EXTRACTA MATHEMATICAE doi:10.17398/2605-5686.34.2.201
Vol. **34**, Num. 2 (2019), 201 – 235 Available online June 3, 2019

Hypo-q-norms on cartesian products of algebras of bounded linear operators on Hilbert spaces

S.S. Dragomir [1,2]

[1] *Mathematics, College of Engineering & Science*
Victoria University, Melbourne City 8001, *Australia*
[2] *DST-NRF Centre of Excellence in the Mathematical and Statistical Sciences*
School of Computer Science & Applied Mathematics
University of the Witwatersrand, Johannesburg 2050, *South Africa*

sever.dragomir@vu.edu.au , http://rgmia.org/dragomir

Received March 6, 2019 Presented by Horst Martini
Accepted May 14, 2019

Abstract: In this paper we introduce the hypo-q-norms on a Cartesian product of algebras of bounded linear operators on Hilbert spaces. A representation of these norms in terms of inner products, the equivalence with the q-norms on a Cartesian product and some reverse inequalities obtained via the scalar reverses of Cauchy-Buniakowski-Schwarz inequality are also given. Several bounds for the norms δ_p, ϑ_p and the real norms $\eta_{r,p}$ and $\theta_{r,p}$ are provided as well.

Key words: Hilbert spaces, bounded linear operators, operator norm and numerical radius, n-tuple of operators, operator inequalities.

AMS *Subject Class.* (2010): 46C05, 26D15.

1. Introduction

In [13], the author has introduced the following norm on the Cartesian product $B^{(n)}(H) := B(H) \times \cdots \times B(H)$, where $B(H)$ denotes the Banach algebra of all bounded linear operators defined on the complex Hilbert space H:

$$\|(T_1, \ldots, T_n)\|_{n,e} := \sup_{(\lambda_1,\ldots,\lambda_n)\in\mathbb{B}_n} \|\lambda_1 T_1 + \cdots + \lambda_n T_n\|, \tag{1.1}$$

where $(T_1, \ldots, T_n) \in B^{(n)}(H)$ and

$$\mathbb{B}_n := \left\{ (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n \,:\, \sum_{i=1}^n |\lambda_i|^2 \le 1 \right\}$$

is the Euclidean closed ball in $\mathbb{C}^n$. It is clear that $\|\cdot\|_{n,e}$ is a norm on $B^{(n)}(H)$ and for any $(T_1, \ldots, T_n) \in B^{(n)}(H)$ we have

$$\|(T_1, \ldots, T_n)\|_{n,e} = \|(T_1^*, \ldots, T_n^*)\|_{n,e}\,, \tag{1.2}$$

ISSN: 0213-8743 (print), 2605-5686 (online)

where T_i^* is the adjoint operator of T_i, $i \in \{1, \dots, n\}$.

It has been shown in [13] that the following inequality holds true:

$$\frac{1}{\sqrt{n}} \left\| \sum_{j=1}^{n} T_j T_j^* \right\|^{\frac{1}{2}} \le \|(T_1, \dots, T_n)\|_{n,e} \le \left\| \sum_{j=1}^{n} T_j T_j^* \right\|^{\frac{1}{2}} \tag{1.3}$$

for any n-tuple $(T_1, \dots, T_n) \in B^{(n)}(H)$ and the constants $\frac{1}{\sqrt{n}}$ and 1 are best possible.

In the same paper [13] the author has introduced the *Euclidean operator radius* of an n-tuple of operators $(T_1, \dots, T_n)$ by

$$w_{n,e}(T_1, \dots, T_n) := \sup_{\|x\|=1} \left(\sum_{j=1}^{n} |\langle T_j x, x \rangle|^2 \right)^{\frac{1}{2}} \tag{1.4}$$

and proved that $w_{n,e}(\cdot)$ is a norm on $B^{(n)}(H)$ and satisfies the double inequality:

$$\frac{1}{2} \|(T_1, \dots, T_n)\|_{n,e} \le w_{n,e}(T_1, \dots, T_n) \le \|(T_1, \dots, T_n)\|_{n,e} \tag{1.5}$$

for each n-tuple $(T_1, \dots, T_n) \in B^{(n)}(H)$.

As pointed out in [13], the Euclidean numerical radius also satisfies the double inequality:

$$\frac{1}{2\sqrt{n}} \left\| \sum_{j=1}^{n} T_j T_j^* \right\|^{\frac{1}{2}} \le w_{n,e}(T_1, \dots, T_n) \le \left\| \sum_{j=1}^{n} T_j T_j^* \right\|^{\frac{1}{2}} \tag{1.6}$$

for any $(T_1, \dots, T_n) \in B^{(n)}(H)$ and the constants $\frac{1}{2\sqrt{n}}$ and 1 are best possible.

Now, let $(E, \|\cdot\|)$ be a normed linear space over the complex number field $\mathbb{C}$. On $\mathbb{C}^n$ endowed with the canonical linear structure we consider a norm $\|\cdot\|_n$. As an example of such norms we should mention the usual *p-norms*

$$\|\lambda\|_{n,p} := \begin{cases} \max\left\{|\lambda_1|, \dots, |\lambda_n|\right\} & \text{if } p = \infty, \\ \left(\sum_{k=1}^{n} |\lambda_k|^p \right)^{\frac{1}{p}} & \text{if } p \in [1, \infty). \end{cases}$$

The *Euclidean norm* is obtained for $p = 2$, i.e.,

$$\|\lambda\|_{n,2} := \left(\sum_{k=1}^{n} |\lambda_k|^2 \right)^{\frac{1}{2}}.$$

It is well known that on $E^n := E \times \cdots \times E$ endowed with the canonical linear structure we can define the following *p-norms*:

$$\|x\|_{n,p} := \begin{cases} \max\left\{|x_1|, \ldots, |x_n|\right\} & \text{if } p = \infty, \\ \left(\sum_{k=1}^{n} |x_k|^p \right)^{\frac{1}{p}} & \text{if } p \in [1, \infty). \end{cases}$$

where $x = (x_1, \ldots, x_n) \in E^n$.

Following the paper [5], for a given norm $\|\cdot\|_n$ on $\mathbb{C}^n$, we define the functional $\|\cdot\|_{h,n} : E^n \to [0, \infty)$ by

$$\|x\|_{h,n} := \sup_{\|\lambda\|_n \leq 1} \left\| \sum_{j=1}^{n} \lambda_j x_j \right\|, \tag{1.7}$$

where $x = (x_1, \ldots, x_n) \in E^n$ and $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$.

It is easy to see that [5]:

(i) $\|x\|_{h,n} \geq 0$ for any $x \in E^n$,

(ii) $\|x + y\|_{h,n} \leq \|x\|_{h,n} + \|y\|_{h,n}$ for any $x, y \in E^n$,

(iii) $\|\alpha x\|_{h,n} = |\alpha| \|x\|_{h,n}$ for each $\alpha \in \mathbb{C}$ and $x \in E^n$,

and therefore $\|\cdot\|_{h,n}$ is a *semi-norm* on E^n. This will be called the *hypo-semi-norm* generated by the norm $\|\cdot\|_n$ on E^n.

We observe that $\|x\|_{h,n} = 0$ if and only if $\sum_{j=1}^{n} \lambda_j x_j = 0$ for any $(\lambda_1, \ldots, \lambda_n) \in B(\|\cdot\|_n)$. If there exists $\lambda_1^0, \ldots, \lambda_n^0 \neq 0$ such that $(\lambda_1^0, 0, \ldots, 0)$, $(0, \lambda_2^0, \ldots, 0)$, $\ldots$, $(0, 0, \ldots, \lambda_n^0) \in B(\|\cdot\|_n)$ then the semi-norm generated by $\|\cdot\|_n$ is a *norm* on E^n.

If $p \in [1, \infty]$ and we consider the p-norms $\|\cdot\|_{n,p}$ on $\mathbb{C}^n$, then we can define the following *hypo-q-norms* on E^n:

$$\|x\|_{h,n,q} := \sup_{\|\lambda\|_{n,p} \leq 1} \left\| \sum_{j=1}^{n} \lambda_j x_j \right\|, \tag{1.8}$$

with $q \in [1, \infty]$. If $p = 1$, then $q = \infty$; if $p = \infty$, then $q = 1$; if $p \in (1, \infty)$, then $\frac{1}{p} + \frac{1}{q} = 1$.

For $p = 2$, we have the *hypo-Euclidean norm* on E^n, i.e.,

$$\|x\|_{h,n,e} := \sup_{\|\lambda\|_{n,2} \leq 1} \left\| \sum_{j=1}^{n} \lambda_j x_j \right\|. \tag{1.9}$$

If we consider now $E = B(H)$ endowed with the *operator norm* $\|\cdot\|$, then we can obtain the following *hypo-q-norms* on $B^{(n)}(H)$

$$\|(T_1, \ldots, T_n)\|_{h,n,q} := \sup_{\|\lambda\|_{n,p} \le 1} \left\| \sum_{j=1}^{n} \lambda_j T_j \right\| \qquad \text{where } p, q \in [1, \infty], \tag{1.10}$$

with the convention that if $p = 1$, $q = \infty$, if $p = \infty$, $q = 1$ and if $p > 1$, then $\frac{1}{p} + \frac{1}{q} = 1$.

For $p = 2$ we obtain the *hypo-Euclidian norm* $\|(\cdot, \ldots, \cdot)\|_{n,e}$ defined in (1.2).

If we consider now $E = B(H)$ endowed with the *operator numerical radius* $w(\cdot)$, which is a norm on $B(H)$, then we can obtain the following *hypo-q-numerical radius* of $(T_1, \ldots, T_n) \in B^{(n)}(H)$ defined by

$$w_{h,n,q}(T_1, \ldots, T_n) := \sup_{\|\lambda\|_{n,p} \le 1} w\left(\sum_{j=1}^{n} \lambda_j T_j \right) \qquad \text{with } p, q \in [1, \infty], \tag{1.11}$$

with the convention that if $p = 1$, $q = \infty$, if $p = \infty$, $q = 1$ and if $p > 1$, then $\frac{1}{p} + \frac{1}{q} = 1$.

For $p = 2$ we obtain the *hypo-Euclidian norm*

$$w_{h,n,e}(T_1, \ldots, T_n) := \sup_{\|\lambda\|_{n,2} \le 1} w\left(\sum_{j=1}^{n} \lambda_j T_j \right) \tag{1.12}$$

and will show further that it coincides with the Euclidean operator radius of an n-tuple of operators $(T_1, \ldots, T_n)$ defined in (1.4).

Using the fundamental inequality between the operator norm and numerical radius $w(T) \le \|T\| \le 2w(T)$ for $T \in B(H)$ we have

$$w\left(\sum_{j=1}^{n} \lambda_j T_j \right) \le \left\| \sum_{j=1}^{n} \lambda_j T_j \right\| \le 2w\left(\sum_{j=1}^{n} \lambda_j T_j \right)$$

for any $(T_1, \ldots, T_n) \in B^{(n)}(H)$ and any $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$. By taking the supremum over λ with $\|\lambda\|_{n,p} \le 1$ we get

$$w_{h,n,q}(T_1, \ldots, T_n) \le \|(T_1, \ldots, T_n)\|_{h,n,q} \le 2w_{h,n,q}(T_1, \ldots, T_n) \tag{1.13}$$

with the convention that if $p = 1$, $q = \infty$, if $p = \infty$, $q = 1$ and if $p > 1$, then $\frac{1}{p} + \frac{1}{q} = 1$.

