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# Hypo- $q$ -norms on cartesian products of algebras of bounded linear operators on Hilbert spaces

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**Abstract:** In this paper we introduce the hypo- $q$ -norms on a Cartesian product of algebras of bounded linear operators on Hilbert spaces. A representation of these norms in terms of inner products, the equivalence with the  $q$ -norms on a Cartesian product and some reverse inequalities obtained via the scalar reverses of Cauchy-Buniakowski-Schwarz inequality are also given. Several bounds for the norms  $\delta_p$ ,  $\vartheta_p$  and the real norms  $\eta_{r,p}$  and  $\theta_{r,p}$  are provided as well.

**Key words:** Hilbert spaces, bounded linear operators, operator norm and numerical radius,  $n$ -tuple of operators, operator inequalities.

**AMS Subject Class.** (2010): 46C05, 26D15.

## 1. INTRODUCTION

In [13], the author has introduced the following norm on the Cartesian product  $B^{(n)}(H) := B(H) \times \cdots \times B(H)$ , where  $B(H)$  denotes the Banach algebra of all bounded linear operators defined on the complex Hilbert space  $H$ :

$$\|(T_1, \dots, T_n)\|_{n,e} := \sup_{(\lambda_1, \dots, \lambda_n) \in \mathbb{B}_n} \|\lambda_1 T_1 + \cdots + \lambda_n T_n\|, \quad (1.1)$$

where  $(T_1, \dots, T_n) \in B^{(n)}(H)$  and

$$\mathbb{B}_n := \left\{ (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n : \sum_{i=1}^n |\lambda_i|^2 \leq 1 \right\}$$

is the Euclidean closed ball in  $\mathbb{C}^n$ . It is clear that  $\|\cdot\|_{n,e}$  is a norm on  $B^{(n)}(H)$  and for any  $(T_1, \dots, T_n) \in B^{(n)}(H)$  we have

$$\|(T_1, \dots, T_n)\|_{n,e} = \|(T_1^*, \dots, T_n^*)\|_{n,e}, \quad (1.2)$$



where  $T_i^*$  is the adjoint operator of  $T_i$ ,  $i \in \{1, \dots, n\}$ .

It has been shown in [13] that the following inequality holds true:

$$\frac{1}{\sqrt{n}} \left\| \sum_{j=1}^n T_j T_j^* \right\|^{\frac{1}{2}} \leq \|(T_1, \dots, T_n)\|_{n,e} \leq \left\| \sum_{j=1}^n T_j T_j^* \right\|^{\frac{1}{2}} \quad (1.3)$$

for any  $n$ -tuple  $(T_1, \dots, T_n) \in B^{(n)}(H)$  and the constants  $\frac{1}{\sqrt{n}}$  and 1 are best possible.

In the same paper [13] the author has introduced the *Euclidean operator radius* of an  $n$ -tuple of operators  $(T_1, \dots, T_n)$  by

$$w_{n,e}(T_1, \dots, T_n) := \sup_{\|x\|=1} \left( \sum_{j=1}^n |\langle T_j x, x \rangle|^2 \right)^{\frac{1}{2}} \quad (1.4)$$

and proved that  $w_{n,e}(\cdot)$  is a norm on  $B^{(n)}(H)$  and satisfies the double inequality:

$$\frac{1}{2} \|(T_1, \dots, T_n)\|_{n,e} \leq w_{n,e}(T_1, \dots, T_n) \leq \|(T_1, \dots, T_n)\|_{n,e} \quad (1.5)$$

for each  $n$ -tuple  $(T_1, \dots, T_n) \in B^{(n)}(H)$ .

As pointed out in [13], the Euclidean numerical radius also satisfies the double inequality:

$$\frac{1}{2\sqrt{n}} \left\| \sum_{j=1}^n T_j T_j^* \right\|^{\frac{1}{2}} \leq w_{n,e}(T_1, \dots, T_n) \leq \left\| \sum_{j=1}^n T_j T_j^* \right\|^{\frac{1}{2}} \quad (1.6)$$

for any  $(T_1, \dots, T_n) \in B^{(n)}(H)$  and the constants  $\frac{1}{2\sqrt{n}}$  and 1 are best possible.

Now, let  $(E, \|\cdot\|)$  be a normed linear space over the complex number field  $\mathbb{C}$ . On  $\mathbb{C}^n$  endowed with the canonical linear structure we consider a norm  $\|\cdot\|_n$ . As an example of such norms we should mention the usual  $p$ -norms

$$\|\lambda\|_{n,p} := \begin{cases} \max \{|\lambda_1|, \dots, |\lambda_n|\} & \text{if } p = \infty, \\ \left( \sum_{k=1}^n |\lambda_k|^p \right)^{\frac{1}{p}} & \text{if } p \in [1, \infty). \end{cases}$$

The *Euclidean norm* is obtained for  $p = 2$ , i.e.,

$$\|\lambda\|_{n,2} := \left( \sum_{k=1}^n |\lambda_k|^2 \right)^{\frac{1}{2}}.$$

It is well known that on  $E^n := E \times \cdots \times E$  endowed with the canonical linear structure we can define the following  $p$ -norms:

$$\|x\|_{n,p} := \begin{cases} \max \{|x_1|, \dots, |x_n|\} & \text{if } p = \infty, \\ \left( \sum_{k=1}^n |x_k|^p \right)^{\frac{1}{p}} & \text{if } p \in [1, \infty). \end{cases}$$

where  $x = (x_1, \dots, x_n) \in E^n$ .

Following the paper [5], for a given norm  $\|\cdot\|_n$  on  $\mathbb{C}^n$ , we define the functional  $\|\cdot\|_{h,n} : E^n \rightarrow [0, \infty)$  by

$$\|x\|_{h,n} := \sup_{\|\lambda\|_n \leq 1} \left\| \sum_{j=1}^n \lambda_j x_j \right\|, \quad (1.7)$$

where  $x = (x_1, \dots, x_n) \in E^n$  and  $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ .

It is easy to see that [5]:

- (i)  $\|x\|_{h,n} \geq 0$  for any  $x \in E^n$ ,
- (ii)  $\|x + y\|_{h,n} \leq \|x\|_{h,n} + \|y\|_{h,n}$  for any  $x, y \in E^n$ ,
- (iii)  $\|\alpha x\|_{h,n} = |\alpha| \|x\|_{h,n}$  for each  $\alpha \in \mathbb{C}$  and  $x \in E^n$ ,

and therefore  $\|\cdot\|_{h,n}$  is a *semi-norm* on  $E^n$ . This will be called the *hypo-semi-norm* generated by the norm  $\|\cdot\|_n$  on  $E^n$ .

We observe that  $\|x\|_{h,n} = 0$  if and only if  $\sum_{j=1}^n \lambda_j x_j = 0$  for any  $(\lambda_1, \dots, \lambda_n) \in B(\|\cdot\|_n)$ . If there exists  $\lambda_1^0, \dots, \lambda_n^0 \neq 0$  such that  $(\lambda_1^0, 0, \dots, 0)$ ,  $(0, \lambda_2^0, \dots, 0)$ ,  $\dots$ ,  $(0, 0, \dots, \lambda_n^0) \in B(\|\cdot\|_n)$  then the semi-norm generated by  $\|\cdot\|_n$  is a *norm* on  $E^n$ .

If  $p \in [1, \infty]$  and we consider the  $p$ -norms  $\|\cdot\|_{n,p}$  on  $\mathbb{C}^n$ , then we can define the following *hypo- $q$ -norms* on  $E^n$ :

$$\|x\|_{h,n,q} := \sup_{\|\lambda\|_{n,p} \leq 1} \left\| \sum_{j=1}^n \lambda_j x_j \right\|, \quad (1.8)$$

with  $q \in [1, \infty]$ . If  $p = 1$ , then  $q = \infty$ ; if  $p = \infty$ , then  $q = 1$ ; if  $p \in (1, \infty)$ , then  $\frac{1}{p} + \frac{1}{q} = 1$ .

For  $p = 2$ , we have the *hypo-Euclidean norm* on  $E^n$ , i.e.,

$$\|x\|_{h,n,e} := \sup_{\|\lambda\|_{n,2} \leq 1} \left\| \sum_{j=1}^n \lambda_j x_j \right\|. \quad (1.9)$$

If we consider now  $E = B(H)$  endowed with the operator norm  $\|\cdot\|$ , then we can obtain the following *hypo- $q$ -norms* on  $B^{(n)}(H)$

$$\|(T_1, \dots, T_n)\|_{h,n,q} := \sup_{\|\lambda\|_{n,p} \leq 1} \left\| \sum_{j=1}^n \lambda_j T_j \right\| \quad \text{where } p, q \in [1, \infty], \quad (1.10)$$

with the convention that if  $p = 1$ ,  $q = \infty$ , if  $p = \infty$ ,  $q = 1$  and if  $p > 1$ , then  $\frac{1}{p} + \frac{1}{q} = 1$ .

For  $p = 2$  we obtain the *hypo-Euclidian norm*  $\|(\cdot, \dots, \cdot)\|_{n,e}$  defined in (1.2).

If we consider now  $E = B(H)$  endowed with the operator numerical radius  $w(\cdot)$ , which is a norm on  $B(H)$ , then we can obtain the following *hypo- $q$ -numerical radius* of  $(T_1, \dots, T_n) \in B^{(n)}(H)$  defined by

$$w_{h,n,q}(T_1, \dots, T_n) := \sup_{\|\lambda\|_{n,p} \leq 1} w\left(\sum_{j=1}^n \lambda_j T_j\right) \quad \text{with } p, q \in [1, \infty], \quad (1.11)$$

with the convention that if  $p = 1$ ,  $q = \infty$ , if  $p = \infty$ ,  $q = 1$  and if  $p > 1$ , then  $\frac{1}{p} + \frac{1}{q} = 1$ .

For  $p = 2$  we obtain the *hypo-Euclidian norm*

$$w_{h,n,e}(T_1, \dots, T_n) := \sup_{\|\lambda\|_{n,2} \leq 1} w\left(\sum_{j=1}^n \lambda_j T_j\right) \quad (1.12)$$

and will show further that it coincides with the Euclidean operator radius of an  $n$ -tuple of operators  $(T_1, \dots, T_n)$  defined in (1.4).

Using the fundamental inequality between the operator norm and numerical radius  $w(T) \leq \|T\| \leq 2w(T)$  for  $T \in B(H)$  we have

$$w\left(\sum_{j=1}^n \lambda_j T_j\right) \leq \left\| \sum_{j=1}^n \lambda_j T_j \right\| \leq 2w\left(\sum_{j=1}^n \lambda_j T_j\right)$$

for any  $(T_1, \dots, T_n) \in B^{(n)}(H)$  and any  $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ . By taking the supremum over  $\lambda$  with  $\|\lambda\|_{n,p} \leq 1$  we get

$$w_{h,n,q}(T_1, \dots, T_n) \leq \|(T_1, \dots, T_n)\|_{h,n,q} \leq 2w_{h,n,q}(T_1, \dots, T_n) \quad (1.13)$$

with the convention that if  $p = 1$ ,  $q = \infty$ , if  $p = \infty$ ,  $q = 1$  and if  $p > 1$ , then  $\frac{1}{p} + \frac{1}{q} = 1$ .

