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## Bounds for the Generalized $(\Phi, f)$ -Mean Difference

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### ABSTRACT

In this paper we establish some bounds for the  $(\Phi, f)$ -mean difference introduced in the general settings of measurable spaces and Lebesgue integral, which is a two functions generalization of *Gini mean difference* that has been widely used by economists and sociologists to measure economic inequality.

### RESUMEN

En este artículo establecemos algunas cotas para la  $(\Phi, f)$ -diferencia media introducida en el contexto general de espacios medibles e integral de Lebesgue, que es una generalización a dos funciones de la *diferencia media de Gini* que ha sido ampliamente utilizada por economistas y sociólogos para medir desigualdad económica.

**Keywords and Phrases:** Gini mean difference, Mean deviation, Lebesgue integral, Expectation, Jensen's integral inequality.

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## 1. Introduction

Let  $(\Omega, \mathcal{A}, \nu)$  be a measurable space consisting of a set  $\Omega$ , a  $\sigma$ -algebra  $\mathcal{A}$  of subsets of  $\Omega$  and a countably additive and positive measure  $\nu$  on  $\mathcal{A}$  with values in  $\mathbb{R} \cup \{\infty\}$ . For a  $\nu$ -measurable function  $w : \Omega \rightarrow \mathbb{R}$ , with  $w(x) \geq 0$  for  $\nu$ -a.e. (almost every)  $x \in \Omega$  and  $\int_{\Omega} w(x) d\nu(x) = 1$ , consider the *Lebesgue space*

$$L_w(\Omega, \nu) := \{f : \Omega \rightarrow \mathbb{R}, f \text{ is } \nu\text{-measurable and } \int_{\Omega} w(x) |f(x)| d\nu(x) < \infty\}.$$

Let  $I$  be an interval of real numbers and  $\Phi : I \rightarrow \mathbb{R}$  a Lebesgue measurable function on  $I$ . For  $f : \Omega \rightarrow I$  a  $\nu$ -measurable function with  $\Phi \circ f \in L_w(\Omega, \nu)$  we define the *generalized  $(\Phi, f)$ -mean difference*  $R_G(\Phi, f; w)$  by

$$R_G(\Phi, f; w) := \frac{1}{2} \int_{\Omega} \int_{\Omega} w(x) w(y) |(\Phi \circ f)(x) - (\Phi \circ f)(y)| d\nu(x) d\nu(y) \quad (1.1)$$

and the *generalized  $(\Phi, f)$ -mean deviation*  $M_D(\Phi, f; w)$  by

$$M_D(\Phi, f; w) := \int_{\Omega} w(x) |(\Phi \circ f)(x) - E(\Phi, f; w)| d\nu(x), \quad (1.2)$$

where

$$E(\Phi, f; w) := \int_{\Omega} (\Phi \circ f)(y) w(y) d\nu(y)$$

the *generalized  $(\Phi, f)$ -expectation*.

If  $\Phi = e$ , where  $e(t) = t$ ,  $t \in \mathbb{R}$  is the *identity mapping*, then we can consider the particular cases of interest, the *generalized  $f$ -mean difference*

$$R_G(f; w) := R_G(e, f; w) = \frac{1}{2} \int_{\Omega} \int_{\Omega} w(x) w(y) |f(x) - f(y)| d\nu(x) d\nu(y) \quad (1.3)$$

and the *generalized  $f$ -mean deviation*

$$M_D(f; w) := M_D(e, f; w) = \int_{\Omega} w(x) |f(x) - E(f; w)| d\nu(x), \quad (1.4)$$

where  $E(f; w) := \int_{\Omega} f(y) w(y) d\nu(y)$  is the *generalized  $f$ -expectation*.

If  $\Omega = [-\infty, \infty]$  and  $f = e$  then we have the usual *mean difference*

$$R_G(w) := R_G(f; w) = \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w(x) w(y) |x - y| dx dy \quad (1.5)$$

and the *mean deviation*

$$M_D(w) := M_D(f; w) = \int_{\Omega} w(x) |x - E(w)| dx, \quad (1.6)$$

where  $w : \mathbb{R} \rightarrow [0, \infty)$  is a *density function*, this means that  $w$  is integrable on  $\mathbb{R}$  and  $\int_{-\infty}^{\infty} w(t) dt = 1$ , and

$$E(w) := \int_{-\infty}^{\infty} xw(x) dx \tag{1.7}$$

denote the *expectation of w* provided that the integral exists and is finite.