For $p = q = 2$ we recapture the inequality (1.5).

In 2012, [8] (see also [9, 10]) the author have introduced the concept of *s-q-numerical radius* of an n-tuple of operators $(T_1, \ldots, T_n)$ for $q \geq 1$ as

$$w_{s,q}(T_1, \ldots, T_n) := \sup_{\|x\|=1} \left(\sum_{j=1}^{n} | \langle T_j x, x \rangle |^q \right)^{1/q} \tag{1.14}$$

and established various inequalities of interest. For more recent results see also [12, 14].

In the same paper [8] we also introduced the concept of *s-q-norm* of an n-tuple of operators $(T_1, \ldots, T_n)$ for $q \geq 1$ as

$$\|(T_1, \ldots, T_n)\|_{s,q} := \sup_{\|x\|=\|y\|=1} \left(\sum_{j=1}^{n} |\langle T_j x, y \rangle|^q \right)^{1/q}. \tag{1.15}$$

In [8], [9] and [10], by utilising *Kato's inequality* [11]

$$|\langle Tx, y \rangle|^2 \leq \left\langle |T|^{2\alpha} x, x \right\rangle \left\langle |T^*|^{2(1-\alpha)} y, y \right\rangle \tag{1.16}$$

for any $x, y \in H$, $\alpha \in [0, 1]$, where *"absolute value"* operator of A is defined by $\|A\| := \sqrt{A^* A}$, the authors have obtained several inequalities for the s-q-numerical radius and s-q-norm.

In this paper we investigate the connections between these norms and establish some fundamental inequalities of interest in multivariate operator theory.

2. Representation results

We start with the following lemma:

LEMMA 1. *Let $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{C}^n$.*

(i) *If $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\sup_{\|\alpha\|_{n,p} \leq 1} \left| \sum_{j=1}^{n} \alpha_j \beta_j \right| = \|\beta\|_{n,q}. \tag{2.1}$$

In particular,

$$\sup_{\|\alpha\|_{n,2} \leq 1} \left| \sum_{j=1}^{n} \alpha_j \beta_j \right| = \|\beta\|_{n,2}. \tag{2.2}$$

(ii) *We have*

$$\sup_{\|\alpha\|_{n,\infty}\leq 1}\left|\sum_{j=1}^{n}\alpha_j\beta_j\right| = \|\beta\|_{n,1} \text{ and } \sup_{\|\alpha\|_{n,1}\leq 1}\left|\sum_{j=1}^{n}\alpha_j\beta_j\right| = \|\beta\|_{n,\infty}. \quad (2.3)$$

Proof. (i) Using Hölder's discrete inequality for $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$ we have

$$\left|\sum_{j=1}^{n}\alpha_j\beta_j\right| \leq \left(\sum_{j=1}^{n}|\alpha_j|^p\right)^{1/p}\left(\sum_{j=1}^{n}|\beta_j|^q\right)^{1/q},$$

which implies that

$$\sup_{\|\alpha\|_{n,p}\leq 1}\left|\sum_{j=1}^{n}\alpha_j\beta_j\right| \leq \|\beta\|_{n,q} \quad (2.4)$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n)$ are n-tuples of complex numbers.

For $(\beta_1, \ldots, \beta_n) \neq 0$, consider $\alpha = (\alpha_1, \ldots, \alpha_n)$ with

$$\alpha_j := \frac{\overline{\beta_j}|\beta_j|^{q-2}}{\left(\sum_{k=1}^{n}|\beta_k|^q\right)^{1/p}}$$

for those j for which $\beta_j \neq 0$ and $\alpha_j = 0$, for the rest.

We observe that

$$\left|\sum_{j=1}^{n}\alpha_j\beta_j\right| = \left|\sum_{j=1}^{n}\frac{\overline{\beta_j}\,|\beta_j|^{q-2}}{\left(\sum_{k=1}^{n}|\beta_k|^q\right)^{1/p}}\beta_j\right| = \frac{\sum_{j=1}^{n}|\beta_j|^q}{\left(\sum_{k=1}^{n}|\beta_k|^q\right)^{1/p}}$$

$$= \left(\sum_{j=1}^{n}|\beta_j|^q\right)^{1/q} = \|\beta\|_{n,q}$$

and

$$\|\alpha\|_{n,p}^p = \sum_{j=1}^{n}|\alpha_j|^p = \sum_{j=1}^{n}\frac{\left|\overline{\beta_j}\,|\beta_j|^{q-2}\right|^p}{\left(\sum_{k=1}^{n}|\beta_k|^q\right)} = \sum_{j=1}^{n}\frac{\left(|\beta_j|^{q-1}\right)^p}{\left(\sum_{k=1}^{n}|\beta_k|^q\right)}$$

$$= \sum_{j=1}^{n}\frac{|\beta_j|^{qp-p}}{\left(\sum_{k=1}^{n}|\beta_k|^q\right)} = \sum_{j=1}^{n}\frac{|\beta_j|^q}{\left(\sum_{k=1}^{n}|\beta_k|^q\right)} = 1.$$

Therefore, by (2.4) we have the representation (2.1).

(ii) Using the properties of the modulus, we have

$$\left|\sum_{j=1}^{n}\alpha_j\beta_j\right| \le \max_{j\in\{1,\dots,n\}}|\alpha_j|\sum_{j=1}^{n}|\beta_j|,$$

which implies that

$$\sup_{\|\alpha\|_{n,\infty}\le 1}\left|\sum_{j=1}^{n}\alpha_j\beta_j\right| \le \|\beta\|_{n,1}, \tag{2.5}$$

where $\alpha=(\alpha_1,\dots,\alpha_n)$ and $\beta=(\beta_1,\dots,\beta_n)$.

For $(\beta_1,\dots,\beta_n)\ne 0$, consider $\alpha=(\alpha_1,\dots,\alpha_n)$ with $\alpha_j := \frac{\overline{\beta_j}}{|\beta_j|}$ for those j for which $\beta_j\ne 0$ and $\alpha_j=0$, for the rest.

We have

$$\left|\sum_{j=1}^{n}\alpha_j\beta_j\right| = \left|\sum_{j=1}^{n}\frac{\overline{\beta_j}}{|\beta_j|}\beta_j\right| = \sum_{j=1}^{n}|\beta_j| = \|\beta\|_{n,1}$$

and

$$\|\alpha\|_{n,\infty} = \max_{j\in\{1,\dots,n\}}|\alpha_j| = \max_{j\in\{1,\dots,n\}}\left|\frac{\overline{\beta_j}}{|\beta_j|}\right| = 1$$

and by (2.5) we get the first representation in (2.3).

Moreover, we have

$$\left|\sum_{j=1}^{n}\alpha_j\beta_j\right| \le \sum_{j=1}^{n}|\alpha_j|\max_{j\in\{1,\dots,n\}}|\beta_j|,$$

which implies that

$$\sup_{\|\alpha\|_{n,1}\le 1}\left|\sum_{j=1}^{n}\alpha_j\beta_j\right| \le \|\beta\|_{n,\infty}, \tag{2.6}$$

where $\alpha=(\alpha_1,\dots,\alpha_n)$ and $\beta=(\beta_1,\dots,\beta_n)$.

For $(\beta_1,\dots,\beta_n)\ne 0$, let $j_0\in\{1,\dots,n\}$ such that

$$\|\beta\|_{\infty} = \max_{j\in\{1,\dots,n\}}|\beta_j| = |\beta_{j_0}|.$$

Consider $\alpha=(\alpha_1,\dots,\alpha_n)$ with $\alpha_{j_0}=\frac{\overline{\beta_{j_0}}}{|\beta_{j_0}|}$ and $\alpha_j=0$ for $j\ne j_0$. For this choice we get

$$\sum_{j=1}^{n}|\alpha_j| = \frac{|\overline{\beta_{j_0}}|}{|\beta_{j_0}|} = 1 \qquad \text{and} \qquad \left|\sum_{j=1}^{n}\alpha_j\beta_j\right| = \left|\frac{\overline{\beta_{j_0}}}{|\beta_{j_0}|}\beta_{j_0}\right| = |\beta_{j_0}| = \|\beta\|_{n,\infty},$$

therefore by (2.6) we obtain the second representation in (4). ∎

Theorem 2. *Let* $(T_1, \ldots, T_n) \in B^{(n)}(H)$ *and* $x, y \in H$, *then for* $p, q > 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$ *we have*

$$\sup_{\|\alpha\|_{n,p} \le 1} \left| \left\langle \left(\sum_{j=1}^{n} \alpha_j T_j \right) x, y \right\rangle \right| = \left(\sum_{j=1}^{n} |\langle T_j x, y \rangle|^q \right)^{1/q} \tag{2.7}$$

and in particular

$$\sup_{\|\alpha\|_{n,2} \le 1} \left| \left\langle \left(\sum_{j=1}^{n} \alpha_j T_j \right) x, y \right\rangle \right| = \left(\sum_{j=1}^{n} |\langle T_j x, y \rangle|^2 \right)^{1/2}. \tag{2.8}$$

We also have

$$\sup_{\|\alpha\|_{n,\infty} \le 1} \left| \left\langle \left(\sum_{j=1}^{n} \alpha_j T_j \right) x, y \right\rangle \right| = \sum_{j=1}^{n} |\langle T_j x, y \rangle| \tag{2.9}$$

and

$$\sup_{\|\alpha\|_{n,1} \le 1} \left| \left\langle \left(\sum_{j=1}^{n} \alpha_j T_j \right) x, y \right\rangle \right| = \max_{j \in \{1,\ldots,n\}} \{ |\langle T_j x, y \rangle| \}. \tag{2.10}$$

Proof. If we take $\beta = (\langle T_1 x, y \rangle, \ldots, \langle T_n x, y \rangle) \in \mathbb{C}^n$ in (2.1), then we get

$$\begin{aligned} \left(\sum_{j=1}^{n} |\langle T_j x, y \rangle|^q \right)^{1/q} &= \|\beta\|_{n,q} = \sup_{\|\alpha\|_p \le 1} \left| \sum_{j=1}^{n} \alpha_j \beta_j \right| \\ &= \sup_{\|\alpha\|_{n,p} \le 1} \left| \sum_{j=1}^{n} \alpha_j \langle T_j x, y \rangle \right| = \sup_{\|\alpha\|_{n,p} \le 1} \left| \left\langle \sum_{j=1}^{n} \alpha_j T_j x, y \right\rangle \right|, \end{aligned}$$

which proves (2.7).