For  $p = q = 2$  we recapture the inequality (1.5).

In 2012, [8] (see also [9, 10]) the author have introduced the concept of  $s$ - $q$ -numerical radius of an  $n$ -tuple of operators  $(T_1, \dots, T_n)$  for  $q \geq 1$  as

$$w_{s,q}(T_1, \dots, T_n) := \sup_{\|x\|=1} \left( \sum_{j=1}^n |\langle T_j x, x \rangle|^q \right)^{1/q} \quad (1.14)$$

and established various inequalities of interest. For more recent results see also [12, 14].

In the same paper [8] we also introduced the concept of  $s$ - $q$ -norm of an  $n$ -tuple of operators  $(T_1, \dots, T_n)$  for  $q \geq 1$  as

$$\|(T_1, \dots, T_n)\|_{s,q} := \sup_{\|x\|=\|y\|=1} \left( \sum_{j=1}^n |\langle T_j x, y \rangle|^q \right)^{1/q}. \quad (1.15)$$

In [8], [9] and [10], by utilising *Kato's inequality* [11]

$$|\langle Tx, y \rangle|^2 \leq \langle |T|^{2\alpha} x, x \rangle \langle |T^*|^{2(1-\alpha)} y, y \rangle \quad (1.16)$$

for any  $x, y \in H$ ,  $\alpha \in [0, 1]$ , where “absolute value” operator of  $A$  is defined by  $\|A\| := \sqrt{A^*A}$ , the authors have obtained several inequalities for the  $s$ - $q$ -numerical radius and  $s$ - $q$ -norm.

In this paper we investigate the connections between these norms and establish some fundamental inequalities of interest in multivariate operator theory.

## 2. REPRESENTATION RESULTS

We start with the following lemma:

LEMMA 1. Let  $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{C}^n$ .

(i) If  $p, q > 1$  and  $\frac{1}{p} + \frac{1}{q} = 1$ , then

$$\sup_{\|\alpha\|_{n,p} \leq 1} \left| \sum_{j=1}^n \alpha_j \beta_j \right| = \|\beta\|_{n,q}. \quad (2.1)$$

In particular,

$$\sup_{\|\alpha\|_{n,2} \leq 1} \left| \sum_{j=1}^n \alpha_j \beta_j \right| = \|\beta\|_{n,2}. \quad (2.2)$$

(ii) We have

$$\sup_{\|\alpha\|_{n,\infty} \leq 1} \left| \sum_{j=1}^n \alpha_j \beta_j \right| = \|\beta\|_{n,1} \quad \text{and} \quad \sup_{\|\alpha\|_{n,1} \leq 1} \left| \sum_{j=1}^n \alpha_j \beta_j \right| = \|\beta\|_{n,\infty}. \quad (2.3)$$

*Proof.* (i) Using Hölder's discrete inequality for  $p, q > 1$  and  $\frac{1}{p} + \frac{1}{q} = 1$  we have

$$\left| \sum_{j=1}^n \alpha_j \beta_j \right| \leq \left( \sum_{j=1}^n |\alpha_j|^p \right)^{1/p} \left( \sum_{j=1}^n |\beta_j|^q \right)^{1/q},$$

which implies that

$$\sup_{\|\alpha\|_{n,p} \leq 1} \left| \sum_{j=1}^n \alpha_j \beta_j \right| \leq \|\beta\|_{n,q} \quad (2.4)$$

where  $\alpha = (\alpha_1, \dots, \alpha_n)$  and  $\beta = (\beta_1, \dots, \beta_n)$  are  $n$ -tuples of complex numbers.

For  $(\beta_1, \dots, \beta_n) \neq 0$ , consider  $\alpha = (\alpha_1, \dots, \alpha_n)$  with

$$\alpha_j := \frac{\overline{\beta_j} |\beta_j|^{q-2}}{\left( \sum_{k=1}^n |\beta_k|^q \right)^{1/p}}$$

for those  $j$  for which  $\beta_j \neq 0$  and  $\alpha_j = 0$ , for the rest.

We observe that

$$\begin{aligned} \left| \sum_{j=1}^n \alpha_j \beta_j \right| &= \left| \sum_{j=1}^n \frac{\overline{\beta_j} |\beta_j|^{q-2}}{\left( \sum_{k=1}^n |\beta_k|^q \right)^{1/p}} \beta_j \right| = \frac{\sum_{j=1}^n |\beta_j|^q}{\left( \sum_{k=1}^n |\beta_k|^q \right)^{1/p}} \\ &= \left( \sum_{j=1}^n |\beta_j|^q \right)^{1/q} = \|\beta\|_{n,q} \end{aligned}$$

and

$$\begin{aligned} \|\alpha\|_{n,p}^p &= \sum_{j=1}^n |\alpha_j|^p = \sum_{j=1}^n \frac{\left| \overline{\beta_j} |\beta_j|^{q-2} \right|^p}{\left( \sum_{k=1}^n |\beta_k|^q \right)} = \sum_{j=1}^n \frac{\left( |\beta_j|^{q-1} \right)^p}{\left( \sum_{k=1}^n |\beta_k|^q \right)} \\ &= \sum_{j=1}^n \frac{|\beta_j|^{qp-p}}{\left( \sum_{k=1}^n |\beta_k|^q \right)} = \sum_{j=1}^n \frac{|\beta_j|^q}{\left( \sum_{k=1}^n |\beta_k|^q \right)} = 1. \end{aligned}$$

Therefore, by (2.4) we have the representation (2.1).

(ii) Using the properties of the modulus, we have

$$\left| \sum_{j=1}^n \alpha_j \beta_j \right| \leq \max_{j \in \{1, \dots, n\}} |\alpha_j| \sum_{j=1}^n |\beta_j|,$$

which implies that

$$\sup_{\|\alpha\|_{n,\infty} \leq 1} \left| \sum_{j=1}^n \alpha_j \beta_j \right| \leq \|\beta\|_{n,1}, \quad (2.5)$$

where  $\alpha = (\alpha_1, \dots, \alpha_n)$  and  $\beta = (\beta_1, \dots, \beta_n)$ .

For  $(\beta_1, \dots, \beta_n) \neq 0$ , consider  $\alpha = (\alpha_1, \dots, \alpha_n)$  with  $\alpha_j := \frac{\overline{\beta_j}}{|\beta_j|}$  for those  $j$  for which  $\beta_j \neq 0$  and  $\alpha_j = 0$ , for the rest.

We have

$$\left| \sum_{j=1}^n \alpha_j \beta_j \right| = \left| \sum_{j=1}^n \frac{\overline{\beta_j}}{|\beta_j|} \beta_j \right| = \sum_{j=1}^n |\beta_j| = \|\beta\|_{n,1}$$

and

$$\|\alpha\|_{n,\infty} = \max_{j \in \{1, \dots, n\}} |\alpha_j| = \max_{j \in \{1, \dots, n\}} \left| \frac{\overline{\beta_j}}{|\beta_j|} \right| = 1$$

and by (2.5) we get the first representation in (2.3).

Moreover, we have

$$\left| \sum_{j=1}^n \alpha_j \beta_j \right| \leq \sum_{j=1}^n |\alpha_j| \max_{j \in \{1, \dots, n\}} |\beta_j|,$$

which implies that

$$\sup_{\|\alpha\|_{n,1} \leq 1} \left| \sum_{j=1}^n \alpha_j \beta_j \right| \leq \|\beta\|_{n,\infty}, \quad (2.6)$$

where  $\alpha = (\alpha_1, \dots, \alpha_n)$  and  $\beta = (\beta_1, \dots, \beta_n)$ .

For  $(\beta_1, \dots, \beta_n) \neq 0$ , let  $j_0 \in \{1, \dots, n\}$  such that

$$\|\beta\|_{\infty} = \max_{j \in \{1, \dots, n\}} |\beta_j| = |\beta_{j_0}|.$$

Consider  $\alpha = (\alpha_1, \dots, \alpha_n)$  with  $\alpha_{j_0} = \frac{\overline{\beta_{j_0}}}{|\beta_{j_0}|}$  and  $\alpha_j = 0$  for  $j \neq j_0$ . For this choice we get

$$\sum_{j=1}^n |\alpha_j| = \frac{|\overline{\beta_{j_0}}|}{|\beta_{j_0}|} = 1 \quad \text{and} \quad \left| \sum_{j=1}^n \alpha_j \beta_j \right| = \left| \frac{\overline{\beta_{j_0}}}{|\beta_{j_0}|} \beta_{j_0} \right| = |\beta_{j_0}| = \|\beta\|_{n,\infty},$$



therefore by (2.6) we obtain the second representation in (4). ■

**THEOREM 2.** *Let  $(T_1, \dots, T_n) \in B^{(n)}(H)$  and  $x, y \in H$ , then for  $p, q > 1$  and  $\frac{1}{p} + \frac{1}{q} = 1$  we have*

$$\sup_{\|\alpha\|_{n,p} \leq 1} \left| \left\langle \left( \sum_{j=1}^n \alpha_j T_j \right) x, y \right\rangle \right| = \left( \sum_{j=1}^n |\langle T_j x, y \rangle|^q \right)^{1/q} \quad (2.7)$$

and in particular

$$\sup_{\|\alpha\|_{n,2} \leq 1} \left| \left\langle \left( \sum_{j=1}^n \alpha_j T_j \right) x, y \right\rangle \right| = \left( \sum_{j=1}^n |\langle T_j x, y \rangle|^2 \right)^{1/2}. \quad (2.8)$$

We also have

$$\sup_{\|\alpha\|_{n,\infty} \leq 1} \left| \left\langle \left( \sum_{j=1}^n \alpha_j T_j \right) x, y \right\rangle \right| = \sum_{j=1}^n |\langle T_j x, y \rangle| \quad (2.9)$$

and

$$\sup_{\|\alpha\|_{n,1} \leq 1} \left| \left\langle \left( \sum_{j=1}^n \alpha_j T_j \right) x, y \right\rangle \right| = \max_{j \in \{1, \dots, n\}} \{|\langle T_j x, y \rangle|\}. \quad (2.10)$$

*Proof.* If we take  $\beta = (\langle T_1 x, y \rangle, \dots, \langle T_n x, y \rangle) \in \mathbb{C}^n$  in (2.1), then we get

$$\begin{aligned} \left( \sum_{j=1}^n |\langle T_j x, y \rangle|^q \right)^{1/q} &= \|\beta\|_{n,q} = \sup_{\|\alpha\|_p \leq 1} \left| \sum_{j=1}^n \alpha_j \beta_j \right| \\ &= \sup_{\|\alpha\|_{n,p} \leq 1} \left| \sum_{j=1}^n \alpha_j \langle T_j x, y \rangle \right| = \sup_{\|\alpha\|_{n,p} \leq 1} \left| \left\langle \sum_{j=1}^n \alpha_j T_j x, y \right\rangle \right|, \end{aligned}$$

which proves (2.7).