The mean difference  $R_G(w)$  was proposed by Gini in 1912 [21], after whom it is usually named, but was discussed by Helmert and other German writers in the 1870's (cf. H. A. David [13], see also [26, p. 48]). It has a certain theoretical attraction, being dependent on the spread of the variate-values among themselves and not on the deviations from some central value ([26, p. 48]). Further, its defining integral (1.5) may converge when that of the *variance*  $\sigma(w)$ ,

$$\sigma(w) := \int_{-\infty}^{\infty} (x - E(w))^2 w(x) dx, \tag{1.8}$$

does not. It is, however, more difficult to compute than the standard deviation.

For some recent results concerning integral representations and bounds for  $R_G(w)$  see [5], [6], [8] and [9].

For instance, if  $w : \mathbb{R} \rightarrow [0, \infty)$  is a density function we define by

$$W(x) := \int_{-\infty}^x w(t) dt, \quad x \in \mathbb{R}$$

its *cumulative function*. Then we have [5], [6]:

$$\begin{aligned} R_G(w) &= 2 \operatorname{Cov}(e, W) = \int_{-\infty}^{\infty} (1 - W(y)) W(y) dy \\ &= 2 \int_{-\infty}^{\infty} xw(x) W(x) dx - E(w) \\ &= 2 \int_{-\infty}^{\infty} (x - E(w)) (W(x) - \gamma) w(x) dx \\ &= 2 \int_{-\infty}^{\infty} (x - \delta) \left( W(x) - \frac{1}{2} \right) w(x) dx \end{aligned} \tag{1.9}$$

for any  $\gamma, \delta \in \mathbb{R}$  and [6]:

$$R_G(w) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x - y) (W(x) - W(y)) w(x) w(y) dx dy. \tag{1.10}$$

With the above assumptions, we have the bounds [5]:

$$\frac{1}{2} M_D(w) \leq R_G(w) \leq 2 \sup_{x \in \mathbb{R}} |W(x) - \gamma| M_D(w) \leq M_D(w), \tag{1.11}$$

for any  $\gamma \in [0, 1]$ , where  $W(\cdot)$  is the cumulative distribution of  $w$  and  $M_D(w)$  is the mean deviation.

Consider the  $n$ -tuple of real numbers  $\mathbf{a} = (a_1, \dots, a_n)$  and  $\mathbf{p} = (p_1, \dots, p_n)$  a probability distribution, i.e.  $p_i \geq 0$  for each  $i \in \{1, \dots, n\}$  with  $\sum_{i=1}^n p_i = 1$ , then by taking  $\Omega = \{1, \dots, n\}$  and the discrete measure, we can consider from (1.1) and (1.2) that (see [7])

$$R_G(\mathbf{a}; \mathbf{p}) := \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n p_i p_j |\Phi(a_i) - \Phi(a_j)|, \quad (1.12)$$

and

$$M_D(\mathbf{a}; \mathbf{p}) := \frac{1}{2} \sum_{i=1}^n p_i \left| \Phi(a_i) - \sum_{j=1}^n p_j \Phi(a_j) \right| \quad (1.13)$$

where  $\mathbf{a} \in I^n := I \times \dots \times I$  and  $\Phi : I \rightarrow \mathbb{R}$ .

The quantity  $R_G(\mathbf{a}; \mathbf{p})$  has been defined in [7] and some results were obtained.

In the case when  $\Phi = e$ , then we get the special case of Gini mean difference and mean deviation of an empirical distribution that is particularly important for applications,

$$R_G(\mathbf{a}; \mathbf{p}) := \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n p_i p_j |a_i - a_j|, \quad (1.14)$$

and

$$M_D(\mathbf{a}; \mathbf{p}) := \frac{1}{2} \sum_{i=1}^n p_i \left| a_i - \sum_{j=1}^n p_j a_j \right|. \quad (1.15)$$

The following result incorporates an upper bound for the weighted Gini mean difference [7]:

For any  $\mathbf{a} \in \mathbb{R}^n$  and any  $\mathbf{p}$  a probability distribution, we have the inequality:

$$\frac{1}{2} M_D(\mathbf{a}; \mathbf{p}) \leq R_G(\mathbf{a}; \mathbf{p}) \leq \inf_{\gamma \in \mathbb{R}} \left[ \sum_{i=1}^n p_i |a_i - \gamma| \right] \leq M_D(\mathbf{a}; \mathbf{p}). \quad (1.16)$$

The constant  $\frac{1}{2}$  in the first inequality in (1.16) is sharp.

For some recent results for discrete Gini mean difference and mean deviation, see [7], [11], [14] and [15].

