z)|^{2}|dz|-2Re\left(\int_{\gamma}h(z)|dz|\int_{\gamma}\overline{h(w)}|dw|\right)\\
&+\ell(\gamma)\int_{\gamma}|h(w)|^{2}|dw|\\
&=2\ell(\gamma)\int_{\gamma}|h(z)|^{2}|dz|-2Re\left(\int_{\gamma}h(z)|dz|\overline{\left(\int_{\gamma}h(w)|dw|\right)}\right)\\
&=2\left[\ell(\gamma)\int_{\gamma}|h(z)|^{2}|dz|-\left|\int_{\gamma}h(z)|dz|\right|^{2}\right]=2\ell^{2}(\gamma)\mathcal{P}_{\gamma}\left(h,\overline{h}\right).
\end{aligned}$$

Therefore, by (10) we get

$$A=L_1L_2\frac{\ell^2(\gamma)}{|w-u|^2}\mathcal{P}_{\gamma}\left(h,\overline{h}\right),$$

and by (9) we get the desired result (8). ☑

Further, for $\gamma\subset\mathbb{C}$ a piecewise smooth path parametrized by $z(t)$, and by taking $h(z)=z$ in (7), we can consider the quantity

$$\begin{aligned}
\mathcal{P}_{\gamma}&:=\frac{1}{\ell(\gamma)}\int_{\gamma}|z|^{2}|dz|-\left|\frac{1}{\ell(\gamma)}\int_{\gamma}z|dz|\right|^{2} \qquad (11)\\
&=\frac{1}{\ell(\gamma)}\int_{\gamma}\left|v-\frac{1}{\ell(\gamma)}\int_{\gamma}z|dz|\right|^{2}|dv|\\
&=\frac{1}{2\ell^{2}(\gamma)}\int_{\gamma}\left(\int_{\gamma}|z-w|^{2}|dw|\right)|dz|\geq 0.
\end{aligned}$$

Corollary 2.3. *Suppose $\gamma\subset\mathbb{C}$ is a piecewise smooth path parametrized by $z(t)$, $t\in[a,b]$ from $z(a)=u$ to $z(b)=w$ with $w\neq u$ and f and g are L_1, L_2-Lipschitzian functions on γ; then*

$$|\mathcal{D}_{\gamma}(f,g)|\leq L_1L_2\frac{\ell^2(\gamma)}{|w-u|^2}\mathcal{P}_{\gamma}. \qquad (12)$$

Remark 2.4. Assume that f is L-h-Lipschitzian on γ. For $g=f$ we have

$$\mathcal{D}_{\gamma}(f,f)=\frac{1}{w-u}\int_{\gamma}f^{2}(z)\,dz-\left(\frac{1}{w-u}\int_{\gamma}f(z)\,dz\right)^{2}, \qquad (13)$$

and by (8) we get

$$\left|\mathcal{D}_{\gamma}\left(f, f\right)\right| \leq L^{2} \frac{\ell^{2}\left(\gamma\right)}{\left|w-u\right|^{2}} \mathcal{P}_{\gamma}\left(h, \overline{h}\right). \tag{14}$$

For $g=\bar{f}$ we have

$$\mathcal{D}_{\gamma}\left(f, \bar{f}\right)=\frac{1}{w-u} \int_{\gamma}\left|f\left(z\right)\right|^{2} dz-\frac{1}{w-u} \int_{\gamma} f\left(z\right) dz \frac{1}{w-u} \int_{\gamma} \overline{f\left(z\right)} dz, \tag{15}$$

and by (8) we get

$$\left|\mathcal{D}_{\gamma}\left(f, \bar{f}\right)\right| \leq L^{2} \frac{\ell^{2}\left(\gamma\right)}{\left|w-u\right|^{2}} \mathcal{P}_{\gamma}\left(h, \overline{h}\right). \tag{16}$$

If f is L-Lipschitzian on γ, then

$$\left|\mathcal{D}_{\gamma}\left(f, f\right)\right| \leq L^{2} \frac{\ell^{2}\left(\gamma\right)}{\left|w-u\right|^{2}} \mathcal{P}_{\gamma} \tag{17}$$

and

$$\left|\mathcal{D}_{\gamma}\left(f, \bar{f}\right)\right| \leq L^{2} \frac{\ell^{2}\left(\gamma\right)}{\left|w-u\right|^{2}} \mathcal{P}_{\gamma}. \tag{18}$$

If the path γ is a segment $[u, w]$ connecting two distinct points u and w in $\mathbb{C}$, then we write $\int_{\gamma} f\left(z\right) dz$ as $\int_{u}^{w} f\left(z\right) dz$.