The equalities (2.9) and (2.10) follow by (2.3). ■

**COROLLARY 3.** *Let  $(T_1, \dots, T_n) \in B^{(n)}(H)$  and  $x \in H$ , then for  $p, q > 1$  and  $\frac{1}{p} + \frac{1}{q} = 1$  we have*

$$\sup_{\|\alpha\|_{n,p} \leq 1} \left| \left\langle \left( \sum_{j=1}^n \alpha_j T_j \right) x, x \right\rangle \right| = \left( \sum_{j=1}^n |\langle T_j x, x \rangle|^q \right)^{1/q} \quad (2.11)$$

and, in particular

$$\sup_{\|\alpha\|_{n,2} \leq 1} \left| \left\langle \left( \sum_{j=1}^n \alpha_j T_j \right) x, x \right\rangle \right| = \left( \sum_{j=1}^n |\langle T_j x, x \rangle|^2 \right)^{1/2}. \quad (2.12)$$

We also have

$$\sup_{\|\alpha\|_{n,\infty} \leq 1} \left| \left\langle \left( \sum_{j=1}^n \alpha_j T_j \right) x, x \right\rangle \right| = \sum_{j=1}^n |\langle T_j x, x \rangle| \quad (2.13)$$

and

$$\sup_{\|\alpha\|_{n,1} \leq 1} \left| \left\langle \left( \sum_{j=1}^n \alpha_j T_j \right) x, x \right\rangle \right| = \max_{j \in \{1, \dots, n\}} \{|\langle T_j x, x \rangle|\}. \quad (2.14)$$

COROLLARY 4. Let  $(T_1, \dots, T_n) \in B^{(n)}(H)$  and  $x \in H$ , then for  $p, q > 1$  and  $\frac{1}{p} + \frac{1}{q} = 1$  we have

$$\sup_{\|\alpha\|_{n,p} \leq 1} \left\| \sum_{j=1}^n \alpha_j T_j x \right\| = \sup_{\|y\|=1} \left( \sum_{j=1}^n |\langle T_j x, y \rangle|^q \right)^{1/q} \quad (2.15)$$

and in particular

$$\sup_{\|\alpha\|_{n,2} \leq 1} \left\| \sum_{j=1}^n \alpha_j T_j x \right\| = \sup_{\|y\|=1} \left( \sum_{j=1}^n |\langle T_j x, y \rangle|^2 \right)^{1/2}. \quad (2.16)$$

We also have

$$\sup_{\|\alpha\|_{n,\infty} \leq 1} \left\| \sum_{j=1}^n \alpha_j T_j x \right\| = \sup_{\|y\|=1} \sum_{j=1}^n |\langle T_j x, y \rangle| \quad (2.17)$$

and

$$\sup_{\|\alpha\|_{n,1} \leq 1} \left\| \sum_{j=1}^n \alpha_j T_j x \right\| = \max_{j \in \{1, \dots, n\}} \{\|T_j x\|\}. \quad (2.18)$$

*Proof.* By the properties of inner product, we have for any  $u \in H$ ,  $u \neq 0$  that

$$\|u\| = \sup_{\|y\|=1} |\langle u, y \rangle|.$$

Let  $x \in H$ , then by taking the supremum over  $\|y\| = 1$  in (2.7) we get for  $p, q > 1$  with  $\frac{1}{p} + \frac{1}{q} = 1$  that

$$\begin{aligned} \sup_{\|y\|=1} \left( \sum_{j=1}^n |\langle T_j x, y \rangle|^q \right)^{1/q} &= \sup_{\|y\|=1} \left( \sup_{\|\alpha\|_{n,p} \leq 1} \left| \left\langle \left( \sum_{j=1}^n \alpha_j T_j \right) x, y \right\rangle \right| \right) \\ &= \sup_{\|\alpha\|_{n,p} \leq 1} \left( \sup_{\|y\|=1} \left| \left\langle \left( \sum_{j=1}^n \alpha_j T_j \right) x, y \right\rangle \right| \right) \\ &= \sup_{\|\alpha\|_{n,p} \leq 1} \left\| \left( \sum_{j=1}^n \alpha_j T_j \right) x \right\|, \end{aligned}$$

which proves the equality (2.15).

The other equalities can be proved in a similar way by using Theorem 2, however the details are omitted. ■

We can state and prove our main result.

**THEOREM 5.** *Let  $(T_1, \dots, T_n) \in B^{(n)}(H)$ .*

(i) *For  $q \geq 1$  we have the representation for the hypo- $q$ -norm*

$$\begin{aligned} \|(T_1, \dots, T_n)\|_{h,n,q} &= \sup_{\|x\|=\|y\|=1} \left( \sum_{j=1}^n |\langle T_j x, y \rangle|^q \right)^{1/q} \\ &= \|(T_1, \dots, T_n)\|_{s,q} \end{aligned} \quad (2.19)$$

*and in particular*

$$\|(T_1, \dots, T_n)\|_{n,e} = \sup_{\|x\|=\|y\|=1} \left( \sum_{j=1}^n |\langle T_j x, y \rangle|^2 \right)^{1/2}. \quad (2.20)$$

*We also have*

$$\|(T_1, \dots, T_n)\|_{h,n,\infty} = \max_{j \in \{1, \dots, n\}} \{\|T_j\|\}. \quad (2.21)$$

(ii) For  $q \geq 1$  we have the representation for the hypo-numerical radius

$$\begin{aligned} w_{h,n,q}(T_1, \dots, T_n) &= \sup_{\|x\|=1} \left( \sum_{j=1}^n |\langle T_j x, x \rangle|^q \right)^{1/q} \\ &= w_{s,q}(T_1, \dots, T_n) \end{aligned} \quad (2.22)$$

and in particular

$$w_{n,e}(T_1, \dots, T_n) := \sup_{\|x\|=1} \left( \sum_{j=1}^n |\langle T_j x, x \rangle|^2 \right)^{1/2}. \quad (2.23)$$

We also have

$$w_{h,n,\infty}(T_1, \dots, T_n) = \max_{j \in \{1, \dots, n\}} \{w(T_j)\}. \quad (2.24)$$

*Proof.* (i) By using the equality (2.15) we have for  $(T_1, \dots, T_n) \in B^{(n)}(H)$  that

$$\begin{aligned} \sup_{\|x\|=\|y\|=1} \left( \sum_{j=1}^n |\langle T_j x, y \rangle|^q \right)^{1/q} &= \sup_{\|x\|=1} \left( \sup_{\|y\|=1} \left( \sum_{j=1}^n |\langle T_j x, y \rangle|^q \right)^{1/q} \right) \\ &= \sup_{\|x\|=1} \left( \sup_{\|\alpha\|_{n,p} \leq 1} \left\| \sum_{j=1}^n \alpha_j T_j x \right\| \right) \\ &= \sup_{\|\alpha\|_{n,p} \leq 1} \left( \sup_{\|x\|=1} \left\| \sum_{j=1}^n \alpha_j T_j x \right\| \right) \\ &= \sup_{\|\alpha\|_{n,p} \leq 1} \left\| \sum_{j=1}^n \alpha_j T_j \right\| \\ &= \|(T_1, \dots, T_n)\|_{h,n,q}, \end{aligned}$$

which proves (2.19). The rest is obvious.

(ii) By using the equality (2.11) we have for  $(T_1, \dots, T_n) \in B^{(n)}(H)$  that

$$\begin{aligned} \sup_{\|x\|=1} \left( \sum_{j=1}^n |\langle T_j x, x \rangle|^q \right)^{1/q} &= \sup_{\|x\|=1} \left( \sup_{\|\alpha\|_{n,p} \leq 1} \left| \left\langle \left( \sum_{j=1}^n \alpha_j T_j \right) x, x \right\rangle \right| \right) \\ &= \sup_{\|\alpha\|_{n,p} \leq 1} \left( \sup_{\|x\|=1} \left| \left\langle \left( \sum_{j=1}^n \alpha_j T_j \right) x, x \right\rangle \right| \right) \\ &= \sup_{\|\alpha\|_{n,p} \leq 1} w \left( \sum_{j=1}^n \alpha_j T_j \right) = w_{h,n,q}(T_1, \dots, T_n), \end{aligned}$$

which proves (2.22). The rest is obvious. ■

*Remark 6.* The case  $q = 2$  was obtained in a different manner in [5] by utilising the rotation-invariant normalised positive Borel measure on the unit sphere.

We can consider on  $B^{(n)}(H)$  the following usual operator and numerical radius  $q$ -norms, for  $q \geq 1$

$$\begin{aligned} \|(T_1, \dots, T_n)\|_{n,q} &:= \left( \sum_{j=1}^n \|T_j\|^q \right)^{1/q}, \\ w_{n,q}(T_1, \dots, T_n) &:= \left( \sum_{j=1}^n w^q(T_j) \right)^{1/q}, \end{aligned}$$

where  $(T_1, \dots, T_n) \in B^{(n)}(H)$ . For  $q = \infty$  we put

$$\begin{aligned} \|(T_1, \dots, T_n)\|_{n,\infty} &:= \max_{j \in \{1, \dots, n\}} \{\|T_j\|\}, \\ w_{n,\infty}(T_1, \dots, T_n) &:= \max_{j \in \{1, \dots, n\}} \{w(T_j)\}. \end{aligned}$$

**COROLLARY 7.** *With the assumptions of Theorem 5 we have for  $q \geq 1$  that*

$$\frac{1}{n^{1/q}} \|(T_1, \dots, T_n)\|_{n,q} \leq \|(T_1, \dots, T_n)\|_{h,n,q} \leq \|(T_1, \dots, T_n)\|_{n,q} \quad (2.25)$$

and

$$\frac{1}{n^{1/q}} w_{n,q}(T_1, \dots, T_n) \leq w_{h,n,q}(T_1, \dots, T_n) \leq w_{n,q}(T_1, \dots, T_n) \quad (2.26)$$

for any  $(T_1, \dots, T_n) \in B^{(n)}(H)$ .

In particular, we have [5]

$$\frac{1}{\sqrt{n}} \|(T_1, \dots, T_n)\|_{n,2} \leq \|(T_1, \dots, T_n)\|_{h,n,e} \leq \|(T_1, \dots, T_n)\|_{n,2} \quad (2.27)$$

and

$$\frac{1}{\sqrt{n}} w_{n,2}(T_1, \dots, T_n) \leq w_{h,n,e}(T_1, \dots, T_n) \leq w_{n,2}(T_1, \dots, T_n) \quad (2.28)$$

for any  $(T_1, \dots, T_n) \in B^{(n)}(H)$ .

*Proof.* Let  $(T_1, \dots, T_n) \in B^{(n)}(H)$  and  $x, y \in H$  with  $\|x\| = \|y\| = 1$ . Then by Schwarz's inequality we have

$$\left( \sum_{j=1}^n |\langle T_j x, y \rangle|^q \right)^{1/q} \leq \left( \sum_{j=1}^n \|T_j x\|^q \|y\|^q \right)^{1/q} = \left( \sum_{j=1}^n \|T_j x\|^q \right)^{1/q}.$$

By the operator norm inequality we also have

$$\left( \sum_{j=1}^n \|T_j x\|^q \right)^{1/q} \leq \left( \sum_{j=1}^n \|T_j\|^q \|x\|^q \right)^{1/q} = \|(T_1, \dots, T_n)\|_{n,q}.$$

Therefore

$$\left( \sum_{j=1}^n |\langle T_j x, y \rangle|^q \right)^{1/q} \leq \|(T_1, \dots, T_n)\|_{n,q}$$

and by taking the supremum over  $\|x\| = \|y\| = 1$  we get the second inequality in (2.25).