## 2. General Bounds

We have:

**Theorem 1.** *Let  $I$  be an interval of real numbers and  $\Phi : I \rightarrow \mathbb{R}$  a Lebesgue measurable function on  $I$ . If  $w : \Omega \rightarrow \mathbb{R}$  is a  $\nu$ -measurable function with  $w(x) \geq 0$  for  $\nu$ -a.e. (almost every)  $x \in \Omega$  and  $\int_{\Omega} w(x) d\nu(x) = 1$  and if  $f : \Omega \rightarrow I$  is a  $\nu$ -measurable function with  $\Phi \circ f \in L_w(\Omega, \nu)$ , then*

$$\frac{1}{2}M_D(\Phi, f; w) \leq R_G(\Phi, f; w) \leq I(\Phi, f; w) \leq M_D(\Phi, f; w), \quad (2.1)$$

where

$$I(\Phi, f; w) := \inf_{\gamma \in \mathbb{R}} \int_{\Omega} w(x) |(\Phi \circ f)(x) - \gamma| d\nu(x). \quad (2.2)$$

*Demostración.* Using the properties of the integral, we have

$$\begin{aligned} R_G(\Phi, f; w) &= \frac{1}{2} \int_{\Omega} \int_{\Omega} w(x) w(y) |(\Phi \circ f)(x) - (\Phi \circ f)(y)| d\nu(x) d\nu(y) \\ &\geq \frac{1}{2} \int_{\Omega} w(x) \left| (\Phi \circ f)(x) \int_{\Omega} w(y) d\nu(y) - \int_{\Omega} w(y) (\Phi \circ f)(y) d\nu(y) \right| d\nu(x) \\ &= \frac{1}{2} \int_{\Omega} w(x) \left| (\Phi \circ f)(x) - \int_{\Omega} w(y) (\Phi \circ f)(y) d\nu(y) \right| d\nu(x) \\ &= \frac{1}{2} M_D(\Phi, f; w) \end{aligned}$$

and the first inequality in (2.1) is proved.

By the triangle inequality for modulus we have

$$\begin{aligned} |(\Phi \circ f)(x) - (\Phi \circ f)(y)| &= |(\Phi \circ f)(x) - \gamma + \gamma - (\Phi \circ f)(y)| \\ &\leq |(\Phi \circ f)(x) - \gamma| + |(\Phi \circ f)(y) - \gamma| \end{aligned} \quad (2.3)$$

for any  $x, y \in \Omega$  and  $\gamma \in \mathbb{R}$ .

Now, if we multiply (2.3) by  $\frac{1}{2}w(x)w(y)$  and integrate, we get

$$\begin{aligned}
 & R_G(\Phi, f; w) \\
 &= \frac{1}{2} \int_{\Omega} \int_{\Omega} w(x)w(y) |(\Phi \circ f)(x) - (\Phi \circ f)(y)| \, d\nu(x) \, d\nu(y) \\
 &\leq \frac{1}{2} \int_{\Omega} \int_{\Omega} w(x)w(y) [ |(\Phi \circ f)(x) - \gamma| + |(\Phi \circ f)(y) - \gamma| ] \, d\nu(x) \, d\nu(y) \\
 &= \frac{1}{2} \int_{\Omega} \int_{\Omega} w(x)w(y) |(\Phi \circ f)(x) - \gamma| \, d\nu(x) \, d\nu(y) \\
 &+ \frac{1}{2} \int_{\Omega} \int_{\Omega} w(x)w(y) |(\Phi \circ f)(y) - \gamma| \, d\nu(x) \, d\nu(y) \\
 &= \frac{1}{2} \int_{\Omega} w(x) |(\Phi \circ f)(x) - \gamma| \, d\nu(x) + \frac{1}{2} \int_{\Omega} w(y) |(\Phi \circ f)(y) - \gamma| \, d\nu(y) \\
 &= \int_{\Omega} w(x) |(\Phi \circ f)(x) - \gamma| \, d\nu(x) \tag{2.4}
 \end{aligned}$$

for any  $\gamma \in \mathbb{R}$ .

Taking the infimum over  $\gamma \in \mathbb{R}$  in (2.4) we get the second part of (2.1).

Since, obviously

$$\begin{aligned}
 I(\Phi, f; w) &= \inf_{\gamma \in \mathbb{R}} \int_{\Omega} w(x) |(\Phi \circ f)(x) - \gamma| \, d\nu(x) \\
 &\leq \int_{\Omega} w(x) \left| (\Phi \circ f)(x) - \int_{\Omega} w(y) (\Phi \circ f)(y) \, d\nu(y) \right| \, d\nu(x) \\
 &= M_D(\Phi, f; w),
 \end{aligned}$$

the last part of (2.1) is thus proved. ■

By the Cauchy-Bunyakowsky-Schwarz (CBS) inequality, if  $(\Phi \circ f)^2 \in L_w(\Omega, \nu)$ , then we have