The equalities (2.9) and (2.10) follow by (2.3). ∎

Corollary 3. *Let* $(T_1, \ldots, T_n) \in B^{(n)}(H)$ *and* $x \in H$, *then for* $p, q > 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$ *we have*

$$\sup_{\|\alpha\|_{n,p} \le 1} \left| \left\langle \left(\sum_{j=1}^{n} \alpha_j T_j \right) x, x \right\rangle \right| = \left(\sum_{j=1}^{n} |\langle T_j x, x \rangle|^q \right)^{1/q} \tag{2.11}$$

and, in particular

$$\sup_{\|\alpha\|_{n,2}\leq 1}\left|\left\langle\left(\sum_{j=1}^{n}\alpha_j T_j\right)x, x\right\rangle\right| = \left(\sum_{j=1}^{n}|\langle T_j x, x\rangle|^2\right)^{1/2}. \tag{2.12}$$

We also have

$$\sup_{\|\alpha\|_{n,\infty}\leq 1}\left|\left\langle\left(\sum_{j=1}^{n}\alpha_j T_j\right)x, x\right\rangle\right| = \sum_{j=1}^{n}|\langle T_j x, x\rangle| \tag{2.13}$$

and

$$\sup_{\|\alpha\|_{n,1}\leq 1}\left|\left\langle\left(\sum_{j=1}^{n}\alpha_j T_j\right)x, x\right\rangle\right| = \max_{j\in\{1,\dots,n\}}\left\{|\langle T_j x, x\rangle|\right\}. \tag{2.14}$$

COROLLARY 4. *Let* $(T_1,\dots,T_n)\in B^{(n)}(H)$ *and* $x\in H$, *then for* $p,q>1$ *and* $\frac{1}{p}+\frac{1}{q}=1$ *we have*

$$\sup_{\|\alpha\|_{n,p}\leq 1}\left\|\sum_{j=1}^{n}\alpha_j T_j x\right\| = \sup_{\|y\|=1}\left(\sum_{j=1}^{n}|\langle T_j x, y\rangle|^q\right)^{1/q} \tag{2.15}$$

and in particular

$$\sup_{\|\alpha\|_{n,2}\leq 1}\left\|\sum_{j=1}^{n}\alpha_j T_j x\right\| = \sup_{\|y\|=1}\left(\sum_{j=1}^{n}|\langle T_j x, y\rangle|^2\right)^{1/2}. \tag{2.16}$$

We also have

$$\sup_{\|\alpha\|_{n,\infty}\leq 1}\left\|\sum_{j=1}^{n}\alpha_j T_j x\right\| = \sup_{\|y\|=1}\sum_{j=1}^{n}|\langle T_j x, y\rangle| \tag{2.17}$$

and

$$\sup_{\|\alpha\|_{n,1}\leq 1}\left\|\sum_{j=1}^{n}\alpha_j T_j x\right\| = \max_{j\in\{1,\dots,n\}}\left\{\|T_j x\|\right\}. \tag{2.18}$$

Proof. By the properties of inner product, we have for any $u\in H$, $u\neq 0$ that

$$\|u\| = \sup_{\|y\|=1}|\langle u, y\rangle|.$$

Let $x \in H$, then by taking the supremum over $\|y\| = 1$ in (2.7) we get for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ that

$$\begin{aligned}
\sup_{\|y\|=1} \left(\sum_{j=1}^{n} |\langle T_j x, y \rangle|^q \right)^{1/q} &= \sup_{\|y\|=1} \left(\sup_{\|\alpha\|_{n,p} \le 1} \left| \left\langle \left(\sum_{j=1}^{n} \alpha_j T_j \right) x, y \right\rangle \right| \right) \\
&= \sup_{\|\alpha\|_{n,p} \le 1} \left(\sup_{\|y\|=1} \left| \left\langle \left(\sum_{j=1}^{n} \alpha_j T_j \right) x, y \right\rangle \right| \right) \\
&= \sup_{\|\alpha\|_{n,p} \le 1} \left\| \left(\sum_{j=1}^{n} \alpha_j T_j \right) x \right\|,
\end{aligned}$$

which proves the equality (2.15).

The other equalities can be proved in a similar way by using Theorem 2, however the details are omitted. ∎

We can state and prove our main result.

THEOREM 5. *Let* $(T_1, \ldots, T_n) \in B^{(n)}(H)$.

(i) *For* $q \ge 1$ *we have the representation for the hypo-q-norm*

$$\begin{aligned}
\|(T_1, \ldots, T_n)\|_{h,n,q} &= \sup_{\|x\|=\|y\|=1} \left(\sum_{j=1}^{n} |\langle T_j x, y \rangle|^q \right)^{1/q} \\
&= \|(T_1, \ldots, T_n)\|_{s,q}
\end{aligned} \tag{2.19}$$

and in particular

$$\|(T_1, \ldots, T_n)\|_{n,e} = \sup_{\|x\|=\|y\|=1} \left(\sum_{j=1}^{n} |\langle T_j x, y \rangle|^2 \right)^{1/2}. \tag{2.20}$$

We also have

$$\|(T_1, \ldots, T_n)\|_{h,n,\infty} = \max_{j \in \{1,\ldots,n\}} \left\{ \|T_j\| \right\}. \tag{2.21}$$

(ii) *For $q \geq 1$ we have the representation for the hypo--numerical radius*

$$w_{h,n,q}(T_1,\dots,T_n) = \sup_{\|x\|=1}\left(\sum_{j=1}^{n}|\langle T_j x, x\rangle|^q\right)^{1/q} = w_{s,q}(T_1,\dots,T_n) \tag{2.22}$$

and in particular

$$w_{n,e}\left(T_1,\dots,T_n\right) := \sup_{\|x\|=1}\left(\sum_{j=1}^{n}|\langle T_j x, x\rangle|^2\right)^{1/2}. \tag{2.23}$$

We also have

$$w_{h,n,\infty}\left(T_1,\dots,T_n\right) = \max_{j\in\{1,\dots,n\}}\{w(T_j)\}. \tag{2.24}$$

Proof. (i) By using the equality (2.15) we have for $(T_1,\dots,T_n) \in B^{(n)}(H)$ that

$$\begin{aligned}\sup_{\|x\|=\|y\|=1}\left(\sum_{j=1}^{n}|\langle T_j x, y\rangle|^q\right)^{1/q} &= \sup_{\|x\|=1}\left(\sup_{\|y\|=1}\left(\sum_{j=1}^{n}|\langle T_j x, y\rangle|^q\right)^{1/q}\right)\\ &= \sup_{\|x\|=1}\left(\sup_{\|\alpha\|_{n,p}\leq 1}\left\|\sum_{j=1}^{n}\alpha_j T_j x\right\|\right)\\ &= \sup_{\|\alpha\|_{n,p}\leq 1}\left(\sup_{\|x\|=1}\left\|\sum_{j=1}^{n}\alpha_j T_j x\right\|\right)\\ &= \sup_{\|\alpha\|_{n,p}\leq 1}\left\|\sum_{j=1}^{n}\alpha_j T_j\right\|\\ &= \|(T_1,\dots,T_n)\|_{h,n,q},\end{aligned}$$

which proves (2.19). The rest is obvious.

(ii) By using the equality (2.11) we have for $(T_1,\ldots,T_n)\in B^{(n)}(H)$ that

$$\begin{aligned}\sup_{\|x\|=1}\left(\sum_{j=1}^{n}|\langle T_j x,x\rangle|^q\right)^{1/q} &= \sup_{\|x\|=1}\left(\sup_{\|\alpha\|_{n,p}\le 1}\left|\left\langle\left(\sum_{j=1}^{n}\alpha_j T_j\right)x,x\right\rangle\right|\right)\\ &= \sup_{\|\alpha\|_{n,p}\le 1}\left(\sup_{\|x\|=1}\left|\left\langle\left(\sum_{j=1}^{n}\alpha_j T_j\right)x,x\right\rangle\right|\right)\\ &= \sup_{\|\alpha\|_{n,p}\le 1} w\left(\sum_{j=1}^{n}\alpha_j T_j\right) = w_{h,n,q}(T_1,\ldots,T_n),\end{aligned}$$

which proves (2.22). The rest is obvious. ∎

Remark 6. The case $q=2$ was obtained in a different manner in [5] by utilising the rotation-invariant normalised positive Borel measure on the unit sphere.

We can consider on $B^{(n)}(H)$ the following usual operator and numerical radius q-norms, for $q\ge 1$

$$\|(T_1,\ldots,T_n)\|_{n,q} := \left(\sum_{j=1}^{n}\|T_j\|^q\right)^{1/q},$$

$$w_{n,q}(T_1,\ldots,T_n) := \left(\sum_{j=1}^{n}w^q(T_j)\right)^{1/q},$$

where $(T_1,\ldots,T_n)\in B^{(n)}(H)$. For $q=\infty$ we put

$$\|(T_1,\ldots,T_n)\|_{n,\infty} := \max_{j\in\{1,\ldots,n\}}\left\{\|T_j\|\right\},$$

$$w_{n,\infty}(T_1,\ldots,T_n) := \max_{j\in\{1,\ldots,n\}}\left\{w(T_j)\right\}.$$

Corollary 7. *With the assumptions of Theorem* 5 *we have for* $q\ge 1$ *that*

$$\frac{1}{n^{1/q}}\|(T_1,\ldots,T_n)\|_{n,q} \le \|(T_1,\ldots,T_n)\|_{h,n,q} \le \|(T_1,\ldots,T_n)\|_{n,q} \tag{2.25}$$

and

$$\frac{1}{n^{1/q}} w_{n,q}(T_1,\ldots,T_n) \le w_{h,n,q}(T_1,\ldots,T_n) \le w_{n,q}(T_1,\ldots,T_n) \tag{2.26}$$

for any $(T_1,\ldots,T_n) \in B^{(n)}(H)$.

In particular, we have [5]

$$\frac{1}{\sqrt{n}} \|(T_1,\ldots,T_n)\|_{n,2} \le \|(T_1,\ldots,T_n)\|_{h,n,e} \le \|(T_1,\ldots,T_n)\|_{n,2} \tag{2.27}$$

and

$$\frac{1}{\sqrt{n}} w_{n,2}(T_1,\ldots,T_n) \le w_{h,n,e}(T_1,\ldots,T_n) \le w_{n,2}(T_1,\ldots,T_n) \tag{2.28}$$

for any $(T_1,\ldots,T_n) \in B^{(n)}(H)$.

Proof. Let $(T_1,\ldots,T_n) \in B^{(n)}(H)$ and $x, y \in H$ with $\|x\| = \|y\| = 1$. Then by Schwarz's inequality we have

$$\left(\sum_{j=1}^{n} |\langle T_j x, y\rangle|^q\right)^{1/q} \le \left(\sum_{j=1}^{n} \|T_j x\|^q \|y\|^q\right)^{1/q} = \left(\sum_{j=1}^{n} \|T_j x\|^q\right)^{1/q}.$$

By the operator norm inequality we also have

$$\left(\sum_{j=1}^{n} \|T_j x\|^q\right)^{1/q} \le \left(\sum_{j=1}^{n} \|T_j\|^q \|x\|^q\right)^{1/q} = \|(T_1,\ldots,T_n)\|_{n,q}.$$

Therefore

$$\left(\sum_{j=1}^{n} |\langle T_j x, y\rangle|^q\right)^{1/q} \le \|(T_1,\ldots,T_n)\|_{n,q}$$

and by taking the supremum over $\|x\| = \|y\| = 1$ we get the second inequality in (2.25).

By the properties of complex numbers, we have

$$\max_{j\in\{1,\ldots,n\}} \left\{|\langle T_j x, y\rangle|\right\} \le \left(\sum_{j=1}^{n} |\langle T_j x, y\rangle|^q\right)^{1/q}$$

$x, y \in H$ with $\|x\| = \|y\| = 1$.