By the properties of complex numbers, we have

$$\max_{j \in \{1, \dots, n\}} \{|\langle T_j x, y \rangle|\} \leq \left( \sum_{j=1}^n |\langle T_j x, y \rangle|^q \right)^{1/q}$$

$x, y \in H$  with  $\|x\| = \|y\| = 1$ .

By taking the supremum over  $\|x\| = \|y\| = 1$  we get

$$\sup_{\|x\|=\|y\|=1} \left( \max_{j \in \{1, \dots, n\}} \{|\langle T_j x, y \rangle|\} \right) \leq \|(T_1, \dots, T_n)\|_{h,n,q} \quad (2.29)$$

and since

$$\begin{aligned} \sup_{\|x\|=\|y\|=1} \left( \max_{j \in \{1, \dots, n\}} \{|\langle T_j x, y \rangle|\} \right) &= \max_{j \in \{1, \dots, n\}} \left\{ \sup_{\|x\|=\|y\|=1} |\langle T_j x, y \rangle| \right\} \\ &= \max_{j \in \{1, \dots, n\}} \{\|T_j\|\} = \|(T_1, \dots, T_n)\|_{n,\infty}, \end{aligned}$$

then by (2.29) we get

$$\|(T_1, \dots, T_n)\|_{n,\infty} \leq \|(T_1, \dots, T_n)\|_{h,n,q} \quad (2.30)$$

for any  $(T_1, \dots, T_n) \in B^{(n)}(H)$ .

Since

$$\begin{aligned} \|(T_1, \dots, T_n)\|_{n,q} &:= \left( \sum_{j=1}^n \|T_j\|^q \right)^{1/q} \leq (n \|(T_1, \dots, T_n)\|_{n,\infty}^q)^{1/q} \\ &= n^{1/q} \|(T_1, \dots, T_n)\|_{n,\infty}, \end{aligned} \quad (2.31)$$

then by (2.30) and (2.31) we get

$$\frac{1}{n^{1/q}} \|(T_1, \dots, T_n)\|_{n,q} \leq \|(T_1, \dots, T_n)\|_{h,n,q}$$

for any  $(T_1, \dots, T_n) \in B^{(n)}(H)$ .

The inequality (2.26) follows in a similar way and we omit the details. ■

**COROLLARY 8.** *With the assumptions of Theorem 5 we have for  $r \geq q \geq 1$  that*

$$\|(T_1, \dots, T_n)\|_{h,n,r} \leq \|(T_1, \dots, T_n)\|_{h,n,q} \leq n^{\frac{r-q}{rq}} \|(T_1, \dots, T_n)\|_{h,n,r} \quad (2.32)$$

and [14]

$$w_{h,n,r}(T_1, \dots, T_n) \leq w_{h,n,q}(T_1, \dots, T_n) \leq n^{\frac{r-q}{rq}} w_{h,n,r}(T_1, \dots, T_n) \quad (2.33)$$

for any  $(T_1, \dots, T_n) \in B^{(n)}(H)$ .

*Proof.* We use the following elementary inequalities for the nonnegative numbers  $a_j$ ,  $j = 1, \dots, n$  and  $r \geq q > 0$  (see for instance [14])

$$\left( \sum_{j=1}^n a_j^r \right)^{1/r} \leq \left( \sum_{j=1}^n a_j^q \right)^{1/q} \leq n^{\frac{r-q}{rq}} \left( \sum_{j=1}^n a_j^r \right)^{1/r}. \quad (2.34)$$

Let  $(T_1, \dots, T_n) \in B^{(n)}(H)$  and  $x, y \in H$  with  $\|x\| = \|y\| = 1$ . Then by (2.34) we get

$$\left( \sum_{j=1}^n |\langle T_j x, y \rangle|^r \right)^{1/r} \leq \left( \sum_{j=1}^n |\langle T_j x, y \rangle|^q \right)^{1/q} \leq n^{\frac{r-q}{rq}} \left( \sum_{j=1}^n |\langle T_j x, y \rangle|^r \right)^{1/r}.$$

By taking the supremum over  $\|x\| = \|y\| = 1$  we get (2.32).

The inequality (2.33) follows in a similar way and we omit the details. ■

*Remark 9.* For  $q \geq 2$  we have by (2.32) and (2.33)

$$\|(T_1, \dots, T_n)\|_{h,n,q} \leq \|(T_1, \dots, T_n)\|_{h,n,e} \leq n^{\frac{q-2}{2q}} \|(T_1, \dots, T_n)\|_{h,n,q} \quad (2.35)$$

and

$$w_{h,n,q}(T_1, \dots, T_n) \leq w_{h,n,e}(T_1, \dots, T_n) \leq n^{\frac{q-2}{2q}} w_{h,n,q}(T_1, \dots, T_n) \quad (2.36)$$

and for  $1 \leq q \leq 2$  we have

$$\|(T_1, \dots, T_n)\|_{h,n,e} \leq \|(T_1, \dots, T_n)\|_{h,n,q} \leq n^{\frac{2-q}{2q}} \|(T_1, \dots, T_n)\|_{h,n,e} \quad (2.37)$$

and

$$w_{h,n,e}(T_1, \dots, T_n) \leq w_{h,n,q}(T_1, \dots, T_n) \leq n^{\frac{2-q}{2q}} w_{h,n,e}(T_1, \dots, T_n) \quad (2.38)$$

for any  $(T_1, \dots, T_n) \in B^{(n)}(H)$ .

Also, if we take  $q = 1$  and  $r \geq 1$  in (2.32) and (2.33), then we get

$$\|(T_1, \dots, T_n)\|_{h,n,r} \leq \|(T_1, \dots, T_n)\|_{h,n,1} \leq n^{\frac{r-1}{r}} \|(T_1, \dots, T_n)\|_{h,n,r} \quad (2.39)$$

and

$$w_{h,n,r}(T_1, \dots, T_n) \leq w_{h,n,1}(T_1, \dots, T_n) \leq n^{\frac{r-1}{r}} w_{h,n,r}(T_1, \dots, T_n) \quad (2.40)$$

for any  $(T_1, \dots, T_n) \in B^{(n)}(H)$ .



In particular, for  $r = 2$  we get

$$\|(T_1, \dots, T_n)\|_{h,n,e} \leq \|(T_1, \dots, T_n)\|_{h,n,1} \leq \sqrt{n} \|(T_1, \dots, T_n)\|_{h,n,e} \quad (2.41)$$

and

$$w_{n,e}(T_1, \dots, T_n) \leq w_{h,n,1}(T_1, \dots, T_n) \leq \sqrt{n} w_{n,e}(T_1, \dots, T_n) \quad (2.42)$$

for any  $(T_1, \dots, T_n) \in B^{(n)}(H)$ .

We have:

PROPOSITION 10. For any  $(T_1, \dots, T_n) \in B^{(n)}(H)$  and  $p, q > 1$  with  $\frac{1}{p} + \frac{1}{q} = 1$ , then we have

$$\|(T_1, \dots, T_n)\|_{h,n,q} \geq \frac{1}{n^{1/p}} \left\| \sum_{j=1}^n T_j \right\| \quad (2.43)$$

and

$$w_{h,n,q}(T_1, \dots, T_n) \geq \frac{1}{n^{1/p}} w \left( \sum_{j=1}^n T_j \right). \quad (2.44)$$

*Proof.* Let  $\lambda_j = \frac{1}{n^{1/p}}$  for  $j \in \{1, \dots, n\}$ , then  $\sum_{j=1}^n |\lambda_j|^p = 1$ . Therefore by (1.8) we get

$$\|(T_1, \dots, T_n)\|_{h,n,q} = \sup_{\|\lambda\|_{n,p} \leq 1} \left\| \sum_{j=1}^n \lambda_j T_j \right\| \geq \left\| \sum_{j=1}^n \frac{1}{n^{1/p}} T_j \right\| = \frac{1}{n^{1/p}} \left\| \sum_{j=1}^n T_j \right\|.$$

The inequality (2.44) follows in a similar way. ■

We can also introduce the following norms for  $(T_1, \dots, T_n) \in B^{(n)}(H)$ ,

$$\|(T_1, \dots, T_n)\|_{s,n,p} := \sup_{\|x\|=1} \left( \sum_{j=1}^n \|T_j x\|^p \right)^{1/p}$$

where  $p \geq 1$  and

$$\|(T_1, \dots, T_n)\|_{s,n,\infty} := \sup_{\|x\|=1} \left( \max_{j \in \{1, \dots, n\}} \|T_j x\| \right) = \max_{j \in \{1, \dots, n\}} \{\|T_j\|\}.$$

The triangle inequality  $\|\cdot\|_{s,n,q}$  follows by Minkowski inequality, while the other properties of the norm are obvious.

PROPOSITION 11. Let  $(T_1, \dots, T_n) \in B^{(n)}(H)$ .

(i) We have for  $p \geq 1$ , that

$$\begin{aligned} \|(T_1, \dots, T_n)\|_{h,n,p} &\leq \|(T_1, \dots, T_n)\|_{s,n,p} \\ &\leq \|(T_1, \dots, T_n)\|_{n,p}, \end{aligned} \quad (2.45)$$

(ii) For  $p \geq 2$  we also have

$$\|(T_1, \dots, T_n)\|_{s,n,p} = \left[ w_{h,n,p/2}(|T_1|^2, \dots, |T_n|^2) \right]^{1/2}, \quad (2.46)$$

where the absolute value  $|T|$  is defined by  $|T| := (T^*T)^{1/2}$ .

*Proof.* (i) We have for  $p \geq 2$  and  $x, y \in H$  with  $\|x\| = \|y\| = 1$ , that

$$\begin{aligned} |\langle T_j x, y \rangle|^p &\leq \|T_j x\|^p \|y\|^p \\ &= \|T_j x\|^p \leq \|T_j\|^p \|x\|^p = \|T_j\|^p \end{aligned}$$

for  $j \in \{1, \dots, n\}$ .

This implies

$$\sum_{j=1}^n |\langle T_j x, y \rangle|^p \leq \sum_{j=1}^n \|T_j x\|^p \leq \sum_{j=1}^n \|T_j\|^p,$$

namely

$$\left( \sum_{j=1}^n |\langle T_j x, y \rangle|^p \right)^{1/p} \leq \left( \sum_{j=1}^n \|T_j x\|^p \right)^{1/p} \leq \left( \sum_{j=1}^n \|T_j\|^p \right)^{1/p}, \quad (2.47)$$

for any  $x, y \in H$  with  $\|x\| = \|y\| = 1$ .

Taking the supremum over  $\|x\| = \|y\| = 1$  in (2.47), we get the desired result (2.45).