$$\begin{aligned}
 & \left[ \int_{\Omega} w(x) \left| (\Phi \circ f)(x) - \int_{\Omega} w(y) (\Phi \circ f)(y) \, d\nu(y) \right| \, d\nu(x) \right]^2 \\
 &\leq \int_{\Omega} w(x) \left[ (\Phi \circ f)(x) - \int_{\Omega} w(y) (\Phi \circ f)(y) \, d\nu(y) \right]^2 \, d\nu(x) \\
 &= \int_{\Omega} w(x) (\Phi \circ f)^2(x) \, d\nu(x) \\
 &- 2 \int_{\Omega} w(y) (\Phi \circ f)(y) \, d\nu(y) \int_{\Omega} w(x) (\Phi \circ f)(x) \, d\nu(x) \\
 &+ \left[ \int_{\Omega} w(y) (\Phi \circ f)(y) \, d\nu(y) \right]^2 \int_{\Omega} w(x) \, d\nu(x) \\
 &= \int_{\Omega} w(x) (\Phi \circ f)^2(x) \, d\nu(x) - \left[ \int_{\Omega} w(x) (\Phi \circ f)(x) \, d\nu(x) \right]^2.
 \end{aligned}$$

By considering the *generalized  $(\Phi, f)$ -dispersion*

$$\sigma(\Phi, f; w) := \left( \int_{\Omega} w(x) (\Phi \circ f)^2(x) \, d\nu(x) - \left[ \int_{\Omega} w(x) (\Phi \circ f)(x) \, d\nu(x) \right]^2 \right)^{1/2},$$

then we have

$$M_D(\Phi, f; w) \leq \sigma(\Phi, f; w) \tag{2.5}$$

provided  $(\Phi \circ f)^2 \in L_w(\Omega, \nu)$ .

If there exists the constants  $m, M$  so that

$$-\infty < m \leq \Phi(t) \leq M < \infty \text{ for almost any } t \in I \tag{2.6}$$

then by the reverse CBS inequality

$$\sigma(\Phi, f; w) \leq \frac{1}{2}(M - m), \tag{2.7}$$

by (2.1) and by (2.5) we can state the following result:

**Corollary 1.** *Let  $I$  be an interval of real numbers and  $\Phi : I \rightarrow \mathbb{R}$  a Lebesgue measurable function on  $I$  satisfying the condition (2.6) for some constants  $m, M$ . If  $w : \Omega \rightarrow \mathbb{R}$  is a  $\nu$ -measurable function with  $w(x) \geq 0$  for  $\nu$ -a.e.  $x \in \Omega$  and  $\int_{\Omega} w(x) \, d\nu(x) = 1$  and if  $f : \Omega \rightarrow I$  is a  $\nu$ -measurable function with  $(\Phi \circ f)^2 \in L_w(\Omega, \nu)$ , then we have the chain of inequalities*

$$\begin{aligned} \frac{1}{2}M_D(\Phi, f; w) &\leq R_G(\Phi, f; w) \leq I(\Phi, f; w) \leq M_D(\Phi, f; w) \\ &\leq \sigma(\Phi, f; w) \leq \frac{1}{2}(M - m). \end{aligned} \tag{2.8}$$

We observe that, in the discrete case we obtain from (2.1) the inequality (1.16) while for the univariate case with  $\int_{-\infty}^{\infty} w(t) \, dt = 1$  we have

$$\frac{1}{2}M_D(w) \leq R_G(w) \leq I(w) \leq M_D(w) \leq \sigma(\Phi, f; w) \tag{2.9}$$

where

$$I(w) := \inf_{\gamma \in \mathbb{R}} \int_{-\infty}^{\infty} w(x) |x - \gamma| \, dx. \tag{2.10}$$

If  $w$  is supported on the finite interval  $[a, b]$ , namely  $\int_a^b w(x) \, dx = 1$ , then we have the chain of inequalities

$$\frac{1}{2}M_D(w) \leq R_G(w) \leq I(w) \leq M_D(w) \leq \sigma(\Phi, f; w) \leq \frac{1}{2}(M - m). \tag{2.11}$$



### 3. Bounds for Various Classes of Functions

In the case of functions of bounded variation we have:

**Theorem 2.** Let  $\Phi : [a, b] \rightarrow \mathbb{R}$  be a function of bounded variation on the closed interval  $[a, b]$ . If  $w : \Omega \rightarrow \mathbb{R}$  is a  $\nu$ -measurable function with  $w(x) \geq 0$  for  $\nu$ -a.e.  $x \in \Omega$  and  $\int_{\Omega} w(x) d\nu(x) = 1$  and if  $f : \Omega \rightarrow [a, b]$  is a  $\nu$ -measurable function with  $\Phi \circ f \in L_w(\Omega, \nu)$ , then

$$R_G(\Phi, f; w) \leq \frac{1}{2} \bigvee_a^b(\Phi), \quad (3.1)$$

where  $\bigvee_a^b(\Phi)$  is the total variation of  $\Phi$  on  $[a, b]$ .