By taking the supremum over $\|x\| = \|y\| = 1$ we get

$$\sup_{\|x\|=\|y\|=1} \left(\max_{j\in\{1,\dots,n\}} \left\{ |\langle T_j x, y\rangle| \right\} \right) \le \|(T_1, \dots, T_n)\|_{h,n,q} \tag{2.29}$$

and since

$$\begin{aligned} \sup_{\|x\|=\|y\|=1} \left(\max_{j\in\{1,\dots,n\}} \left\{ |\langle T_j x, y\rangle| \right\} \right) &= \max_{j\in\{1,\dots,n\}} \left\{ \sup_{\|x\|=\|y\|=1} |\langle T_j x, y\rangle| \right\} \\ &= \max_{j\in\{1,\dots,n\}} \left\{ \|T_j\| \right\} = \|(T_1, \dots, T_n t)\|_{n,\infty}, \end{aligned}$$

then by (2.29) we get

$$\|(T_1, \dots, T_n)\|_{n,\infty} \le \|(T_1, \dots, T_n)\|_{h,n,q} \tag{2.30}$$

for any $(T_1, \dots, T_n) \in B^{(n)}(H)$.

Since

$$\begin{aligned} \|(T_1, \dots, T_n)\|_{n,q} := \left(\sum_{j=1}^{n} \|T_j\|^q \right)^{1/q} &\le \left(n \|(T_1, \dots, T_n)\|_{n,\infty}^q \right)^{1/q} \\ &= n^{1/q} \|(T_1, \dots, T_n)\|_{n,\infty}, \end{aligned} \tag{2.31}$$

then by (2.30) and (2.31) we get

$$\frac{1}{n^{1/q}} \|(T_1, \dots, T_n)\|_{n,q} \le \|(T_1, \dots, T_n)\|_{h,n,q}$$

for any $(T_1, \dots, T_n) \in B^{(n)}(H)$.

The inequality (2.26) follows in a similar way and we omit the details. ∎

Corollary 8. *With the assumptions of Theorem 5 we have for $r \ge q \ge 1$ that*

$$\|(T_1, \dots, T_n)\|_{h,n,r} \le \|(T_1, \dots, T_n)\|_{h,n,q} \le n^{\frac{r-q}{rq}} \|(T_1, \dots, T_n)\|_{h,n,r} \tag{2.32}$$

and [14]

$$w_{h,n,r}(T_1, \dots, T_n) \le w_{h,n,q}(T_1, \dots, T_n) \le n^{\frac{r-q}{rq}} w_{h,n,r}(T_1, \dots, T_n) \tag{2.33}$$

for any $(T_1, \dots, T_n) \in B^{(n)}(H)$.

Proof. We use the following elementary inequalities for the nonnegative numbers a_j, $j = 1, \dots, n$ and $r \geq q > 0$ (see for instance [14])

$$\left(\sum_{j=1}^{n} a_j^r \right)^{1/r} \leq \left(\sum_{j=1}^{n} a_j^q \right)^{1/q} \leq n^{\frac{r-q}{rq}} \left(\sum_{j=1}^{n} a_j^r \right)^{1/r}. \tag{2.34}$$

Let $(T_1, \dots, T_n) \in B^{(n)}(H)$ and $x, y \in H$ with $\|x\| = \|y\| = 1$. Then by (2.34) we get

$$\left(\sum_{j=1}^{n} |\langle T_j x, y \rangle|^r \right)^{1/r} \leq \left(\sum_{j=1}^{n} |\langle T_j x, y \rangle|^q \right)^{1/q} \leq n^{\frac{r-q}{rq}} \left(\sum_{j=1}^{n} |\langle T_j x, y \rangle|^r \right)^{1/r}.$$

By taking the supremum over $\|x\| = \|y\| = 1$ we get (2.32).

The inequality (2.33) follows in a similar way and we omit the details. ∎

Remark 9. For $q \geq 2$ we have by (2.32) and (2.33)

$$\|(T_1, \dots, T_n)\|_{h,n,q} \leq \|(T_1, \dots, T_n)\|_{h,n,e} \leq n^{\frac{q-2}{2q}} \|(T_1, \dots, T_n)\|_{h,n,q} \tag{2.35}$$

and

$$w_{h,n,q}(T_1, \dots, T_n) \leq w_{h,n,e}(T_1, \dots, T_n) \leq n^{\frac{q-2}{2q}} w_{h,n,q}(T_1, \dots, T_n) \tag{2.36}$$

and for $1 \leq q \leq 2$ we have

$$\|(T_1, \dots, T_n)\|_{h,n,e} \leq \|(T_1, \dots, T_n)\|_{h,n,q} \leq n^{\frac{2-q}{2q}} \|(T_1, \dots, T_n)\|_{h,n,e} \tag{2.37}$$

and

$$w_{h,n,e}(T_1, \dots, T_n) \leq w_{h,n,e}(T_1, \dots, T_n) \leq n^{\frac{2-q}{2q}} w_{h,n,e}(T_1, \dots, T_n) \tag{2.38}$$

for any $(T_1, \dots, T_n) \in B^{(n)}(H)$.

Also, if we take $q = 1$ and $r \geq 1$ in (2.32) and (2.33), then we get

$$\|(T_1, \dots, T_n)\|_{h,n,r} \leq \|(T_1, \dots, T_n)\|_{h,n,1} \leq n^{\frac{r-1}{r}} \|(T_1, \dots, T_n)\|_{h,n,r} \tag{2.39}$$

and

$$w_{h,n,r}(T_1, \dots, T_n) \leq w_{h,n,1}(T_1, \dots, T_n) \leq n^{\frac{r-1}{r}} w_{h,n,r}(T_1, \dots, T_n) \tag{2.40}$$

for any $(T_1, \dots, T_n) \in B^{(n)}(H)$.

In particular, for $r = 2$ we get

$$\|(T_1,\dots,T_n)\|_{h,n,e} \le \|(T_1,\dots,T_n)\|_{h,n,1} \le \sqrt{n}\|(T_1,\dots,T_n)\|_{h,n,e} \tag{2.41}$$

and

$$w_{n,e}(T_1,\dots,T_n) \le w_{h,n,1}(T_1,\dots,T_n) \le \sqrt{n} w_{n,e}(T_1,\dots,T_n) \tag{2.42}$$

for any $(T_1,\dots,T_n) \in B^{(n)}(H)$.

We have:

PROPOSITION 10. *For any $(T_1,\dots,T_n) \in B^{(n)}(H)$ and $p, q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$, then we have*

$$\|(T_1,\dots,T_n)\|_{h,n,q} \ge \frac{1}{n^{1/p}}\left\|\sum_{j=1}^n T_j\right\| \tag{2.43}$$

and

$$w_{h,n,q}(T_1,\dots,T_n) \ge \frac{1}{n^{1/p}} w\left(\sum_{j=1}^n T_j\right). \tag{2.44}$$

Proof. Let $\lambda_j = \frac{1}{n^{1/p}}$ for $j \in \{1,\dots,n\}$, then $\sum_{j=1}^n |\lambda_j|^p = 1$. Therefore by (1.8) we get

$$\|(T_1,\dots,T_n)\|_{h,n,q} = \sup_{\|\lambda\|_{n,p}\le 1}\left\|\sum_{j=1}^n \lambda_j T_j\right\| \ge \left\|\sum_{j=1}^n \frac{1}{n^{1/p}} T_j\right\| = \frac{1}{n^{1/p}}\left\|\sum_{j=1}^n T_j\right\|.$$

The inequality (2.44) follows in a similar way. ∎

We can also introduce the following norms for $(T_1,\dots,T_n) \in B^{(n)}(H)$,

$$\|(T_1,\dots,T_n)\|_{s,n,p} := \sup_{\|x\|=1}\left(\sum_{j=1}^n \|T_j x\|^p\right)^{1/p}$$

where $p \ge 1$ and

$$\|(T_1,\dots,T_n)\|_{s,n,\infty} := \sup_{\|x\|=1}\left(\max_{j\in\{1,\dots,n\}} \|T_j x\|\right) = \max_{j\in\{1,\dots,n\}}\{\|T_j\|\}.$$

The triangle inequality $\|\cdot\|_{s,n,q}$ follows by Minkowski inequality, while the other properties of the norm are obvious.

PROPOSITION 11. *Let* $(T_1, \ldots, T_n) \in B^{(n)}(H)$.

(i) *We have for* $p \geq 1$, *that*

$$\begin{aligned} \|(T_1, \ldots, T_n)\|_{h,n,p} &\leq \|(T_1, \ldots, T_n)\|_{s,n,p} \\ &\leq \|(T_1, \ldots, T_n)\|_{n,p}, \end{aligned} \tag{2.45}$$

(ii) *For* $p \geq 2$ *we also have*

$$\|(T_1, \ldots, T_n)\|_{s,n,p} = \left[w_{h,n,p/2}\left(|T_1|^2, \ldots, |T_n|^2\right)\right]^{1/2}, \tag{2.46}$$

where the absolute value $|T|$ *is defined by* $|T| := (T^*T)^{1/2}$.

Proof. (i) We have for $p \geq 2$ and $x, y \in H$ with $\|x\| = \|y\| = 1$, that

$$\begin{aligned} |\langle T_j x, y\rangle|^p &\leq \|T_j x\|^p \|y\|^p \\ &= \|T_j x\|^p \leq \|T_j\|^p \|x\|^p = \|T_j\|^p \end{aligned}$$

for $j \in \{1, \ldots, n\}$.

This implies

$$\sum_{j=1}^n |\langle T_j x, y\rangle|^p \leq \sum_{j=1}^n \|T_j x\|^p \leq \sum_{j=1}^n \|T_j\|^p,$$

namely

$$\left(\sum_{j=1}^n |\langle T_j x, y\rangle|^p\right)^{1/p} \leq \left(\sum_{j=1}^n \|T_j x\|^p\right)^{1/p} \leq \left(\sum_{j=1}^n \|T_j\|^p\right)^{1/p}, \tag{2.47}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$.

Taking the supremum over $\|x\| = \|y\| = 1$ in (2.47), we get the desired result (2.45).

(ii) We have

$$\begin{aligned}
&\|(T_1,\dots,T_n)\|_{s,n,p}\\
&=\sup_{\|x\|=1}\left(\sum_{j=1}^{n}\|T_jx\|^p\right)^{1/p}=\sup_{\|x\|=1}\left(\sum_{j=1}^{n}\left(\|T_jx\|^2\right)^{p/2}\right)^{1/p}\\
&=\sup_{\|x\|=1}\left(\sum_{j=1}^{n}\langle T_jx,T_jx\rangle^{p/2}\right)^{1/p}=\sup_{\|x\|=1}\left(\sum_{j=1}^{n}\left\langle T_j^*T_jx,x\right\rangle^{p/2}\right)^{1/p}\\
&=\sup_{\|x\|=1}\left(\sum_{j=1}^{n}\left\langle |T_j|^2x,x\right\rangle^{p/2}\right)^{1/p}=\left[\sup_{\|x\|=1}\left(\sum_{j=1}^{n}\left\langle |T_j|^2x,x\right\rangle^{p/2}\right)^{1/(p/2)}\right]^{1/2}\\
&=\left[w_{h,n,p/2}\left(|T_1|^2,\dots,|T_n|^2\right)\right]^{1/2},
\end{aligned}$$

which proves the equality (2.46). ∎

3. Some reverse inequalities

Recall the following reverse of Cauchy-Buniakowski-Schwarz inequality [2] (see also [3, Theorem 5.14]):

Lemma 12. *Let $a,A\in\mathbb{R}$ and $\mathbf{z}=(z_1,\dots,z_n)$, $\mathbf{y}=(y_1,\dots,y_n)$ be two sequences of real numbers with the property that:*

$$ay_j\le z_j\le Ay_j \qquad \text{for each } j\in\{1,\dots,n\}. \tag{3.1}$$

Then for any $\mathbf{w}=(w_1,\dots,w_n)$ a sequence of positive real numbers, one has the inequality

$$0\le\sum_{j=1}^{n}w_jz_j^2\sum_{j=1}^{n}w_jy_j^2-\left(\sum_{j=1}^{n}w_jz_jy_j\right)^2\le\frac{1}{4}(A-a)^2\left(\sum_{j=1}^{n}w_jy_j^2\right)^2. \tag{3.2}$$

The constant $\frac{1}{4}$ is sharp in (3.2).