(ii) We have

$$\begin{aligned}
& \|(T_1, \dots, T_n)\|_{s,n,p} \\
&= \sup_{\|x\|=1} \left( \sum_{j=1}^n \|T_j x\|^p \right)^{1/p} = \sup_{\|x\|=1} \left( \sum_{j=1}^n (\|T_j x\|^2)^{p/2} \right)^{1/p} \\
&= \sup_{\|x\|=1} \left( \sum_{j=1}^n \langle T_j x, T_j x \rangle^{p/2} \right)^{1/p} = \sup_{\|x\|=1} \left( \sum_{j=1}^n \langle T_j^* T_j x, x \rangle^{p/2} \right)^{1/p} \\
&= \sup_{\|x\|=1} \left( \sum_{j=1}^n \langle |T_j|^2 x, x \rangle^{p/2} \right)^{1/p} = \left[ \sup_{\|x\|=1} \left( \sum_{j=1}^n \langle |T_j|^2 x, x \rangle^{p/2} \right)^{1/(p/2)} \right]^{1/2} \\
&= \left[ w_{h,n,p/2}(|T_1|^2, \dots, |T_n|^2) \right]^{1/2},
\end{aligned}$$

which proves the equality (2.46). ■

### 3. SOME REVERSE INEQUALITIES

Recall the following reverse of Cauchy-Buniakowski-Schwarz inequality [2] (see also [3, Theorem 5.14]):

LEMMA 12. Let  $a, A \in \mathbb{R}$  and  $\mathbf{z} = (z_1, \dots, z_n)$ ,  $\mathbf{y} = (y_1, \dots, y_n)$  be two sequences of real numbers with the property that:

$$ay_j \leq z_j \leq Ay_j \quad \text{for each } j \in \{1, \dots, n\}. \quad (3.1)$$

Then for any  $\mathbf{w} = (w_1, \dots, w_n)$  a sequence of positive real numbers, one has the inequality

$$0 \leq \sum_{j=1}^n w_j z_j^2 \sum_{j=1}^n w_j y_j^2 - \left( \sum_{j=1}^n w_j z_j y_j \right)^2 \leq \frac{1}{4} (A - a)^2 \left( \sum_{j=1}^n w_j y_j^2 \right)^2. \quad (3.2)$$

The constant  $\frac{1}{4}$  is sharp in (3.2).

O. Shisha and B. Mond obtained in 1967 (see [15]) the following counterparts of (CBS)-inequality (see also [3, Theorem 5.20 & 5.21]):

LEMMA 13. Assume that  $\mathbf{a} = (a_1, \dots, a_n)$  and  $\mathbf{b} = (b_1, \dots, b_n)$  are such that there exists  $a, A, b, B$  with the property that:

$$0 \leq a \leq a_j \leq A \quad \text{and} \quad 0 < b \leq b_j \leq B \quad \text{for any } j \in \{1, \dots, n\}, \quad (3.3)$$

then we have the inequality

$$\sum_{j=1}^n a_j^2 \sum_{j=1}^n b_j^2 - \left( \sum_{j=1}^n a_j b_j \right)^2 \leq \left( \sqrt{\frac{A}{b}} - \sqrt{\frac{a}{B}} \right)^2 \sum_{j=1}^n a_j b_j \sum_{j=1}^n b_j^2. \quad (3.4)$$

and

LEMMA 14. Assume that  $\mathbf{a}, \mathbf{b}$  are nonnegative sequences and there exists  $\gamma, \Gamma$  with the property that

$$0 \leq \gamma \leq \frac{a_j}{b_j} \leq \Gamma < \infty \quad \text{for any } j \in \{1, \dots, n\}. \quad (3.5)$$

Then we have the inequality

$$0 \leq \left( \sum_{j=1}^n a_j^2 \sum_{j=1}^n b_j^2 \right)^{\frac{1}{2}} - \sum_{j=1}^n a_j b_j \leq \frac{(\Gamma - \gamma)^2}{4(\gamma + \Gamma)} \sum_{j=1}^n b_j^2. \quad (3.6)$$

We have:

THEOREM 15. Let  $(T_1, \dots, T_n) \in B^{(n)}(H)$ .

(i) We have

$$\begin{aligned} 0 &\leq \|(T_1, \dots, T_n)\|_{h,n,e}^2 - \frac{1}{n} \|(T_1, \dots, T_n)\|_{h,n,1}^2 \\ &\leq \frac{1}{4} n \|(T_1, \dots, T_n)\|_{n,\infty}^2 \end{aligned} \quad (3.7)$$

and

$$\begin{aligned} 0 &\leq w_{n,e}^2(T_1, \dots, T_n) - \frac{1}{n} w_{h,n,1}^2(T_1, \dots, T_n) \\ &\leq \frac{1}{4} n \|(T_1, \dots, T_n)\|_{n,\infty}^2. \end{aligned} \quad (3.8)$$

(ii) We have

$$\begin{aligned} 0 &\leq \|(T_1, \dots, T_n)\|_{h,n,e}^2 - \frac{1}{n} \|(T_1, \dots, T_n)\|_{h,n,1}^2 \\ &\leq \|(T_1, \dots, T_n)\|_{n,\infty} \|(T_1, \dots, T_n)\|_{h,n,1} \end{aligned} \quad (3.9)$$

and

$$\begin{aligned} 0 &\leq w_{n,e}^2(T_1, \dots, T_n) - \frac{1}{n} w_{h,n,1}^2(T_1, \dots, T_n) \\ &\leq \|(T_1, \dots, T_n)\|_{n,\infty} w_{h,n,1}(T_1, \dots, T_n). \end{aligned} \quad (3.10)$$

(iii) We have

$$\begin{aligned} 0 &\leq \|(T_1, \dots, T_n)\|_{h,n,e} - \frac{1}{\sqrt{n}} \|(T_1, \dots, T_n)\|_{h,n,1} \\ &\leq \frac{1}{4} \sqrt{n} \|(T_1, \dots, T_n)\|_{n,\infty} \end{aligned} \quad (3.11)$$

and

$$\begin{aligned} 0 &\leq w_{n,e}(T_1, \dots, T_n) - \frac{1}{\sqrt{n}} w_{h,n,1}(T_1, \dots, T_n) \\ &\leq \frac{1}{4} \sqrt{n} \|(T_1, \dots, T_n)\|_{n,\infty}. \end{aligned} \quad (3.12)$$

*Proof.* (i) Let  $(T_1, \dots, T_n) \in B^{(n)}(H)$  and put

$$R = \max_{j \in \{1, \dots, n\}} \{\|T_j\|\} = \|(T_1, \dots, T_n)\|_{n,\infty}.$$

If  $x, y \in H$ , with  $\|x\| = \|y\| = 1$  then  $|\langle T_j x, y \rangle| \leq \|T_j x\| \leq \|T_j\| \leq R$  for any  $j \in \{1, \dots, n\}$ .

If we write the inequality (3.2) for  $z_j = |\langle T_j x, y \rangle|$ ,  $w_j = y_j = 1$ ,  $A = R$  and  $a = 0$ , we get

$$0 \leq n \sum_{j=1}^n |\langle T_j x, y \rangle|^2 - \left( \sum_{j=1}^n |\langle T_j x, y \rangle| \right)^2 \leq \frac{1}{4} n^2 R^2$$

for any  $x, y \in H$ , with  $\|x\| = \|y\| = 1$ .

This implies that

$$\sum_{j=1}^n |\langle T_j x, y \rangle|^2 \leq \frac{1}{n} \left( \sum_{j=1}^n |\langle T_j x, y \rangle| \right)^2 + \frac{1}{4} n R^2 \quad (3.13)$$

for any  $x, y \in H$ , with  $\|x\| = \|y\| = 1$  and, in particular

$$\sum_{j=1}^n |\langle T_j x, x \rangle|^2 \leq \frac{1}{n} \left( \sum_{j=1}^n |\langle T_j x, x \rangle| \right)^2 + \frac{1}{4} n R^2 \quad (3.14)$$

for any  $x \in H$ , with  $\|x\| = 1$ .

Taking the supremum over  $\|x\| = \|y\| = 1$  in (3.13) and  $\|x\| = 1$  in (3.14), then we get (3.7) and (3.8).

(ii) Let  $(T_1, \dots, T_n) \in B^{(n)}(H)$ . If we write the inequality (3.4) for  $a_j = |\langle T_j x, y \rangle|$ ,  $b_j = 1$ ,  $b = B = 1$ ,  $a = 0$  and  $A = R$ , then we get

$$0 \leq n \sum_{j=1}^n |\langle T_j x, y \rangle|^2 - \left( \sum_{j=1}^n |\langle T_j x, y \rangle| \right)^2 \leq n R \sum_{j=1}^n |\langle T_j x, y \rangle|,$$

for any  $x, y \in H$ , with  $\|x\| = \|y\| = 1$ .

This implies that

$$\sum_{j=1}^n |\langle T_j x, y \rangle|^2 \leq \frac{1}{n} \left( \sum_{j=1}^n |\langle T_j x, y \rangle| \right)^2 + R \sum_{j=1}^n |\langle T_j x, y \rangle|, \quad (3.15)$$

for any  $x, y \in H$ , with  $\|x\| = \|y\| = 1$  and, in particular

$$\sum_{j=1}^n |\langle T_j x, x \rangle|^2 \leq \frac{1}{n} \left( \sum_{j=1}^n |\langle T_j x, x \rangle| \right)^2 + R \sum_{j=1}^n |\langle T_j x, x \rangle|, \quad (3.16)$$

for any  $x \in H$  with  $\|x\| = 1$ .

Taking the supremum over  $\|x\| = \|y\| = 1$  in (3.15) and  $\|x\| = 1$  in (3.16), then we get (3.9) and (3.10).

(iii) If we write the inequality (3.6) for  $a_j = |\langle T_j x, y \rangle|$ ,  $b_j = 1$ ,  $b = B = 1$ ,  $\gamma = 0$  and  $\Gamma = R$  we have

$$0 \leq \left( n \sum_{j=1}^n |\langle T_j x, y \rangle|^2 \right)^{\frac{1}{2}} - \sum_{j=1}^n |\langle T_j x, y \rangle| \leq \frac{1}{4} n R,$$

for any  $x, y \in H$ , with  $\|x\| = \|y\| = 1$ .

This implies that

$$\left( \sum_{j=1}^n |\langle T_j x, y \rangle|^2 \right)^{\frac{1}{2}} \leq \frac{1}{\sqrt{n}} \sum_{j=1}^n |\langle T_j x, y \rangle| + \frac{1}{4} \sqrt{n} R, \quad (3.17)$$

for any  $x, y \in H$ , with  $\|x\| = \|y\| = 1$  and, in particular

$$\left( \sum_{j=1}^n |\langle T_j x, x \rangle|^2 \right)^{\frac{1}{2}} \leq \frac{1}{\sqrt{n}} \sum_{j=1}^n |\langle T_j x, x \rangle| + \frac{1}{4} \sqrt{n} R, \quad (3.18)$$

for any  $x \in H$  with  $\|x\| = 1$ .