*Demostración.* Using the inequality (2.4) we have

$$R_G(\Phi, f; w) \leq \int_{\Omega} w(x) |(\Phi \circ f)(x) - \gamma| d\nu(x) \quad (3.2)$$

for any  $\gamma \in \mathbb{R}$ .

By the triangle inequality, we have

$$\begin{aligned} & \left| (\Phi \circ f)(x) - \frac{1}{2} [\Phi(a) + \Phi(b)] \right| \\ & \leq \frac{1}{2} |\Phi(a) - \Phi(f(x))| + \frac{1}{2} |\Phi(b) - \Phi(f(x))| \end{aligned} \quad (3.3)$$

for any  $x \in \Omega$ .

Since  $\Phi : [a, b] \rightarrow \mathbb{R}$  is of bounded variation and  $d$  is a division of  $[a, b]$ , namely

$$d \in \mathcal{D}([a, b]) := \{d := \{a = t_0 < t_1 < \dots < t_n = b\}\},$$

then

$$\bigvee_a^b(\Phi) = \sup_{d \in \mathcal{D}([a, b])} \sum_{i=0}^{n-1} |\Phi(t_{i+1}) - \Phi(t_i)| < \infty.$$

Taking the division  $d_0 := \{a = t_0 < t < t_2 = b\}$  we then have

$$|\Phi(t) - \Phi(a)| + |\Phi(b) - \Phi(t)| \leq \bigvee_a^b(\Phi)$$

for any  $t \in [a, b]$  and then

$$|\Phi(f(x)) - \Phi(a)| + |\Phi(b) - \Phi(f(x))| \leq \bigvee_a^b(\Phi) \quad (3.4)$$

for any  $x \in \Omega$ .

On making use of (3.3) and (3.4) we get

$$\left| (\Phi \circ f)(x) - \frac{1}{2} [\Phi(a) + \Phi(b)] \right| \leq \frac{1}{2} \bigvee_a^b(\Phi) \tag{3.5}$$

for any  $x \in \Omega$ .

If we multiply (3.5) by  $w(x)$  and integrate, then we obtain

$$\int_{\Omega} w(x) \left| (\Phi \circ f)(x) - \frac{1}{2} [\Phi(a) + \Phi(b)] \right| \leq \frac{1}{2} \bigvee_a^b(\Phi). \tag{3.6}$$

Finally, by choosing  $\gamma = \frac{1}{2} [\Phi(a) + \Phi(b)]$  in (3.2) and making use of (3.6) we deduce the desired result (3.1). ■

In the case of absolutely continuous functions we have:

**Theorem 3.** *Let  $\Phi : [a, b] \rightarrow \mathbb{R}$  be an absolutely continuous function on the closed interval  $[a, b]$ . If  $w : \Omega \rightarrow \mathbb{R}$  is a  $\nu$ -measurable function with  $w(x) \geq 0$  for  $\nu$ -a.e.  $x \in \Omega$  and  $\int_{\Omega} w(x) d\nu(x) = 1$  and if  $f : \Omega \rightarrow [a, b]$  is a  $\nu$ -measurable function with  $\Phi \circ f \in L_w(\Omega, \nu)$ , then*

$$R_G(\Phi, f; w) \leq \begin{cases} \|\Phi'\|_{[a,b],\infty} R_G(f; w) & \text{if } \Phi' \in L_{\infty}([a, b]), \\ \frac{1}{2^{1/p}} \|\Phi'\|_{[a,b],p} R_G^{1/q}(f; w) & \text{if } \Phi' \in L_p([a, b]), \\ p > 1, \frac{1}{p} + \frac{1}{q} = 1, \end{cases} \tag{3.7}$$

where the Lebesgue norms are defined by

$$\|g\|_{[\alpha,\beta],p} := \begin{cases} \operatorname{ess\,sup}_{t \in [\alpha,\beta]} |g(t)| & \text{if } p = \infty, \\ \left( \int_{\alpha}^{\beta} |g(t)|^p dt \right)^{1/p} & \text{if } p \geq 1 \end{cases}$$

and  $L_p([\alpha, \beta]) := \{g \mid g \text{ measurable and } \|g\|_{[\alpha,\beta],p} < \infty\}$ ,  $p \in [1, \infty]$ .