O. Shisha and B. Mond obtained in 1967 (see [15]) the following counterparts of (*CBS*)–inequality (see also [3, Theorem 5.20 & 5.21]):

LEMMA 13. *Assume that* $\mathbf{a} = (a_1, \ldots, a_n)$ *and* $\mathbf{b} = (b_1, \ldots, b_n)$ *are such that there exists* a, A, b, B *with the property that:*

$$0 \le a \le a_j \le A \quad \text{and} \quad 0 < b \le b_j \le B \quad \text{for any} \quad j \in \{1, \ldots, n\}, \tag{3.3}$$

then we have the inequality

$$\sum_{j=1}^{n} a_j^2 \sum_{j=1}^{n} b_j^2 - \left(\sum_{j=1}^{n} a_j b_j \right)^2 \le \left(\sqrt{\frac{A}{b}} - \sqrt{\frac{a}{B}} \right)^2 \sum_{j=1}^{n} a_j b_j \sum_{j=1}^{n} b_j^2. \tag{3.4}$$

and

LEMMA 14. *Assume that* $\mathbf{a}$, $\mathbf{b}$ *are nonnegative sequences and there exists* γ, Γ *with the property that*

$$0 \le \gamma \le \frac{a_j}{b_j} \le \Gamma < \infty \qquad \text{for any } j \in \{1, \ldots, n\}. \tag{3.5}$$

Then we have the inequality

$$0 \le \left(\sum_{j=1}^{n} a_j^2 \sum_{j=1}^{n} b_j^2 \right)^{\frac{1}{2}} - \sum_{j=1}^{n} a_j b_j \le \frac{(\Gamma - \gamma)^2}{4(\gamma + \Gamma)} \sum_{j=1}^{n} b_j^2. \tag{3.6}$$

We have:

THEOREM 15. *Let* $(T_1, \ldots, T_n) \in B^{(n)}(H)$.

(i) *We have*

$$\begin{aligned} 0 &\le \|(T_1, \ldots, T_n)\|_{h,n,e}^2 - \frac{1}{n} \|(T_1, \ldots, T_n)\|_{h,n,1}^2 \\ &\le \frac{1}{4} n \|(T_1, \ldots, T_n)\|_{n,\infty}^2 \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} 0 &\le w_{n,e}^2(T_1, \ldots, T_n) - \frac{1}{n} w_{h,n,1}^2(T_1, \ldots, T_n) \\ &\le \frac{1}{4} n \|(T_1, \ldots, T_n)\|_{n,\infty}^2. \end{aligned} \tag{3.8}$$

(ii) *We have*

$$
\begin{aligned}
0 &\le \|(T_1,\dots,T_n)\|_{h,n,e}^2 - \frac{1}{n}\|(T_1,\dots,T_n)\|_{h,n,1}^2 \\
&\le \|(T_1,\dots,T_n)\|_{n,\infty}\|(T_1,\dots,T_n)\|_{h,n,1}
\end{aligned}
\tag{3.9}
$$

and

$$
\begin{aligned}
0 &\le w_{n,e}^2(T_1,\dots,T_n) - \frac{1}{n}w_{h,n,1}^2(T_1,\dots,T_n) \\
&\le \|(T_1,\dots,T_n)\|_{n,\infty}w_{h,n,1}(T_1,\dots,T_n).
\end{aligned}
\tag{3.10}
$$

(iii) *We have*

$$
\begin{aligned}
0 &\le \|(T_1,\dots,T_n)\|_{h,n,e} - \frac{1}{\sqrt{n}}\|(T_1,\dots,T_n)\|_{h,n,1} \\
&\le \frac{1}{4}\sqrt{n}\|(T_1,\dots,T_n)\|_{n,\infty}
\end{aligned}
\tag{3.11}
$$

and

$$
\begin{aligned}
0 &\le w_{n,e}(T_1,\dots,T_n) - \frac{1}{\sqrt{n}}w_{h,n,1}(T_1,\dots,T_n) \\
&\le \frac{1}{4}\sqrt{n}\|(T_1,\dots,T_n)\|_{n,\infty}.
\end{aligned}
\tag{3.12}
$$

Proof. (i) Let $(T_1,\dots,T_n) \in B^{(n)}(H)$ and put

$$R = \max_{j\in\{1,\dots,n\}} \left\{\|T_j\|\right\} = \|(T_1,\dots,T_n)\|_{n,\infty}.$$

If $x, y \in H$, with $\|x\| = \|y\| = 1$ then $|\langle T_j x, y\rangle| \le \|T_j x\| \le \|T_j\| \le R$ for any $j \in \{1,\dots,n\}$.

If we write the inequality (3.2) for $z_j = |\langle T_j x, y\rangle|$, $w_j = y_j = 1$, $A = R$ and $a = 0$, we get

$$0 \le n\sum_{j=1}^{n} |\langle T_j x, y\rangle|^2 - \left(\sum_{j=1}^{n} |\langle T_j x, y\rangle|\right)^2 \le \frac{1}{4}n^2R^2$$

for any $x, y \in H$, with $\|x\| = \|y\| = 1$.

This implies that

$$\sum_{j=1}^{n}|\langle T_jx,y\rangle|^2 \le \frac{1}{n}\left(\sum_{j=1}^{n}|\langle T_jx,y\rangle|\right)^2 + \frac{1}{4}nR^2 \tag{3.13}$$

for any $x, y \in H$, with $\|x\| = \|y\| = 1$ and, in particular

$$\sum_{j=1}^{n}|\langle T_jx,x\rangle|^2 \le \frac{1}{n}\left(\sum_{j=1}^{n}|\langle T_jx,x\rangle|\right)^2 + \frac{1}{4}nR^2 \tag{3.14}$$

for any $x \in H$, with $\|x\| = 1$.

Taking the supremum over $\|x\| = \|y\| = 1$ in (3.13) and $\|x\| = 1$ in (3.14), then we get (3.7) and (3.8).

(ii) Let $(T_1, \ldots, T_n) \in B^{(n)}(H)$. If we write the inequality (3.4) for $a_j = |\langle T_jx,y\rangle|$, $b_j = 1$, $b = B = 1$, $a = 0$ and $A = R$, then we get

$$0 \le n\sum_{j=1}^{n}|\langle T_jx,y\rangle|^2 - \left(\sum_{j=1}^{n}|\langle T_jx,y\rangle|\right)^2 \le nR\sum_{j=1}^{n}|\langle T_jx,y\rangle|,$$

for any $x, y \in H$, with $\|x\| = \|y\| = 1$.

This implies that

$$\sum_{j=1}^{n}|\langle T_jx,y\rangle|^2 \le \frac{1}{n}\left(\sum_{j=1}^{n}|\langle T_jx,y\rangle|\right)^2 + R\sum_{j=1}^{n}|\langle T_jx,y\rangle|, \tag{3.15}$$

for any $x, y \in H$, with $\|x\| = \|y\| = 1$ and, in particular

$$\sum_{j=1}^{n}|\langle T_jx,x\rangle|^2 \le \frac{1}{n}\left(\sum_{j=1}^{n}|\langle T_jx,x\rangle|\right)^2 + R\sum_{j=1}^{n}|\langle T_jx,x\rangle|, \tag{3.16}$$

for any $x \in H$ with $\|x\| = 1$.

Taking the supremum over $\|x\| = \|y\| = 1$ in (3.15) and $\|x\| = 1$ in (3.16), then we get (3.9) and (3.10).

(iii) If we write the inequality (3.6) for $a_j = |\langle T_jx,y\rangle|$, $b_j = 1$, $b = B = 1$, $\gamma = 0$ and $\Gamma = R$ we have

$$0 \le \left(n\sum_{j=1}^{n}|\langle T_jx,y\rangle|^2\right)^{\frac{1}{2}} - \sum_{j=1}^{n}|\langle T_jx,y\rangle| \le \frac{1}{4}nR,$$

for any $x, y \in H$, with $\|x\| = \|y\| = 1$.

This implies that

$$\left(\sum_{j=1}^{n} |\langle T_j x, y \rangle|^2 \right)^{\frac{1}{2}} \leq \frac{1}{\sqrt{n}} \sum_{j=1}^{n} |\langle T_j x, y \rangle| + \frac{1}{4} \sqrt{n} R, \tag{3.17}$$

for any $x, y \in H$, with $\|x\| = \|y\| = 1$ and, in particular

$$\left(\sum_{j=1}^{n} |\langle T_j x, x \rangle|^2 \right)^{\frac{1}{2}} \leq \frac{1}{\sqrt{n}} \sum_{j=1}^{n} |\langle T_j x, x \rangle| + \frac{1}{4} \sqrt{n} R, \tag{3.18}$$

for any $x \in H$ with $\|x\| = 1$.

Taking the supremum over $\|x\| = \|y\| = 1$ in (3.17) and $\|x\| = 1$ in (3.18), then we get (3.11) and (3.12). ∎

Before we proceed with establishing some reverse inequalities for the hypo-Euclidean numerical radius, we recall some reverse results of the Cauchy-Bunyakovsky-Schwarz inequality for complex numbers as follows:

If $\gamma, \Gamma \in \mathbb{C}$ and $\alpha_j \in \mathbb{C}$, $j \in \{1, \dots, n\}$ with the property that

$$\begin{aligned} 0 &\leq \operatorname{Re}\left[(\Gamma - \alpha_j) \left(\overline{\alpha_j} - \overline{\gamma} \right) \right] \\ &= (\operatorname{Re}\Gamma - \operatorname{Re}\alpha_j)(\operatorname{Re}\alpha_j - \operatorname{Re}\gamma) + (\operatorname{Im}\Gamma - \operatorname{Im}\alpha_j)(\operatorname{Im}\alpha_j - \operatorname{Im}\gamma) \end{aligned} \tag{3.19}$$

or, equivalently,

$$\left| \alpha_j - \frac{\gamma + \Gamma}{2} \right| \leq \frac{1}{2} |\Gamma - \gamma| \tag{3.20}$$

for each $j \in \{1, \dots, n\}$, then (see for instance [4, p. 9])

$$n \sum_{j=1}^{n} |\alpha_j|^2 - \left| \sum_{j=1}^{n} \alpha_j \right|^2 \leq \frac{1}{4} n^2 |\Gamma - \gamma|^2 . \tag{3.21}$$

In addition, if $\operatorname{Re}\left(\Gamma\bar{\gamma}\right) > 0$, then (see for example [4, p. 26]):

$$\begin{aligned} n \sum_{j=1}^{n} |\alpha_j|^2 &\leq \frac{1}{4} \frac{\left\{ \operatorname{Re}\left[\left(\overline{\Gamma} + \overline{\gamma} \right) \sum_{j=1}^{n} \alpha_j \right] \right\}^2}{\operatorname{Re}} (\Gamma\overline{\gamma}) \\ &\leq \frac{1}{4} \frac{|\Gamma + \gamma|^2}{\operatorname{Re}} (\Gamma\overline{\gamma}) \left| \sum_{j=1}^{n} \alpha_j \right|^2 . \end{aligned} \tag{3.22}$$

Also, if $\Gamma \neq -\gamma$, then (see for instance [4, p. 32]):

$$\left(n\sum_{j=1}^{n}|\alpha_j|^2\right)^{\frac{1}{2}} - \left|\sum_{j=1}^{n}\alpha_j\right| \leq \frac{1}{4}n\frac{|\Gamma-\gamma|^2}{|\Gamma+\gamma|}. \tag{3.23}$$

Finally, from [7] we can also state that

$$n\sum_{j=1}^{n}|\alpha_j|^2 - \left|\sum_{j=1}^{n}\alpha_j\right|^2 \leq n\Big[|\Gamma+\gamma| - 2\sqrt{\operatorname{Re}(\Gamma\overline{\gamma})}\Big]\left|\sum_{j=1}^{n}\alpha_j\right|, \tag{3.24}$$

provided $\operatorname{Re}(\Gamma\overline{\gamma}) > 0$.