Taking the supremum over  $\|x\| = \|y\| = 1$  in (3.17) and  $\|x\| = 1$  in (3.18), then we get (3.11) and (3.12). ■

Before we proceed with establishing some reverse inequalities for the hypo-Euclidean numerical radius, we recall some reverse results of the Cauchy-Bunyakovsky-Schwarz inequality for complex numbers as follows:

If  $\gamma, \Gamma \in \mathbb{C}$  and  $\alpha_j \in \mathbb{C}$ ,  $j \in \{1, \dots, n\}$  with the property that

$$\begin{aligned} 0 &\leq \operatorname{Re}[(\Gamma - \alpha_j)(\overline{\alpha_j} - \overline{\gamma})] \\ &= (\operatorname{Re} \Gamma - \operatorname{Re} \alpha_j)(\operatorname{Re} \alpha_j - \operatorname{Re} \gamma) + (\operatorname{Im} \Gamma - \operatorname{Im} \alpha_j)(\operatorname{Im} \alpha_j - \operatorname{Im} \gamma) \end{aligned} \quad (3.19)$$

or, equivalently,

$$\left| \alpha_j - \frac{\gamma + \Gamma}{2} \right| \leq \frac{1}{2} |\Gamma - \gamma| \quad (3.20)$$

for each  $j \in \{1, \dots, n\}$ , then (see for instance [4, p. 9])

$$n \sum_{j=1}^n |\alpha_j|^2 - \left| \sum_{j=1}^n \alpha_j \right|^2 \leq \frac{1}{4} n^2 |\Gamma - \gamma|^2. \quad (3.21)$$

In addition, if  $\operatorname{Re}(\Gamma \overline{\gamma}) > 0$ , then (see for example [4, p. 26]):

$$\begin{aligned} n \sum_{j=1}^n |\alpha_j|^2 &\leq \frac{1}{4} \frac{\left\{ \operatorname{Re}[(\overline{\Gamma} + \overline{\gamma}) \sum_{j=1}^n \alpha_j] \right\}^2}{\operatorname{Re}(\Gamma \overline{\gamma})} (\Gamma \overline{\gamma}) \\ &\leq \frac{1}{4} \frac{|\Gamma + \gamma|^2}{\operatorname{Re}(\Gamma \overline{\gamma})} \left| \sum_{j=1}^n \alpha_j \right|^2. \end{aligned} \quad (3.22)$$

Also, if  $\Gamma \neq -\gamma$ , then (see for instance [4, p. 32]):

$$\left( n \sum_{j=1}^n |\alpha_j|^2 \right)^{\frac{1}{2}} - \left| \sum_{j=1}^n \alpha_j \right| \leq \frac{1}{4} n \frac{|\Gamma - \gamma|^2}{|\Gamma + \gamma|}. \quad (3.23)$$

Finally, from [7] we can also state that

$$n \sum_{j=1}^n |\alpha_j|^2 - \left| \sum_{j=1}^n \alpha_j \right|^2 \leq n \left[ |\Gamma + \gamma| - 2\sqrt{\operatorname{Re}(\Gamma\bar{\gamma})} \right] \left| \sum_{j=1}^n \alpha_j \right|, \quad (3.24)$$

provided  $\operatorname{Re}(\Gamma\bar{\gamma}) > 0$ .

We notice that a simple sufficient condition for (3.19) to hold is that

$$\operatorname{Re} \Gamma \geq \operatorname{Re} \alpha_j \geq \operatorname{Re} \gamma \quad \text{and} \quad \operatorname{Im} \Gamma \geq \operatorname{Im} \alpha_j \geq \operatorname{Im} \gamma \quad (3.25)$$

for each  $j \in \{1, \dots, n\}$ .

**THEOREM 16.** *Let  $(T_1, \dots, T_n) \in B^{(n)}(H)$  and  $\gamma, \Gamma \in \mathbb{C}$  with  $\Gamma \neq \gamma$ . Assume that*

$$w\left(T_j - \frac{\gamma + \Gamma}{2} I\right) \leq \frac{1}{2} |\Gamma - \gamma| \quad \text{for any } j \in \{1, \dots, n\}. \quad (3.26)$$

(i) *We have*

$$w_{h,n,e}^2(T_1, \dots, T_n) \leq \frac{1}{n} w^2\left(\sum_{j=1}^n T_j\right) + \frac{1}{4} n |\Gamma - \gamma|^2. \quad (3.27)$$

(ii) *If  $\operatorname{Re}(\Gamma\bar{\gamma}) > 0$ , then*

$$w_{h,n,e}(T_1, \dots, T_n) \leq \frac{1}{2\sqrt{n}} \frac{|\Gamma + \gamma|}{\sqrt{\operatorname{Re}(\Gamma\bar{\gamma})}} w\left(\sum_{j=1}^n T_j\right) \quad (3.28)$$

and

$$\begin{aligned} & w_{h,n,e}^2(T_1, \dots, T_n) \\ & \leq \left[ \frac{1}{n} w\left(\sum_{j=1}^n T_j\right) + \left[ |\Gamma + \gamma| - 2\sqrt{\operatorname{Re}(\Gamma\bar{\gamma})} \right] \right] \cdot w\left(\sum_{j=1}^n T_j\right). \end{aligned} \quad (3.29)$$



(iii) If  $\Gamma \neq -\gamma$ , then

$$w_{h,n,e}(T_1, \dots, T_n) \leq \frac{1}{\sqrt{n}} \left( w \left( \sum_{j=1}^n T_j \right) + \frac{1}{4} \frac{|\Gamma - \gamma|^2}{|\Gamma + \gamma|} \right). \quad (3.30)$$

*Proof.* Let  $x \in H$  with  $\|x\| = 1$  and  $(T_1, \dots, T_n) \in B^{(n)}(H)$  with the property (3.26). By taking  $\alpha_j = \langle T_j x, x \rangle$  we have

$$\begin{aligned} \left| \alpha_j - \frac{\gamma + \Gamma}{2} \right| &= \left| \langle T_j x, x \rangle - \frac{\gamma + \Gamma}{2} \langle x, x \rangle \right| = \left| \left\langle \left( T_j - \frac{\gamma + \Gamma}{2} I \right) x, x \right\rangle \right| \\ &\leq \sup_{\|x\|=1} \left| \left\langle \left( T_j - \frac{\gamma + \Gamma}{2} I \right) x, x \right\rangle \right| \\ &= w \left( T_j - \frac{\gamma + \Gamma}{2} I \right) \leq \frac{1}{2} |\Gamma - \gamma| \end{aligned}$$

for any  $j \in \{1, \dots, n\}$ .

(i) By using the inequality (3.21), we have

$$\begin{aligned} \sum_{j=1}^n |\langle T_j x, x \rangle|^2 &\leq \frac{1}{n} \left| \sum_{j=1}^n \langle T_j x, x \rangle \right|^2 + \frac{1}{4} n |\Gamma - \gamma|^2 \\ &= \frac{1}{n} \left| \left\langle \sum_{j=1}^n T_j x, x \right\rangle \right|^2 + \frac{1}{4} n |\Gamma - \gamma|^2 \end{aligned} \quad (3.31)$$

for any  $x \in H$  with  $\|x\| = 1$ .

By taking the supremum over  $\|x\| = 1$  in (3.31) we get

$$\begin{aligned} \sup_{\|x\|=1} \left( \sum_{j=1}^n |\langle T_j x, x \rangle|^2 \right) &\leq \frac{1}{n} \sup_{\|x\|=1} \left| \left\langle \sum_{j=1}^n T_j x, x \right\rangle \right|^2 + \frac{1}{4} n |\Gamma - \gamma|^2 \\ &= \frac{1}{n} w^2 \left( \sum_{j=1}^n T_j \right) + \frac{1}{4} n |\Gamma - \gamma|^2, \end{aligned}$$

which proves (3.27).

(ii) If  $\operatorname{Re}(\Gamma\bar{\gamma}) > 0$ , then by (3.22) we have for  $\alpha_j = \langle T_j x, x \rangle$ ,  $j \in \{1, \dots, n\}$  that

$$\begin{aligned} \sum_{j=1}^n |\langle T_j x, x \rangle|^2 &\leq \frac{1}{4n} \frac{|\Gamma + \gamma|^2}{\operatorname{Re}(\Gamma\bar{\gamma})} \left| \sum_{j=1}^n \langle T_j x, x \rangle \right|^2 \\ &= \frac{1}{4n} \frac{|\Gamma + \gamma|^2}{\operatorname{Re}(\Gamma\bar{\gamma})} \left| \left\langle \sum_{j=1}^n T_j x, x \right\rangle \right|^2 \end{aligned} \quad (3.32)$$

for any  $x \in H$  with  $\|x\| = 1$ .

On taking the supremum over  $\|x\| = 1$  in (3.32) we get (3.32).

Also, by (3.24) we get

$$\sum_{j=1}^n |\langle T_j x, x \rangle|^2 \leq \frac{1}{n} \left| \sum_{j=1}^n \langle T_j x, x \rangle \right|^2 + \left[ |\Gamma + \gamma| - 2\sqrt{\operatorname{Re}(\Gamma\bar{\gamma})} \right] \left| \sum_{j=1}^n \langle T_j x, x \rangle \right|,$$

for any  $x \in H$  with  $\|x\| = 1$ .

By taking the supremum over  $\|x\| = 1$  in this inequality, we have

$$\begin{aligned} &\sup_{\|x\|=1} \sum_{j=1}^n |\langle T_j x, x \rangle|^2 \\ &\leq \sup_{\|x\|=1} \left[ \frac{1}{n} \left| \sum_{j=1}^n \langle T_j x, x \rangle \right|^2 + \left[ |\Gamma + \gamma| - 2\sqrt{\operatorname{Re}(\Gamma\bar{\gamma})} \right] \left| \sum_{j=1}^n \langle T_j x, x \rangle \right| \right] \\ &\leq \frac{1}{n} \sup_{\|x\|=1} \left| \left\langle \sum_{j=1}^n T_j x, x \right\rangle \right|^2 + \left[ |\Gamma + \gamma| - 2\sqrt{\operatorname{Re}(\Gamma\bar{\gamma})} \right] \sup_{\|x\|=1} \left| \left\langle \sum_{j=1}^n T_j x, x \right\rangle \right| \\ &= \frac{1}{n} w^2 \left( \sum_{j=1}^n T_j \right) + \left[ |\Gamma + \gamma| - 2\sqrt{\operatorname{Re}(\Gamma\bar{\gamma})} \right] w \left( \sum_{j=1}^n T_j \right), \end{aligned}$$

which proves (3.29).

(iii) By the inequality (3.23) we have

$$\begin{aligned} \left( \sum_{j=1}^n |\langle T_j x, x \rangle|^2 \right)^{\frac{1}{2}} &\leq \frac{1}{\sqrt{n}} \left( \left| \sum_{j=1}^n \langle T_j x, x \rangle \right| + \frac{1}{4} \frac{|\Gamma - \gamma|^2}{|\Gamma + \gamma|} \right) \\ &= \frac{1}{\sqrt{n}} \left( \left| \left\langle \sum_{j=1}^n T_j x, x \right\rangle \right| + \frac{1}{4} \frac{|\Gamma - \gamma|^2}{|\Gamma + \gamma|} \right) \end{aligned}$$

for any  $x \in H$  with  $\|x\| = 1$ .