*Demostración.* Since  $f$  is absolutely continuous, then we have

$$\Phi(t) - \Phi(s) = \int_s^t \Phi'(u) du$$

for any  $t, s \in [a, b]$ .

Using the Hölder integral inequality we have

$$\begin{aligned} |\Phi(t) - \Phi(s)| &= \left| \int_s^t \Phi'(u) du \right| \\ &\leq \begin{cases} \|\Phi'\|_{[a,b],\infty} |t-s| & \text{if } p = \infty, \\ \|\Phi'\|_{[a,b],p} |t-s|^{1/q} & \text{if } p > 1, \frac{1}{p} + \frac{1}{q} = 1 \end{cases} \end{aligned} \tag{3.8}$$

for any  $t, s \in [a, b]$ .

Using (3.8) we then have

$$\begin{aligned}
 & |(\Phi \circ f)(x) - (\Phi \circ f)(y)| \\
 & \leq \begin{cases} \|\Phi'\|_{[a,b],\infty} |f(x) - f(y)| & \text{if } p = \infty, \\ \|\Phi'\|_{[a,b],p} |f(x) - f(y)|^{1/q} & \text{if } p > 1, \frac{1}{p} + \frac{1}{q} = 1 \end{cases} \quad (3.9)
 \end{aligned}$$

for any  $x, y \in \Omega$ .

If we multiply (3.9) by  $\frac{1}{2}w(x)w(y)$  and integrate, then we get

$$\begin{aligned}
 & \frac{1}{2} \int_{\Omega} \int_{\Omega} w(x)w(y) |(\Phi \circ f)(x) - (\Phi \circ f)(y)| \, d\nu(x) \, d\nu(y) \\
 & \leq \begin{cases} \frac{1}{2} \|\Phi'\|_{[a,b],\infty} \int_{\Omega} \int_{\Omega} w(x)w(y) |f(x) - f(y)| \, d\nu(x) \, d\nu(y) & \text{if } p = \infty, \\ \frac{1}{2} \|\Phi'\|_{[a,b],p} \int_{\Omega} \int_{\Omega} w(x)w(y) |f(x) - f(y)|^{1/q} \, d\nu(x) \, d\nu(y) & \\ \text{if } p > 1, \frac{1}{p} + \frac{1}{q} = 1. \end{cases} \quad (3.10)
 \end{aligned}$$

This proves the first branch of (3.7).

Using Jensen's integral inequality for concave function  $\Psi(t) = t^s$ ,  $s \in (0, 1)$  we have for  $s = \frac{1}{q} < 1$  that

$$\begin{aligned}
 & \int_{\Omega} \int_{\Omega} w(x)w(y) |f(x) - f(y)|^{1/q} \, d\nu(x) \, d\nu(y) \\
 & \leq \left( \int_{\Omega} \int_{\Omega} w(x)w(y) |f(x) - f(y)| \, d\nu(x) \, d\nu(y) \right)^{1/q},
 \end{aligned}$$

which implies that

$$\begin{aligned}
 & \frac{1}{2} \|\Phi'\|_{[a,b],p} \int_{\Omega} \int_{\Omega} w(x)w(y) |f(x) - f(y)|^{1/q} \, d\nu(x) \, d\nu(y) \\
 & \leq \frac{1}{2} \|\Phi'\|_{[a,b],p} \left( \int_{\Omega} \int_{\Omega} w(x)w(y) |f(x) - f(y)| \, d\nu(x) \, d\nu(y) \right)^{1/q} \\
 & = \|\Phi'\|_{[a,b],p} \left( \frac{1}{2^q} \int_{\Omega} \int_{\Omega} w(x)w(y) |f(x) - f(y)| \, d\nu(x) \, d\nu(y) \right)^{1/q} \\
 & = \|\Phi'\|_{[a,b],p} \left( \frac{1}{2^{q-1}} \frac{1}{2} \int_{\Omega} \int_{\Omega} w(x)w(y) |f(x) - f(y)| \, d\nu(x) \, d\nu(y) \right)^{1/q} \\
 & = \frac{1}{2^{\frac{q-1}{q}}} \|\Phi'\|_{[a,b],p} (\mathcal{R}_G(f; w))^{1/q} = \frac{1}{2^{1/p}} \|\Phi'\|_{[a,b],p} \mathcal{R}_G^{1/q}(f; w)
 \end{aligned}$$

and the second part of (3.7) is proved. ■

The function  $\Phi : [a, b] \rightarrow \mathbb{R}$  is called of  $r$ -H-Hölder type with the given constants  $r \in (0, 1]$  and  $H > 0$  if

$$|\Phi(t) - \Phi(s)| \leq H|t - s|^r$$

for any  $t, s \in [a, b]$ .