We notice that a simple sufficient condition for (3.19) to hold is that

$$\operatorname{Re}\Gamma \geq \operatorname{Re}\alpha_j \geq \operatorname{Re}\gamma \qquad \text{and} \qquad \operatorname{Im}\Gamma \geq \operatorname{Im}\alpha_j \geq \operatorname{Im}\gamma \tag{3.25}$$

for each $j \in \{1,\dots,n\}$.

THEOREM 16. *Let* $(T_1,\dots,T_n) \in B^{(n)}(H)$ *and* γ, $\Gamma \in \mathbb{C}$ *with* $\Gamma \neq \gamma$. *Assume that*

$$w\left(T_j - \frac{\gamma+\Gamma}{2}I\right) \leq \frac{1}{2}|\Gamma-\gamma| \qquad \textit{for any } j \in \{1,\dots,n\}. \tag{3.26}$$

(i) *We have*

$$w_{h,n,e}^2(T_1,\dots,T_n) \leq \frac{1}{n}w^2\left(\sum_{j=1}^{n}T_j\right) + \frac{1}{4}n|\Gamma-\gamma|^2. \tag{3.27}$$

(ii) *If* $\operatorname{Re}(\Gamma\overline{\gamma}) > 0$, *then*

$$w_{h,n,e}(T_1,\dots,T_n) \leq \frac{1}{2\sqrt{n}}\frac{|\Gamma+\gamma|}{\sqrt{(\Gamma\overline{\gamma})}}w\left(\sum_{j=1}^{n}T_j\right) \tag{3.28}$$

and

$$w_{h,n,e}^2(T_1,\dots,T_n) \leq \left[\frac{1}{n}w\left(\sum_{j=1}^{n}T_j\right) + \Big[|\Gamma+\gamma| - 2\sqrt{(\Gamma\overline{\gamma})}\Big]\right]\cdot w\left(\sum_{j=1}^{n}T_j\right). \tag{3.29}$$

(iii) *If* $\Gamma \neq -\gamma$, *then*

$$w_{h,n,e}(T_1, \dots, T_n) \leq \frac{1}{\sqrt{n}} \left(w\left(\sum_{j=1}^{n} T_j \right) + \frac{1}{4} \frac{|\Gamma - \gamma|^2}{|\Gamma + \gamma|} \right). \tag{3.30}$$

Proof. Let $x \in H$ with $\|x\| = 1$ and $(T_1, \dots, T_n) \in B^{(n)}(H)$ with the property (3.26). By taking $\alpha_j = \langle T_j x, x \rangle$ we have

$$\begin{aligned} \left| \alpha_j - \frac{\gamma + \Gamma}{2} \right| &= \left| \langle T_j x, x \rangle - \frac{\gamma + \Gamma}{2} \langle x, x \rangle \right| = \left| \left\langle \left(T_j - \frac{\gamma + \Gamma}{2} I \right) x, x \right\rangle \right| \\ &\leq \sup_{\|x\|=1} \left| \left\langle \left(T_j - \frac{\gamma + \Gamma}{2} I \right) x, x \right\rangle \right| \\ &= w\left(T_j - \frac{\gamma + \Gamma}{2} \right) \leq \frac{1}{2} |\Gamma - \gamma| \end{aligned}$$

for any $j \in \{1, \dots, n\}$.

(i) By using the inequality (3.21), we have

$$\begin{aligned} \sum_{j=1}^{n} |\langle T_j x, x \rangle|^2 &\leq \frac{1}{n} \left| \sum_{j=1}^{n} \langle T_j x, x \rangle \right|^2 + \frac{1}{4} n |\Gamma - \gamma|^2 \\ &= \frac{1}{n} \left| \left\langle \sum_{j=1}^{n} T_j x, x \right\rangle \right|^2 + \frac{1}{4} n |\Gamma - \gamma|^2 \end{aligned} \tag{3.31}$$

for any $x \in H$ with $\|x\| = 1$.

By taking the supremum over $\|x\| = 1$ in (3.31) we get

$$\begin{aligned} \sup_{\|x\|=1} \left(\sum_{j=1}^{n} |\langle T_j x, x \rangle|^2 \right) &\leq \frac{1}{n} \sup_{\|x\|=1} \left| \left\langle \sum_{j=1}^{n} T_j x, x \right\rangle \right|^2 + \frac{1}{4} n |\Gamma - \gamma|^2 \\ &= \frac{1}{n} w^2 \left(\sum_{j=1}^{n} T_j \right) + \frac{1}{4} n |\Gamma - \gamma|^2, \end{aligned}$$

which proves (3.27).

(ii) If $\operatorname{Re}(\Gamma\overline{\gamma}) > 0$, then by (3.22) we have for $\alpha_j = \langle T_j x, x\rangle$, $j \in \{1, \ldots, n\}$ that

$$\begin{aligned}\sum_{j=1}^{n} |\langle T_j x, x\rangle|^2 &\leq \frac{1}{4n} \frac{|\Gamma+\gamma|^2}{\operatorname{Re}(\Gamma\overline{\gamma})} \left| \sum_{j=1}^{n} \langle T_j x, x\rangle \right|^2 \\ &= \frac{1}{4n} \frac{|\Gamma+\gamma|^2}{\operatorname{Re}(\Gamma\overline{\gamma})} \left| \left\langle \sum_{j=1}^{n} T_j x, x \right\rangle \right|^2 \end{aligned} \tag{3.32}$$

for any $x \in H$ with $\|x\| = 1$.

On taking the supremum over $\|x\| = 1$ in (3.32) we get (3.32).

Also, by (3.24) we get

$$\sum_{j=1}^{n} |\langle T_j x, x\rangle|^2 \leq \frac{1}{n} \left| \sum_{j=1}^{n} \langle T_j x, x\rangle \right|^2 + \left[|\Gamma+\gamma| - 2\sqrt{\operatorname{Re}(\Gamma\overline{\gamma})} \right] \left| \sum_{j=1}^{n} \langle T_j x, x\rangle \right|,$$

for any $x \in H$ with $\|x\| = 1$.

By taking the supremum over $\|x\| = 1$ in this inequality, we have

$$\begin{aligned}&\sup_{\|x\|=1} \sum_{j=1}^{n} |\langle T_j x, x\rangle|^2 \\ &\quad\leq \sup_{\|x\|=1} \left[\frac{1}{n} \left| \sum_{j=1}^{n} \langle T_j x, x\rangle \right|^2 + \left[|\Gamma+\gamma| - 2\sqrt{(\Gamma\overline{\gamma})} \right] \left| \sum_{j=1}^{n} \langle T_j x, x\rangle \right| \right] \\ &\quad\leq \frac{1}{n} \sup_{\|x\|=1} \left| \left\langle \sum_{j=1}^{n} T_j x, x \right\rangle \right|^2 + \left[|\Gamma+\gamma| - 2\sqrt{(\Gamma\overline{\gamma})} \right] \sup_{\|x\|=1} \left| \left\langle \sum_{j=1}^{n} T_j x, x \right\rangle \right| \\ &\quad= \frac{1}{n} w^2 \left(\sum_{j=1}^{n} T_j \right) + \left[|\Gamma+\gamma| - 2\sqrt{(\Gamma\overline{\gamma})} \right] w \left(\sum_{j=1}^{n} T_j \right),\end{aligned}$$

which proves (3.29).

(iii) By the inequality (3.23) we have

$$\left(\sum_{j=1}^{n}|\langle T_j x,x\rangle|^2\right)^{\frac{1}{2}} \le \frac{1}{\sqrt{n}}\left(\left|\sum_{j=1}^{n}\langle T_j x,x\rangle\right| + \frac{1}{4}\frac{|\Gamma-\gamma|^2}{|\Gamma+\gamma|}\right) = \frac{1}{\sqrt{n}}\left(\left|\left\langle\sum_{j=1}^{n}T_j x,x\right\rangle\right| + \frac{1}{4}\frac{|\Gamma-\gamma|^2}{|\Gamma+\gamma|}\right)$$

for any $x\in H$ with $\|x\|=1$.

By taking the supremum over $\|x\|=1$ in this inequality, we get (3.30). ∎

Remark 17. By the use of the elementary inequality $w(T)\le\|T\|$ that holds for any $T\in B(H)$, a sufficient condition for (3.26) to hold is that

$$\left\|T_j-\frac{\gamma+\Gamma}{2}\right\|\le\frac{1}{2}|\Gamma-\gamma| \qquad \text{for any } j\in\{1,\dots,n\}. \tag{3.33}$$

4. Inequalities for δ_p and ϑ_p norms

For $T\in B(H)$ and $p\ge 1$ we can consider the functionals

$$\delta_p(T) := \sup_{\|x\|=\|y\|=1}\left(|\langle Tx,y\rangle|^p+|\langle T^*x,y\rangle|^p\right)^{1/p} = \|(T,T^*)\|_{h,2,p} \tag{4.1}$$

and

$$\vartheta_p(T) := \sup_{\|x\|=1}\left(\|Tx\|^p+\|T^*x\|^p\right)^{1/p} = \|(T,T^*)\|_{s,2,p}\,. \tag{4.2}$$

It is easy to see that both δ_p and ϑ_p are *norms* on $B(H)$. The case $p=2$ for the norm $\delta:=\delta_2$ was considered and studied in [5].

Observe that, for any $T\in B(H)$ and $p\ge 1$, we have

$$\begin{aligned} w_{h,2,p}((T,T^*)) &= \sup_{\|x\|=1}\left(|\langle Tx,x\rangle|^p+|\langle T^*x,x\rangle|^p\right)^{1/p} \\ &= \sup_{\|x\|=1}\left(|\langle Tx,x\rangle|^p+|\langle Tx,x\rangle|^p\right)^{1/p} \\ &= 2^{1/p}\sup_{\|x\|=1}|\langle Tx,x\rangle| = 2^{1/p}w(T). \end{aligned} \tag{4.3}$$

Using the inequality (1.13) we have

$$2^{1/p}w(T)\le\delta_p(T)\le 2^{1+1/p}w(T) \tag{4.4}$$

for any $T \in B(H)$ and $p \geq 1$.

For $p = 2$, we get

$$\sqrt{2}w(T) \leq \delta(T) \leq \sqrt{8}w(T) \tag{4.5}$$

while for $p = 1$ we get

$$2w(T) \leq \delta_1(T) \leq 4w(T) \tag{4.6}$$

for any $T \in B(H)$.