By taking the supremum over  $\|x\|=1$  in this inequality, we get (3.30). ■

*Remark 17.* By the use of the elementary inequality  $w(T) \leq \|T\|$  that holds for any  $T \in B(H)$ , a sufficient condition for (3.26) to hold is that

$$\left\| T_j - \frac{\gamma + \Gamma}{2} \right\| \leq \frac{1}{2} |\Gamma - \gamma| \quad \text{for any } j \in \{1, \dots, n\}. \quad (3.33)$$

#### 4. INEQUALITIES FOR $\delta_p$ AND $\vartheta_p$ NORMS

For  $T \in B(H)$  and  $p \geq 1$  we can consider the functionals

$$\delta_p(T) := \sup_{\|x\|=\|y\|=1} \left( |\langle Tx, y \rangle|^p + |\langle T^*x, y \rangle|^p \right)^{1/p} = \|(T, T^*)\|_{h,2,p} \quad (4.1)$$

and

$$\vartheta_p(T) := \sup_{\|x\|=1} \left( \|Tx\|^p + \|T^*x\|^p \right)^{1/p} = \|(T, T^*)\|_{s,2,p}. \quad (4.2)$$

It is easy to see that both  $\delta_p$  and  $\vartheta_p$  are norms on  $B(H)$ . The case  $p = 2$  for the norm  $\delta := \delta_2$  was considered and studied in [5].

Observe that, for any  $T \in B(H)$  and  $p \geq 1$ , we have

$$\begin{aligned} w_{h,2,p}((T, T^*)) &= \sup_{\|x\|=1} \left( |\langle Tx, x \rangle|^p + |\langle T^*x, x \rangle|^p \right)^{1/p} \\ &= \sup_{\|x\|=1} \left( |\langle Tx, x \rangle|^p + |\langle Tx, x \rangle|^p \right)^{1/p} \\ &= 2^{1/p} \sup_{\|x\|=1} |\langle Tx, x \rangle| = 2^{1/p} w(T). \end{aligned} \quad (4.3)$$

Using the inequality (1.13) we have

$$2^{1/p} w(T) \leq \delta_p(T) \leq 2^{1+1/p} w(T) \quad (4.4)$$

for any  $T \in B(H)$  and  $p \geq 1$ .

For  $p = 2$ , we get

$$\sqrt{2}w(T) \leq \delta(T) \leq \sqrt{8}w(T) \quad (4.5)$$

while for  $p = 1$  we get

$$2w(T) \leq \delta_1(T) \leq 4w(T) \quad (4.6)$$

for any  $T \in B(H)$ .

We have for any  $T \in B(H)$  and  $p \geq 1$  that

$$\|(T, T^*)\|_{2,p} = (\|T\|^p + \|T^*\|^p)^{1/p} = 2^{1/p}\|T\|$$

and by (2.25) we get

$$\|T\| \leq \delta_p(T) \leq 2^{1/p}\|T\| \quad (4.7)$$

for any  $T \in B(H)$  and  $p \geq 1$ .

For  $p = 2$ , we get

$$\|T\| \leq \delta(T) \leq \sqrt{2}\|T\| \quad (4.8)$$

while for  $p = 1$  we get

$$\|T\| \leq \delta_1(T) \leq 2\|T\| \quad (4.9)$$

for any  $T \in B(H)$ .

From (2.32) we get for  $r \geq q \geq 1$  that

$$\delta_r(T) \leq \delta_q(T) \leq 2^{\frac{r-q}{rq}} \delta_r(T) \quad (4.10)$$

for any  $T \in B(H)$ .

For any  $T \in B(H)$  and  $p, q > 1$  with  $\frac{1}{p} + \frac{1}{q} = 1$ , then by (2.43) we have

$$\delta_q(T) \geq \frac{1}{2^{1/p}}\|T + T^*\|. \quad (4.11)$$

In particular, for  $p = q = 2$  we get

$$\delta(T) \geq \frac{\sqrt{2}}{2}\|T + T^*\|, \quad (4.12)$$

for any  $T \in B(H)$ .

By using the inequality (2.45) we get

$$\delta_p(T) \leq \vartheta_p(T) \leq 2^{1/p}\|T\| \quad (4.13)$$

for any  $T \in B(H)$  and  $p \geq 1$ .

For  $p = 1$  we get

$$\delta_1(T) \leq \vartheta_1(T) \leq 2\|T\| \quad (4.14)$$

for any  $T \in B(H)$ .

For  $p \geq 2$ , by employing the equality (2.46) we get

$$\vartheta_p(T) = \left[ w_{h,2,p/2}(|T|^2, |T^*|^2) \right]^{1/2} = \left[ 2^{2/p} w(|T|^2) \right]^{1/2} = 2^{1/p} \|T\| \quad (4.15)$$

for any  $T \in B(H)$ .

On utilising (3.7), (3.9) and (3.11) we get

$$0 \leq \delta^2(T) - \frac{1}{2}\delta_1^2(T) \leq \frac{1}{2}\|T\|^2, \quad (4.16)$$

$$0 \leq \delta^2(T) - \frac{1}{2}\delta_1^2(T) \leq \|T\|\delta_1(T) \quad (4.17)$$

and

$$0 \leq \delta(T) - \frac{1}{\sqrt{2}}\delta_1(T) \leq \frac{\sqrt{2}}{4}\|T\| \quad (4.18)$$

for any  $T \in B(H)$ .

Observe, by (4.3) we have that

$$w_{h,2,e}((T, T^*)) = \sqrt{2}w(T),$$

for any  $T \in B(H)$ .

Assume that  $T \in B(H)$  and  $\gamma, \Gamma \in \mathbb{C}$  with  $\Gamma \neq \gamma$  such that

$$w\left(T - \frac{\gamma + \Gamma}{2}I\right), w\left(T^* - \frac{\gamma + \Gamma}{2}I\right) \leq \frac{1}{2}|\Gamma - \gamma|, \quad (4.19)$$

then by (3.27) we get

$$w^2(T) \leq \|\operatorname{Re}(T)\|^2 + \frac{1}{4}|\Gamma - \gamma|^2, \quad (4.20)$$

where  $\operatorname{Re}(T) := \frac{T+T^*}{2}$ .

If  $\operatorname{Re}(\Gamma\bar{\gamma}) > 0$ , then by (3.28) and (3.29)

$$w(T) \leq \frac{1}{2} \frac{|\Gamma + \gamma|}{\sqrt{\operatorname{Re}(\Gamma\bar{\gamma})}} \|\operatorname{Re}(T)\| \quad (4.21)$$

and

$$w^2(T) \leq \left[ \|\operatorname{Re}(T)\| + \left[ |\Gamma + \gamma| - 2\sqrt{(\Gamma\bar{\gamma})} \right] \right] \|\operatorname{Re}(T)\|. \quad (4.22)$$

If  $\Gamma \neq -\gamma$ , then by (3.30) we get

$$w(T) \leq \|\operatorname{Re}(T)\| + \frac{1}{8} \frac{|\Gamma - \gamma|^2}{|\Gamma + \gamma|}. \quad (4.23)$$

Due to the fact that  $w(A) = w(A^*)$  for any  $A \in B(H)$ , the condition (4.19) can be simplified as follows.

If  $m, M$  are real numbers with  $M > m$  and if

$$w\left(T - \frac{m+M}{2}I\right) \leq \frac{1}{2}(M-m),$$

then

$$w^2(T) \leq \|\operatorname{Re}(T)\|^2 + \frac{1}{4}(M-m)^2. \quad (4.24)$$

If  $m > 0$ , then

$$w(T) \leq \frac{1}{2} \frac{m+M}{\sqrt{mM}} \|\operatorname{Re}(T)\| \quad (4.25)$$

and

$$w^2(T) \leq \left[ \|\operatorname{Re}(T)\| + \left( \sqrt{M} - \sqrt{m} \right)^2 \right] \|\operatorname{Re}(T)\|. \quad (4.26)$$

If  $M \neq -m$ , then

$$w(T) \leq \|\operatorname{Re}(T)\| + \frac{1}{8} \frac{(M-m)^2}{m+M}. \quad (4.27)$$

For other numerical radius and norm inequalities, the interested reader may also consult [1] and [6] and compare these results. The details are not provided here.

## 5. INEQUALITIES FOR REAL NORMS

If  $X$  is a complex linear space, then the functional  $\|\cdot\|$  is a real norm, if the homogeneity property in the definition of the norms is satisfied only for real numbers, namely we have

$$\|\alpha x\| = |\alpha| \|x\| \quad \text{for any } \alpha \in \mathbb{R} \text{ and } x \in X.$$

For instance if we consider the complex linear space of complex numbers  $\mathbb{C}$  then the functionals

$$|z|_p := (|\operatorname{Re}(z)|^p + |\operatorname{Im}(z)|^p)^{1/p}, \quad p \geq 1,$$

$$|z|_\infty := \max\{|\operatorname{Re}(z)|, |\operatorname{Im}(z)|\}, \quad p = \infty,$$

are real norms on  $\mathbb{C}$ .

For  $T \in B(H)$  we consider the Cartesian decomposition

$$T = \operatorname{Re}(T) + i \operatorname{Im}(T)$$

where the selfadjoint operators  $\operatorname{Re}(T)$  and  $\operatorname{Im}(T)$  are uniquely defined by

$$\operatorname{Re}(T) = \frac{T + T^*}{2} \quad \text{and} \quad \operatorname{Im}(T) = \frac{T - T^*}{2i}.$$

We can introduce the following functionals

$$\|T\|_{r,p} := (\|\operatorname{Re}(T)\|^p + \|\operatorname{Im}(T)\|^p)^{1/p}, \quad p \geq 1,$$

and

$$\|T\|_{r,\infty} := \max\{\|\operatorname{Re}(T)\|, \|\operatorname{Im}(T)\|\}, \quad p = \infty,$$

where  $\|\cdot\|$  is the usual operator norm on  $B(H)$ . *The definition can be extended for any other norms on  $B(H)$  or its subspaces.*

Using the properties of the norm  $\|\cdot\|$  and the Minkowski's inequality

$$(|a+b|^p + |c+d|^p)^{1/p} \leq (|a|^p + |c|^p)^{1/p} + (|b|^p + |d|^p)^{1/p}$$

for  $p \geq 1$  and  $a, b, c, d \in \mathbb{C}$ , we observe that  $\|\cdot\|_{r,p}$ ,  $p \in [1, \infty]$  is a real norm on  $B(H)$ .