In the case when  $r = 1$ , namely, there is the constant  $L > 0$  such that

$$|\Phi(t) - \Phi(s)| \leq L|t - s|$$

for any  $t, s \in [a, b]$ , the function  $\Phi$  is called  $L$ -Lipschitzian on  $[a, b]$ .

We have:

**Theorem 4.** *Let  $\Phi : [a, b] \rightarrow \mathbb{R}$  be a function of  $r$ -H-Hölder type on the closed interval  $[a, b]$ . If  $w : \Omega \rightarrow \mathbb{R}$  is a  $\nu$ -measurable function with  $w(x) \geq 0$  for  $\nu$ -a.e.  $x \in \Omega$  and  $\int_{\Omega} w(x) d\nu(x) = 1$  and if  $f : \Omega \rightarrow [a, b]$  is a  $\nu$ -measurable function with  $\Phi \circ f \in L_w(\Omega, \nu)$ , then*

$$R_G(\Phi, f; w) \leq \frac{1}{2^{1-r}} HR_G^r(f; w). \tag{3.11}$$

*In particular, if  $\Phi$  is  $L$ -Lipschitzian on  $[a, b]$ , then*

$$R_G(\Phi, f; w) \leq LR_G(f; w). \tag{3.12}$$

*Demostración.* We have

$$|(\Phi \circ f)(x) - (\Phi \circ f)(y)| \leq H|f(x) - f(y)|^r \tag{3.13}$$

for any  $x, y \in \Omega$ .

If we multiply (3.13) by  $\frac{1}{2}w(x)w(y)$  and integrate, then we get

$$\begin{aligned} & \frac{1}{2} \int_{\Omega} \int_{\Omega} w(x)w(y) |(\Phi \circ f)(x) - (\Phi \circ f)(y)| d\nu(x) d\nu(y) \\ & \leq \frac{1}{2} H \int_{\Omega} \int_{\Omega} w(x)w(y) |f(x) - f(y)|^r d\nu(x) d\nu(y). \end{aligned} \tag{3.14}$$

By Jensen's integral inequality for concave functions we also have

$$\begin{aligned} & \int_{\Omega} \int_{\Omega} w(x)w(y) |f(x) - f(y)|^r d\nu(x) d\nu(y) \\ & \leq \left( \int_{\Omega} \int_{\Omega} w(x)w(y) |f(x) - f(y)| d\nu(x) d\nu(y) \right)^r. \end{aligned} \tag{3.15}$$

Therefore, by (3.14) and (3.15) we get

$$\begin{aligned} R_G(\Phi, f; w) & \leq \frac{1}{2} H \left( \int_{\Omega} \int_{\Omega} w(x)w(y) |f(x) - f(y)| d\nu(x) d\nu(y) \right)^r \\ & = \frac{1}{2^{1-r}} H \left( \frac{1}{2} \int_{\Omega} \int_{\Omega} w(x)w(y) |f(x) - f(y)| d\nu(x) d\nu(y) \right)^r \\ & = \frac{1}{2^{1-r}} HR_G^r(f; w) \end{aligned}$$

and the inequality (3.11) is proved. ■

We have:

**Theorem 5.** Let  $\Phi, \Psi : [a, b] \rightarrow \mathbb{R}$  be continuous functions on  $[a, b]$  and differentiable on  $(a, b)$  with  $\Psi'(t) \neq 0$  for  $t \in (a, b)$ . If  $w : \Omega \rightarrow \mathbb{R}$  is a  $\nu$ -measurable function with  $w(x) \geq 0$  for  $\nu$ -a.e.  $x \in \Omega$  and  $\int_{\Omega} w(x) d\nu(x) = 1$  and if  $f : \Omega \rightarrow [a, b]$  is a  $\nu$ -measurable function with  $\Phi \circ f \in L_w(\Omega, \nu)$ , then

$$\inf_{t \in (a, b)} \left| \frac{\Phi'(t)}{\Psi'(t)} \right| R_G(\Psi, f; w) \leq R_G(\Phi, f; w) \leq \sup_{t \in (a, b)} \left| \frac{\Phi'(t)}{\Psi'(t)} \right| R_G(\Psi, f; w). \quad (3.16)$$

*Demostración.* By the Cauchy's mean value theorem, for any  $t, s \in [a, b]$  with  $t \neq s$  there exists a  $\xi$  between  $t$  and  $s$  such that

$$\frac{\Phi(t) - \Phi(s)}{\Psi(t) - \Psi(s)} = \frac{\Phi'(\xi)}{\Psi'(\xi)}.$$

This implies that

$$\begin{aligned} \inf_{\tau \in (a, b)} \left| \frac{\Phi'(\tau)}{\Psi'(\tau)} \right| |\Psi(t) - \Psi(s)| &\leq |\Phi(t) - \Phi(s)| \\ &\leq \sup_{\tau \in (a, b)} \left| \frac{\Phi'(\tau)}{\Psi'(\tau)} \right| |\Psi(t) - \Psi(s)| \end{aligned} \quad (3.17)$$

for any  $t, s \in [a, b]$ .