We have for any $T \in B(H)$ and $p \geq 1$ that

$$\|(T, T^*)\|_{2,p} = \left(\|T\|^p + \|T^*\|^p \right)^{1/p} = 2^{1/p}\|T\|$$

and by (2.25) we get

$$\|T\| \leq \delta_p(T) \leq 2^{1/p}\|T\| \tag{4.7}$$

for any $T \in B(H)$ and $p \geq 1$.

For $p = 2$, we get

$$\|T\| \leq \delta(T) \leq \sqrt{2}\|T\| \tag{4.8}$$

while for $p = 1$ we get

$$\|T\| \leq \delta_1(T) \leq 2\|T\| \tag{4.9}$$

for any $T \in B(H)$.

From (2.32) we get for $r \geq q \geq 1$ that

$$\delta_r(T) \leq \delta_q(T) \leq 2^{\frac{r-q}{rq}}\delta_r(T) \tag{4.10}$$

for any $T \in B(H)$.

For any $T \in B(H)$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then by (2.43) we have

$$\delta_q(T) \geq \frac{1}{2^{1/p}}\|T + T^*\|. \tag{4.11}$$

In particular, for $p = q = 2$ we get

$$\delta(T) \geq \frac{\sqrt{2}}{2}\|T + T^*\|, \tag{4.12}$$

for any $T \in B(H)$.

By using the inequality (2.45) we get

$$\delta_p(T) \leq \vartheta_p(T) \leq 2^{1/p}\|T\| \tag{4.13}$$

for any $T \in B(H)$ and $p \geq 1$.

For $p = 1$ we get

$$\delta_1(T) \leq \vartheta_1(T) \leq 2\|T\| \tag{4.14}$$

for any $T \in B(H)$.

For $p \geq 2$, by employing the equality (2.46) we get

$$\vartheta_p(T) = \left[w_{h,2,p/2}\Big(|T|^2, |T^*|^2\Big)\right]^{1/2} = \left[2^{2/p} w\Big(|T|^2\Big)\right]^{1/2} = 2^{1/p}\|T\| \tag{4.15}$$

for any $T \in B(H)$.

On utilising (3.7), (3.9) and (3.11) we get

$$0 \leq \delta^2(T) - \frac{1}{2}\delta_1^2(T) \leq \frac{1}{2}\|T\|^2, \tag{4.16}$$

$$0 \leq \delta^2(T) - \frac{1}{2}\delta_1^2(T) \leq \|T\|\delta_1(T) \tag{4.17}$$

and

$$0 \leq \delta(T) - \frac{1}{\sqrt{2}}\delta_1(T) \leq \frac{\sqrt{2}}{4}\|T\| \tag{4.18}$$

for any $T \in B(H)$.

Observe, by (4.3) we have that

$$w_{h,2,e}((T, T^*)) = \sqrt{2}w(T),$$

for any $T \in B(H)$.

Assume that $T \in B(H)$ and $\gamma, \Gamma \in \mathbb{C}$ with $\Gamma \neq \gamma$ such that

$$w\left(T - \frac{\gamma + \Gamma}{2}I\right),\ w\left(T^* - \frac{\gamma + \Gamma}{2}I\right) \leq \frac{1}{2}|\Gamma - \gamma|, \tag{4.19}$$

then by (3.27) we get

$$w^2(T) \leq \|\mathrm{Re}\,(T)\|^2 + \frac{1}{4}|\Gamma - \gamma|^2, \tag{4.20}$$

where $\mathrm{Re}\,(T) := \frac{T+T^*}{2}$.

If $\mathrm{Re}\,(\Gamma\overline{\gamma}) > 0$, then by (3.28) and (3.29)

$$w(T) \leq \frac{1}{2}\frac{|\Gamma + \gamma|}{\sqrt{\mathrm{Re}\,(\Gamma\overline{\gamma})}}\|\mathrm{Re}\,(T)\| \tag{4.21}$$

and

$$w^2(T) \le \Big[\|\mathrm{Re}\,(T)\| + \Big[|\Gamma+\gamma| - 2\sqrt{(\Gamma\overline{\gamma})}\Big]\Big]\|\mathrm{Re}\,(T)\|. \tag{4.22}$$

If $\Gamma \neq -\gamma$, then by (3.30) we get

$$w(T) \le \|\mathrm{Re}\,(T)\| + \frac{1}{8}\frac{|\Gamma-\gamma|^2}{|\Gamma+\gamma|}. \tag{4.23}$$

Due to the fact that $w(A) = w(A^*)$ for any $A \in B(H)$, the condition (4.19) can be simplified as follows.

If m, M are real numbers with $M > m$ and if

$$w\Big(T - \frac{m+M}{2}I\Big) \le \frac{1}{2}(M-m),$$

then

$$w^2(T) \le \|\mathrm{Re}\,(T)\|^2 + \frac{1}{4}(M-m)^2. \tag{4.24}$$

If $m > 0$, then

$$w(T) \le \frac{1}{2}\frac{m+M}{\sqrt{mM}}\|\mathrm{Re}\,(T)\| \tag{4.25}$$

and

$$w^2(T) \le \Big[\|\mathrm{Re}\,(T)\| + \Big(\sqrt{M} - \sqrt{m}\Big)^2\Big]\|\mathrm{Re}\,(T)\|. \tag{4.26}$$

If $M \neq -m$, then

$$w(T) \le \|\mathrm{Re}\,(T)\| + \frac{1}{8}\frac{(M-m)^2}{m+M}. \tag{4.27}$$

For other numerical radius and norm inequalities, the interested reader may also consult [1] and [6] and compare these results. The details are not provided here.

5. Inequalities for real norms

If X is a complex linear space, then the functional $\|\cdot\|$ is a real norm, if the homogeneity property in the definition of the norms is satisfied only for real numbers, namely we have

$$\|\alpha x\| = |\alpha|\|x\| \qquad \text{for any } \alpha \in \mathbb{R} \text{ and } x \in X.$$

For instance if we consider the complex linear space of complex numbers $\mathbb{C}$ then the functionals

$$|z|_p := \left(|\mathrm{Re}\,(z)|^p + |\mathrm{Im}\,(z)|^p \right)^{1/p}, \quad p \geq 1,$$
$$|z|_\infty := \max\{|\mathrm{Re}\,(z)|, |\mathrm{Im}\,(z)|\}, \qquad p = \infty,$$

are real norms on $\mathbb{C}$.

For $T \in B(H)$ we consider the Cartesian decomposition

$$T = \mathrm{Re}\,(T) + i\,\mathrm{Im}\,(T)$$

where the selfadjoint operators $\mathrm{Re}\,(T)$ and $\mathrm{Im}\,(T)$ are uniquely defined by

$$\mathrm{Re}\,(T) = \frac{T + T^*}{2} \qquad \text{and} \qquad \mathrm{Im}\,(T) = \frac{T - T^*}{2i}.$$

We can introduce the following functionals

$$\|T\|_{r,p} := \left(\|\mathrm{Re}\,(T)\|^p + \|\mathrm{Im}\,(T)\|^p \right)^{1/p}, \qquad p \geq 1,$$

and

$$\|T\|_{r,\infty} := \max\left\{ \|\mathrm{Re}\,(T)\|, \|\mathrm{Im}\,(T)\| \right\}, \qquad p = \infty,$$

where $\|\cdot\|$ is the usual operator norm on $B(H)$. *The definition can be extended for any other norms on $B(H)$ or its subspaces.*

Using the properties of the norm $\|\cdot\|$ and the Minkowski's inequality

$$\left(|a+b|^p + |c+d|^p \right)^{1/p} \leq \left(|a|^p + |c|^p \right)^{1/p} + \left(|b|^p + |d|^p \right)^{1/p}$$

for $p \geq 1$ and $a, b, c, d \in \mathbb{C}$, we observe that $\|\cdot\|_{r,p}$, $p \in [1, \infty]$ is a real norm on $B(H)$.

For $p \geq 1$ and $T \in B$ we can introduce the following functionals

$$\begin{aligned}
\eta_{r,p}(T) &:= \sup_{\|x\|=\|y\|=1} \left(|\mathrm{Re}\,\langle Tx, y\rangle|^p + |\mathrm{Im}\,\langle Tx, y\rangle|^p \right)^{1/p} \\
&= \sup_{\|x\|=\|y\|=1} \left(|\langle \mathrm{Re}\,Tx, y\rangle|^p + |\langle \mathrm{Im}\,Tx, y\rangle|^p \right)^{1/p} \\
&= \|(\mathrm{Re}\,T, \mathrm{Im}\,T)\|_{h,2,p},
\end{aligned}$$

$$\begin{aligned}\theta_{r,p}(T) &:= \sup_{\|x\|=1}\left(|\operatorname{Re}\langle Tx,x\rangle|^p+|\operatorname{Im}\langle Tx,x\rangle|^p\right)^{1/p}\\ &= \sup_{\|x\|=1}\left(|\langle \operatorname{Re}Tx,x\rangle|^p+|\langle \operatorname{Im}Tx,x\rangle|^p\right)^{1/p}\\ &= w_{h,2,p}(\operatorname{Re}T,\operatorname{Im}T)\end{aligned}$$

and

$$\kappa_{r,p}(T) := \sup_{\|x\|=1}\left(\|\operatorname{Re}Tx\|^p+\|\operatorname{Im}Tx\|^p\right)^{1/p} = \|(\operatorname{Re}T,\operatorname{Im}T)\|_{s,2,p}\,.$$

The case $p=2$ is of interest since for $T\in B(H)$ we have

$$\begin{aligned}\eta_{r,2}(T) &:= \sup_{\|x\|=\|y\|=1}\left(|\operatorname{Re}\langle Tx,y\rangle|^2+|\operatorname{Im}\langle Tx,y\rangle|^2\right)^{1/2}\\ &= \sup_{\|x\|=\|y\|=1}|\langle Tx,y\rangle| \;=\; \|T\|\,,\\ \theta_{r,2}(T) &:= \sup_{\|x\|=1}\left(|\operatorname{Re}\langle Tx,x\rangle|^2+|\operatorname{Im}\langle Tx,x\rangle|^2\right)^{1/2}\\ &= \sup_{\|x\|=1}|\langle Tx,x\rangle| \;=\; w(T)\end{aligned}$$

and

$$\begin{aligned}\kappa_{r,2}(T) &:= \sup_{\|x\|=1}\left(\|\operatorname{Re}Tx\|^2+\|\operatorname{Im}Tx\|^2\right)^{1/2}\\ &= \sup_{\|x\|=1}\left(\left\langle(\operatorname{Re}T)^2x,x\right\rangle+\left\langle(\operatorname{Im}T)^2x,x\right\rangle\right)^{1/2}\\ &= \sup_{\|x\|=1}\left(\left\langle\left[\,(\operatorname{Re}T)^2+(\operatorname{Im}T)^2\,\right]x,x\right\rangle\right)^{1/2}\\ &= \left\|(\operatorname{Re}T)^2+(\operatorname{Im}T)^2\right\|^{1/2} = \left\|\frac{|T|^2+|T^*|^2}{2}\right\|^{1/2}.\end{aligned}$$

For $p = \infty$ we have

$$\begin{aligned}
\eta_{r,\infty}(T) &:= \sup_{\|x\|=\|y\|=1} \Big(\max \big\{ |\operatorname{Re}\langle Tx, y\rangle| , |\operatorname{Im}\langle Tx, y\rangle| \big\} \Big) \\
&= \max \left\{ \sup_{\|x\|=\|y\|=1} |\langle \operatorname{Re} Tx, y\rangle| , \sup_{\|x\|=\|y\|=1} |\langle \operatorname{Im} Tx, y\rangle| \right\} \\
&= \max \big\{ \|\operatorname{Re} T\|, \|\operatorname{Im} T\| \big\},
\end{aligned}$$

and in a similar way

$$\theta_{r,\infty}(T) = \kappa_{r,\infty}(T) = \max \big\{ \|\operatorname{Re} T\|, \|\operatorname{Im} T\| \big\} = \|T\|_{r,\infty} .$$

The functionals $\eta_{r,p}$, $\theta_{r,p}$ and $\kappa_{r,p}$ with $p \in [1, \infty]$ are real norms on $B(H)$.