For  $p \geq 1$  and  $T \in B$  we can introduce the following functionals

$$\begin{aligned} \eta_{r,p}(T) &:= \sup_{\|x\|=\|y\|=1} (|\operatorname{Re}\langle Tx, y \rangle|^p + |\operatorname{Im}\langle Tx, y \rangle|^p)^{1/p} \\ &= \sup_{\|x\|=\|y\|=1} (|\langle \operatorname{Re} T x, y \rangle|^p + |\langle \operatorname{Im} T x, y \rangle|^p)^{1/p} \\ &= \|(\operatorname{Re} T, \operatorname{Im} T)\|_{h,2,p}, \end{aligned}$$

$$\begin{aligned}
\theta_{r,p}(T) &:= \sup_{\|x\|=1} \left( |\operatorname{Re} \langle Tx, x \rangle|^p + |\operatorname{Im} \langle Tx, x \rangle|^p \right)^{1/p} \\
&= \sup_{\|x\|=1} \left( |\langle \operatorname{Re} T x, x \rangle|^p + |\langle \operatorname{Im} T x, x \rangle|^p \right)^{1/p} \\
&= w_{h,2,p}(\operatorname{Re} T, \operatorname{Im} T)
\end{aligned}$$

and

$$\kappa_{r,p}(T) := \sup_{\|x\|=1} \left( \|\operatorname{Re} T x\|^p + \|\operatorname{Im} T x\|^p \right)^{1/p} = \|(\operatorname{Re} T, \operatorname{Im} T)\|_{s,2,p}.$$

The case  $p = 2$  is of interest since for  $T \in B(H)$  we have

$$\begin{aligned}
\eta_{r,2}(T) &:= \sup_{\|x\|=\|y\|=1} \left( |\operatorname{Re} \langle Tx, y \rangle|^2 + |\operatorname{Im} \langle Tx, y \rangle|^2 \right)^{1/2} \\
&= \sup_{\|x\|=\|y\|=1} |\langle Tx, y \rangle| = \|T\|, \\
\theta_{r,2}(T) &:= \sup_{\|x\|=1} \left( |\operatorname{Re} \langle Tx, x \rangle|^2 + |\operatorname{Im} \langle Tx, x \rangle|^2 \right)^{1/2} \\
&= \sup_{\|x\|=1} |\langle Tx, x \rangle| = w(T)
\end{aligned}$$

and

$$\begin{aligned}
\kappa_{r,2}(T) &:= \sup_{\|x\|=1} \left( \|\operatorname{Re} T x\|^2 + \|\operatorname{Im} T x\|^2 \right)^{1/2} \\
&= \sup_{\|x\|=1} \left( \left\langle (\operatorname{Re} T)^2 x, x \right\rangle + \left\langle (\operatorname{Im} T)^2 x, x \right\rangle \right)^{1/2} \\
&= \sup_{\|x\|=1} \left( \left\langle [(\operatorname{Re} T)^2 + (\operatorname{Im} T)^2] x, x \right\rangle \right)^{1/2} \\
&= \left\| (\operatorname{Re} T)^2 + (\operatorname{Im} T)^2 \right\|^{1/2} = \left\| \frac{|T|^2 + |T^*|^2}{2} \right\|^{1/2}.
\end{aligned}$$



For  $p = \infty$  we have

$$\begin{aligned}\eta_{r,\infty}(T) &:= \sup_{\|x\|=\|y\|=1} \left( \max \{ |\operatorname{Re} \langle Tx, y \rangle|, |\operatorname{Im} \langle Tx, y \rangle| \} \right) \\ &= \max \left\{ \sup_{\|x\|=\|y\|=1} |\langle \operatorname{Re} T x, y \rangle|, \sup_{\|x\|=\|y\|=1} |\langle \operatorname{Im} T x, y \rangle| \right\} \\ &= \max \{ \|\operatorname{Re} T\|, \|\operatorname{Im} T\| \},\end{aligned}$$

and in a similar way

$$\theta_{r,\infty}(T) = \kappa_{r,\infty}(T) = \max \{ \|\operatorname{Re} T\|, \|\operatorname{Im} T\| \} = \|T\|_{r,\infty}.$$

The functionals  $\eta_{r,p}$ ,  $\theta_{r,p}$  and  $\kappa_{r,p}$  with  $p \in [1, \infty]$  are real norms on  $B(H)$ . We have

$$\begin{aligned}\eta_{r,p}(T) &= \sup_{\|x\|=\|y\|=1} \left( |\operatorname{Re} \langle Tx, y \rangle|^p + |\operatorname{Im} \langle Tx, y \rangle|^p \right)^{1/p} \\ &\leq \left( \sup_{\|x\|=\|y\|=1} |\operatorname{Re} \langle Tx, y \rangle|^p + \sup_{\|x\|=\|y\|=1} |\operatorname{Im} \langle Tx, y \rangle|^p \right)^{1/p} \\ &= (\|\operatorname{Re}(T)\|^p + \|\operatorname{Im}(T)\|^p)^{1/p} = \|T\|_{r,p}\end{aligned}$$

and

$$\begin{aligned}\|T\|_{r,\infty} &= \sup_{\|x\|=\|y\|=1} \left( \max \{ |\operatorname{Re} \langle Tx, y \rangle|, |\operatorname{Im} \langle Tx, y \rangle| \} \right) \\ &\leq \sup_{\|x\|=\|y\|=1} \left( |\operatorname{Re} \langle Tx, y \rangle|^p + |\operatorname{Im} \langle Tx, y \rangle|^p \right)^{1/p} = \eta_{r,p}(T)\end{aligned}$$

for any  $p \geq 1$  and  $T \in B(H)$ .

In a similar way we have

$$\|T\|_{r,\infty} \leq \theta_{r,p}(T) \leq \|T\|_{r,p}$$

and

$$\|T\|_{r,\infty} \leq \kappa_{r,p}(T) \leq \|T\|_{r,p}$$

for any  $p \geq 1$  and  $T \in B(H)$ .

If we write the inequality (1.13) for  $n = 2$ ,  $T_1 = \operatorname{Re} T$  and  $T_2 = \operatorname{Im} T$  then we get

$$\theta_{r,p}(T) \leq \eta_{r,p}(T) \leq 2\theta_{r,p}(T) \quad (5.1)$$

for any  $p \geq 1$  and  $T \in B(H)$ .

Using the inequalities (2.25) and (2.26) for  $n = 2$ ,  $T_1 = \operatorname{Re} T$  and  $T_2 = \operatorname{Im} T$  then we get

$$\frac{1}{2^{1/p}} \|T\|_{r,p} \leq \eta_{r,p}(T) \leq \|T\|_{r,p} \quad (5.2)$$

and

$$\frac{1}{2^{1/p}} \|T\|_{r,p} \leq \theta_{r,p}(T) \leq \|T\|_{r,p} \quad (5.3)$$

for any  $p \geq 1$  and  $T \in B(H)$ .

If we use the inequalities (2.32) and (2.33) for  $n = 2$ ,  $T_1 = \operatorname{Re} T$  and  $T_2 = \operatorname{Im} T$  then we get for  $t \geq p \geq 1$  that

$$\eta_{r,t}(T) \leq \eta_{r,p}(T) \leq 2^{\frac{t-p}{tp}} \eta_{r,t}(T) \quad (5.4)$$

and

$$\theta_{r,t}(T) \leq \theta_{r,p}(T) \leq 2^{\frac{t-p}{tp}} \theta_{r,t}(T) \quad (5.5)$$

for any  $T \in B(H)$ .

For  $p = 1$  we have the functionals

$$\eta_{r,1}(T) = \sup_{\|x\|=\|y\|=1} (|\langle \operatorname{Re} Tx, y \rangle| + |\langle \operatorname{Im} Tx, y \rangle|) = \|(\operatorname{Re} T, \operatorname{Im} T)\|_{h,2,1},$$

$$\theta_{r,1}(T) := \sup_{\|x\|=1} (|\langle \operatorname{Re} Tx, x \rangle| + |\langle \operatorname{Im} Tx, x \rangle|) = w_{h,2,1}(\operatorname{Re} T, \operatorname{Im} T)$$

and

$$\kappa_{r,1}(T) := \sup_{\|x\|=1} (\|\operatorname{Re} Tx\| + \|\operatorname{Im} Tx\|) = \|(\operatorname{Re} T, \operatorname{Im} T)\|_{s,2,1}.$$

By utilising the inequalities (3.7), (3.9) and (3.11) for  $n = 2$ ,  $T_1 = \operatorname{Re} T$  and  $T_2 = \operatorname{Im} T$ , then

$$0 \leq \|T\|^2 - \frac{1}{2} \eta_{r,1}^2(T) \leq \frac{1}{2} (\max \{\|\operatorname{Re} T\|, \|\operatorname{Im} T\|\})^2, \quad (5.6)$$

$$0 \leq \|T\|^2 - \frac{1}{2} \eta_{r,1}^2(T) \leq \max \{\|\operatorname{Re} T\|, \|\operatorname{Im} T\|\} \eta_{r,1}(T) \quad (5.7)$$

and

$$0 \leq \|T\| - \frac{\sqrt{2}}{2} \eta_{r,1}(T) \leq \frac{\sqrt{2}}{4} \max \{ \|\operatorname{Re} T\|, \|\operatorname{Im} T\| \} \quad (5.8)$$

for any  $T \in B(H)$ .

Also, by utilising the inequalities (3.8), (3.10) and (3.12) for  $n = 2$ ,  $T_1 = \operatorname{Re} T$  and  $T_2 = \operatorname{Im} T$ , then

$$0 \leq w^2(T) - \frac{1}{2} \theta_{r,1}^2(T) \leq \frac{1}{2} \left( \max \{ \|\operatorname{Re} T\|, \|\operatorname{Im} T\| \} \right)^2, \quad (5.9)$$

$$0 \leq w^2(T) - \frac{1}{2} \theta_{r,1}^2(T) \leq \max \{ \|\operatorname{Re} T\|, \|\operatorname{Im} T\| \} \theta_{r,1}(T) \quad (5.10)$$

and

$$0 \leq w(T) - \frac{\sqrt{2}}{2} \theta_{r,1}(T) \leq \frac{\sqrt{2}}{4} \max \{ \|\operatorname{Re} T\|, \|\operatorname{Im} T\| \} \quad (5.11)$$

for any  $T \in B(H)$ .

If  $m, M$  are real numbers with  $M > m$  and if

$$\left\| \operatorname{Re} T - \frac{m+M}{2} I \right\|, \left\| \operatorname{Im} T - \frac{m+M}{2} I \right\| \leq \frac{1}{2} (M - m), \quad (5.12)$$

then by (3.27) we get

$$w^2(T) \leq \frac{1}{2} \|\operatorname{Re} T + \operatorname{Im} T\|^2 + \frac{1}{2} (M - m)^2. \quad (5.13)$$

If  $m > 0$ , then (3.28) and (3.29) we have

$$w(T) \leq \frac{1}{2\sqrt{2}} \frac{m+M}{\sqrt{mM}} \|\operatorname{Re} T + \operatorname{Im} T\| \quad (5.14)$$

and

$$w^2(T) \leq \left[ \frac{1}{2} \|\operatorname{Re} T + \operatorname{Im} T\| + \left( \sqrt{M} - \sqrt{m} \right)^2 \right] \|\operatorname{Re} T + \operatorname{Im} T\|. \quad (5.15)$$

If  $M \neq -m$ , then by (3.30) we get

$$w(T) \leq \frac{1}{\sqrt{2}} \left( \|\operatorname{Re} T + \operatorname{Im} T\| + \frac{1}{4} \frac{(M - m)^2}{M + m} \right). \quad (5.16)$$

Finally, we observe that a simple sufficient condition for (5.12) to hold, is that

$$mI \leq \operatorname{Re} T, \quad \operatorname{Im} T \leq MI$$

in the operator order of  $B(H)$ .

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