Now, if f and g are L_1, L_2-Lipschitzian functions on $[u, w]:=\{(1-t) u+t w,\ t \in[0,1]\}$, then by (12) we have

$$\left|\mathcal{D}_{\gamma}\left(f, g\right)\right| \leq L_{1} L_{2} \mathcal{P}_{[u, w]},$$

where

$$\begin{aligned}
\mathcal{P}_{[u, w]} & =\frac{|w-u|^{2}}{2|w-u|^{2}} \int_{0}^{1}\left(\int_{0}^{1}|(1-t) u+t w-(1-s) u-s w|^{2} d t\right) d s \\
& =\frac{1}{2}|w-u|^{2} \int_{0}^{1}\left(\int_{0}^{1}(t-s)^{2} d t\right) d s=\frac{1}{12}|w-u|^{2}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \left|\frac{1}{w-u} \int_{\gamma} f\left(z\right) g\left(z\right) dz-\frac{1}{w-u} \int_{\gamma} f\left(z\right) dz \frac{1}{w-u} \int_{\gamma} g\left(z\right) dz\right| \\
& \leq \frac{1}{12}|w-u|^{2} L_{1} L_{2},
\end{aligned} \tag{19}$$

if f and g are L_1, L_2-Lipschitzian functions on $[u, w]$.

If f is L-Lipschitzian on $[u, w]$, then

$$\left|\frac{1}{w-u} \int_{\gamma} f^{2}\left(z\right) dz-\left(\frac{1}{w-u} \int_{\gamma} f\left(z\right) dz\right)^{2}\right| \leq \frac{1}{12}|w-u|^{2} L^{2} \tag{20}$$

and

$$\begin{aligned}
& \left|\frac{1}{w-u} \int_{\gamma}\left|f\left(z\right)\right|^{2} dz-\frac{1}{w-u} \int_{\gamma} f\left(z\right) dz \frac{1}{w-u} \int_{\gamma} \overline{f\left(z\right)} dz\right| \\
& \leq \frac{1}{12}|w-u|^{2} L^{2}.
\end{aligned} \tag{21}$$

3. Examples for circular paths

Let $[a,b]\subseteq[0,2\pi]$ and the circular path $\gamma_{[a,b],R}$ centered in 0 and with radius $R>0$:

$$z(t)=R\exp(it)=R(\cos t+i\sin t),\ t\in[a,b].$$

If $[a,b]=[0,\pi]$, then we get a half circle, while for $[a,b]=[0,2\pi]$ we get the full circle. Since

$$\begin{aligned}\left|e^{is}-e^{it}\right|^2&=\left|e^{is}\right|^2-2Re\left(e^{i(s-t)}\right)+\left|e^{it}\right|^2\\&=2-2\cos(s-t)=4\sin^2\left(\frac{s-t}{2}\right)\end{aligned}$$

for any $t,\ s\in\mathbb{R}$, then

$$\left|e^{is}-e^{it}\right|^r=2^r\left|\sin\left(\frac{s-t}{2}\right)\right|^r \tag{22}$$

for any $t,\ s\in\mathbb{R}$ and $r>0$. In particular,

$$\left|e^{is}-e^{it}\right|=2\left|\sin\left(\frac{s-t}{2}\right)\right|$$

for any $t,\ s\in\mathbb{R}$.