We have

$$\begin{aligned}
\eta_{r,p}(T) &= \sup_{\|x\|=\|y\|=1} \Big(|\operatorname{Re}\langle Tx, y\rangle|^p + |\operatorname{Im}\langle Tx, y\rangle|^p \Big)^{1/p} \\
&\le \left(\sup_{\|x\|=\|y\|=1} |\operatorname{Re}\langle Tx, y\rangle|^p + \sup_{\|x\|=\|y\|=1} |\operatorname{Im}\langle Tx, y\rangle|^p \right)^{1/p} \\
&= \left(\|\operatorname{Re}(T)\|^p + \|\operatorname{Im}(T)\|^p \right)^{1/p} = \|T\|_{r,p}
\end{aligned}$$

and

$$\begin{aligned}
\|T\|_{r,\infty} &= \sup_{\|x\|=\|y\|=1} \Big(\max \{ |\operatorname{Re}\langle Tx, y\rangle|, |\operatorname{Im}\langle Tx, y\rangle| \} \Big) \\
&\le \sup_{\|x\|=\|y\|=1} \Big(|\operatorname{Re}\langle Tx, y\rangle|^p + |\operatorname{Im}\langle Tx, y\rangle|^p \Big)^{1/p} = \eta_{r,p}(T)
\end{aligned}$$

for any $p \ge 1$ and $T \in B(H)$.

In a similar way we have

$$\|T\|_{r,\infty} \le \theta_{r,p}(T) \le \|T\|_{r,p}$$

and

$$\|T\|_{r,\infty} \le \kappa_{r,p}(T) \le \|T\|_{r,p}$$

for any $p \ge 1$ and $T \in B(H)$.

If we write the inequality (1.13) for $n=2$, $T_1=\operatorname{Re}T$ and $T_2=\operatorname{Im}T$ then we get

$$\theta_{r,p}(T)\le\eta_{r,p}(T)\le 2\theta_{r,p}(T) \tag{5.1}$$

for any $p\ge 1$ and $T\in B(H)$.

Using the inequalities (2.25) and (2.26) for $n=2$, $T_1=\operatorname{Re}T$ and $T_2=\operatorname{Im}T$ then we get

$$\frac{1}{2^{1/p}}\|T\|_{r,p}\le\eta_{r,p}(T)\le\|T\|_{r,p} \tag{5.2}$$

and

$$\frac{1}{2^{1/p}}\|T\|_{r,p}\le\theta_{r,p}(T)\le\|T\|_{r,p} \tag{5.3}$$

for any $p\ge 1$ and $T\in B(H)$.

If we use the inequalities (2.32) and (2.33) for $n=2$, $T_1=\operatorname{Re}T$ and $T_2=\operatorname{Im}T$ then we get for $t\ge p\ge 1$ that

$$\eta_{r,t}(T)\le\eta_{r,p}(T)\le 2^{\frac{t-p}{tp}}\eta_{r,t}(T) \tag{5.4}$$

and

$$\theta_{r,t}(T)\le\theta_{r,p}(T)\le 2^{\frac{t-p}{tp}}\theta_{r,t}(T) \tag{5.5}$$

for any $T\in B(H)$.

For $p=1$ we have the functionals

$$\eta_{r,1}(T)=\sup_{\|x\|=\|y\|=1}\big(|\langle\operatorname{Re}Tx,y\rangle|+|\langle\operatorname{Im}Tx,y\rangle|\big)=\|(\operatorname{Re}T,\operatorname{Im}T)\|_{h,2,1},$$

$$\theta_{r,1}(T):=\sup_{\|x\|=1}\big(|\langle\operatorname{Re}Tx,x\rangle|+|\langle\operatorname{Im}Tx,x\rangle|\big)=w_{h,2,1}(\operatorname{Re}T,\operatorname{Im}T)$$

and

$$\kappa_{r,1}(T):=\sup_{\|x\|=1}\big(\|\operatorname{Re}Tx\|+\|\operatorname{Im}Tx\|\big)=\|(\operatorname{Re}T,\operatorname{Im}T)\|_{s,2,1}.$$

By utilising the inequalities (3.7), (3.9) and (3.11) for $n=2$, $T_1=\operatorname{Re}T$ and $T_2=\operatorname{Im}T$, then

$$0\le\|T\|^2-\frac{1}{2}\eta_{r,1}^2(T)\le\frac{1}{2}\big(\max\{\|\operatorname{Re}T\|,\|\operatorname{Im}T\|\}\big)^2, \tag{5.6}$$

$$0\le\|T\|^2-\frac{1}{2}\eta_{r,1}^2(T)\le\max\big\{\|\operatorname{Re}T\|,\|\operatorname{Im}T\|\big\}\eta_{r,1}(T) \tag{5.7}$$

and

$$0 \leq \|T\| - \frac{\sqrt{2}}{2}\eta_{r,1}(T) \leq \frac{\sqrt{2}}{4}\max\{\|\mathrm{Re}\,T\|, \|\mathrm{Im}\,T\|\} \tag{5.8}$$

for any $T \in B(H)$.

Also, by utilising the inequalities (3.8), (3.10) and (3.12) for $n = 2$, $T_1 = \mathrm{Re}\,T$ and $T_2 = \mathrm{Im}\,T$, then

$$0 \leq w^2(T) - \frac{1}{2}\theta_{r,1}^2(T) \leq \frac{1}{2}\big(\max\{\|\mathrm{Re}\,T\|, \|\mathrm{Im}\,T\|\}\big)^2, \tag{5.9}$$

$$0 \leq w^2(T) - \frac{1}{2}\theta_{r,1}^2(T) \leq \max\big\{\|\mathrm{Re}\,T\|, \|\mathrm{Im}\,T\|\big\}\theta_{r,1}(T) \tag{5.10}$$

and

$$0 \leq w(T) - \frac{\sqrt{2}}{2}\theta_{r,1}(T) \leq \frac{\sqrt{2}}{4}\max\big\{\|\mathrm{Re}\,T\|, \|\mathrm{Im}\,T\|\big\} \tag{5.11}$$

for any $T \in B(H)$.

If m, M are real numbers with $M > m$ and if

$$\left\|\mathrm{Re}\,T - \frac{m+M}{2}I\right\|, \left\|\mathrm{Im}\,T - \frac{m+M}{2}I\right\| \leq \frac{1}{2}(M-m), \tag{5.12}$$

then by (3.27) we get

$$w^2(T) \leq \frac{1}{2}\|\mathrm{Re}\,T + \mathrm{Im}\,T\|^2 + \frac{1}{2}(M-m)^2. \tag{5.13}$$

If $m > 0$, then (3.28) and (3.29) we have

$$w(T) \leq \frac{1}{2\sqrt{2}}\frac{m+M}{\sqrt{mM}}\|\mathrm{Re}\,T + \mathrm{Im}\,T\| \tag{5.14}$$

and

$$w^2(T) \leq \left[\frac{1}{2}\|\mathrm{Re}\,T + \mathrm{Im}\,T\| + \left(\sqrt{M} - \sqrt{m}\right)^2\right]\|\mathrm{Re}\,T + \mathrm{Im}\,T\|. \tag{5.15}$$

If $M \neq -m$, then by (3.30) we get

$$w(T) \leq \frac{1}{\sqrt{2}}\left(\|\mathrm{Re}\,T + \mathrm{Im}\,T\| + \frac{1}{4}\frac{(M-m)^2}{M+m}\right). \tag{5.16}$$

Finally, we observe that a simple sufficient condition for (5.12) to hold, is that

$$mI \leq \mathrm{Re}\,T, \qquad \mathrm{Im}\,T \leq MI$$

in the operator order of $B(H)$.

Acknowledgements

The author would like to thank the anonymous referee for valuable suggestions that have been implemented in the final version of the paper.

References

[1] M. Bakherad, K. Shebrawi, Upper bounds for numerical radius inequalities involving off-diagonal operator matrices, *Ann. Funct. Anal.* **9** (3) (2018), 297 – 309.

[2] S.S. Dragomir, A counterpart of Schwarz's inequality in inner product spaces, *East Asian Math. J.* **20** (1) (2004), 1 – 10, preprint *RGMIA Res. Rep. Coll.* **6** (2003), `http://rgmia.org/papers/v6e/CSIIPS.pdf`.

[3] S.S. Dragomir, A survey on Cauchy-Bunyakovsky-Schwarz type discrete inequalities, *J. Inequal. Pure Appl. Math.* **4** (3) (2003), Article 63, 142 pp., `https://www.emis.de/journals/JIPAM/article301.html?sid=301`.

[4] S.S. Dragomir, "Advances in Inequalities of the Schwarz, Grüss and Bessel Type in Inner Product Spaces", Nova Science Publishers Inc., NY, 2005.

[5] S.S. Dragomir, The hypo-Euclidean norm of an n-tuple of vectors in inner product spaces and applications, *J. Ineq. Pure & Appl. Math.* **8** (2) (2007), Article 52, 22 pp., `https://www.emis.de/journals/JIPAM/article854.html?sid=854`.

[6] S.S. Dragomir, Some inequalities for the Euclidean operator radius of two operators in Hilbert spaces, *Linear Algebra Appl.* **419** (1) (2006), 256 – 264.

[7] S.S. Dragomir, Reverses of the Schwarz inequality generalising the Klamkin-McLeneghan result, *Bull. Austral. Math. Soc.* **73** (1) (2006), 69 – 78.

[8] S. S. Dragomir, Some inequalities of Kato type for sequences of operators in Hilbert spaces, *Publ. Res. Inst. Math. Sci.* **48** (4) (2012), 937 – 955.

[9] S. S. Dragomir, Y. J. Cho, Y.-H. Kim, Applications of Kato's inequality for n-tuples of operators in Hilbert spaces, (I), *J. Inequal. Appl.* 2013:21 (2013), 16pp.

[10] S. S. Dragomir, Y. J. Cho, Y.-H. Kim, Applications of Kato's inequality for n-tuples of operators in Hilbert spaces, (II), *J. Inequal. Appl.* 2013:464 (2013), 20 pp.

[11] T. Kato, Notes on some inequalities for linear operators, *Math. Ann.* **125** (1952), 208 – 212.

[12] M.S. Moslehian, M. Sattari, K. Shebrawi, Extensions of Euclidean operator radius inequalitie, *Math. Scand.* **120** (1) (2017), 129 – 144.

[13] G. Popescu, "Unitary Invariants in Multivariable Operator Theory", Mem. Amer. Math. Soc. no. 941, 2009.

[14] A. Sheikhhosseini, M.S. Moslehian, K. Shebrawi, Inequalities for generalized Euclidean operator radius via Young's inequality. *J. Math. Anal. Appl.* **445** (2) (2017), 1516 – 1529.

[15] O. Shisha, B. Mond, Bounds on differences of means, in "Inequalities", Academic Press Inc., New York, 1967, 293 – 308.