If $u=R\exp(ia)$ and $w=R\exp(ib)$, then

$$\begin{aligned}w-u&=R[\exp(ib)-\exp(ia)]=R[\cos b+i\sin b-\cos a-i\sin a]\\&=R[\cos b-\cos a+i(\sin b-\sin a)].\end{aligned}$$

Since

$$\cos b-\cos a=-2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{b-a}{2}\right)$$

and

$$\sin b-\sin a=2\sin\left(\frac{b-a}{2}\right)\cos\left(\frac{a+b}{2}\right),$$

hence

$$\begin{aligned}w-u&=R\left[-2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{b-a}{2}\right)+2i\sin\left(\frac{b-a}{2}\right)\cos\left(\frac{a+b}{2}\right)\right]\\&=2R\sin\left(\frac{b-a}{2}\right)\left[-\sin\left(\frac{a+b}{2}\right)+i\cos\left(\frac{a+b}{2}\right)\right]\\&=2Ri\sin\left(\frac{b-a}{2}\right)\left[\cos\left(\frac{a+b}{2}\right)+i\sin\left(\frac{a+b}{2}\right)\right]\\&=2Ri\sin\left(\frac{b-a}{2}\right)\exp\left[\left(\frac{a+b}{2}\right)i\right].\end{aligned}$$

If $\gamma = \gamma_{[a,b],R}$, then the *circular complex Chebyshev functional* is defined by

$$\begin{aligned}
&\mathcal{C}_{[a,b],R}(f,g) \\
&:= \mathcal{D}_{\gamma_{[a,b],R}}(f,g) \\
&= \frac{1}{2\sin\left(\frac{b-a}{2}\right)\exp\left[\left(\frac{a+b}{2}\right)i\right]}\int_a^b f(R\exp(it))\,g(R\exp(it))\exp(it)\,dt \\
&- \frac{1}{4\sin^2\left(\frac{b-a}{2}\right)\exp\left[2\left(\frac{a+b}{2}\right)i\right]} \\
&\times \int_a^b f(R\exp(it))\exp(it)\,dt\int_a^b g(R\exp(it))\exp(it)\,dt.
\end{aligned} \tag{23}$$

If $\gamma = \gamma_{[a,b],R}$, then

$$\mathcal{P}_\gamma := \frac{1}{2\ell^2(\gamma)}\int_\gamma\left(\int_\gamma |z-w|^2\,|dw|\right)|dz| \tag{24}$$

$$\begin{aligned}
&= \frac{R^4}{2R^2(b-a)^2}\int_a^b\left(\int_a^b \left|e^{is}-e^{it}\right|^2 dt\right)ds \qquad (25)\\
&= \frac{R^4}{2R^2(b-a)^2}\int_a^b\left(\int_a^b [2-2\cos(s-t)]\,dt\right)ds \\
&= \frac{R^2}{(b-a)^2}\int_a^b\left(\int_a^b [1-\cos(s-t)]\,dt\right)ds \\
&= \frac{R^2}{(b-a)^2}\int_a^b (b-a-\sin(b-s)-\sin(s-a))\,ds \\
&= \frac{R^2}{(b-a)^2}\left[(b-a)^2 - 1 + \cos(b-a) + \cos(b-a) - 1\right] \\
&= \frac{R^2}{(b-a)^2}\left[(b-a)^2 - 2(1-\cos(b-a))\right] \\
&= \frac{R^2}{(b-a)^2}\left[(b-a)^2 - 4\sin^2\left(\frac{b-a}{2}\right)\right] \\
&= \frac{4R^2}{(b-a)^2}\left[\left(\frac{b-a}{2}\right)^2 - \sin^2\left(\frac{b-a}{2}\right)\right].
\end{aligned}$$

We have the following result:

Proposition 3.1. *Let $\gamma_{[a,b],R}$ be a circular path centered in 0, with radius $R > 0$ and $[a,b] \subset [0,2\pi]$. If f and g are L_1, L_2-Lipschitzian functions on $\gamma_{[a,b],R}$, then*

$$\left|\mathcal{C}_{[a,b],R}(f,g)\right| \le \frac{R^2}{\sin^2\left(\frac{b-a}{2}\right)}\left[\left(\frac{b-a}{2}\right)^2 - \sin^2\left(\frac{b-a}{2}\right)\right]L_1L_2. \tag{26}$$

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