Therefore, we have

$$\begin{aligned} \inf_{\tau \in (a, b)} \left| \frac{\Phi'(\tau)}{\Psi'(\tau)} \right| |\Psi(f(x)) - \Psi(f(y))| &\leq |\Phi(f(x)) - \Phi(f(y))| \\ &\leq \sup_{\tau \in (a, b)} \left| \frac{\Phi'(\tau)}{\Psi'(\tau)} \right| |\Psi(f(x)) - \Psi(f(y))| \end{aligned} \quad (3.18)$$

for any  $x, y \in \Omega$ .

If we multiply (3.18) by  $\frac{1}{2}w(x)w(y)$  and integrate, we get the desired result (3.16). ■

**Corollary 2.** Let  $\Phi : [a, b] \rightarrow \mathbb{R}$  be a continuous function on  $[a, b]$  and differentiable on  $(a, b)$ . If  $w$  is as in Theorem 5, then we have

$$\inf_{t \in (a, b)} |\Phi'(t)| R_G(f; w) \leq R_G(\Phi, f; w) \leq \sup_{t \in (a, b)} |\Phi'(t)| R_G(f; w). \quad (3.19)$$

We also have:

**Theorem 6.** Let  $\Phi : [a, b] \rightarrow \mathbb{R}$  be an absolutely continuous function on the closed interval  $[a, b]$ . If  $w : \Omega \rightarrow \mathbb{R}$  is a  $\nu$ -measurable function with  $w(x) \geq 0$  for  $\nu$ -a.e.  $x \in \Omega$  and  $\int_{\Omega} w(x) d\nu(x) = 1$

and if  $f : \Omega \rightarrow [a, b]$  is a  $\nu$ -measurable function with  $\Phi \circ f \in L_w(\Omega, \nu)$ , then

$$R_G(\Phi, f; w) \leq \begin{cases} \|\Phi'\|_{[a,b],\infty} M(f; w) & \text{if } p = \infty, \\ \|\Phi'\|_{[a,b],p} M^{1/q}(f; w) & \text{if } p > 1, \frac{1}{p} + \frac{1}{q} = 1 \\ \frac{1}{2}(b-a) \|\Phi'\|_{[a,b],\infty} & \text{if } p = \infty, \\ \frac{1}{2^{1/q}}(b-a)^{1/q} \|\Phi'\|_{[a,b],p} & \text{if } p > 1, \frac{1}{p} + \frac{1}{q} = 1, \end{cases} \quad (3.20)$$

where  $M(f; w)$  is defined by

$$M(f; w) := \int_{\Omega} w(x) \left| f(x) - \frac{a+b}{2} \right| d\nu(x). \quad (3.21)$$

*Demostración.* From the inequality (3.8) we have

$$\left| (\Phi \circ f)(x) - \Phi\left(\frac{a+b}{2}\right) \right| \leq \begin{cases} \|\Phi'\|_{[a,b],\infty} \left| f(x) - \frac{a+b}{2} \right| & \text{if } p = \infty, \\ \|\Phi'\|_{[a,b],p} \left| f(x) - \frac{a+b}{2} \right|^{1/q} & \text{if } p > 1, \frac{1}{p} + \frac{1}{q} = 1 \end{cases} \quad (3.22)$$

for any  $x \in \Omega$ .

Now, if we multiply (3.22) by  $w(x)$  and integrate, then we get

$$\int_{\Omega} w(x) \left| (\Phi \circ f)(x) - \Phi\left(\frac{a+b}{2}\right) \right| d\nu(x) \leq \begin{cases} \|\Phi'\|_{[a,b],\infty} \int_{\Omega} w(x) \left| f(x) - \frac{a+b}{2} \right| d\nu(x) & \text{if } p = \infty, \\ \|\Phi'\|_{[a,b],p} \int_{\Omega} w(x) \left| f(x) - \frac{a+b}{2} \right|^{1/q} d\nu(x) & \text{if } p > 1, \frac{1}{p} + \frac{1}{q} = 1. \end{cases} \quad (3.23)$$

By Jensen's integral inequality for concave functions we have

$$\int_{\Omega} w(x) \left| f(x) - \frac{a+b}{2} \right|^{1/q} d\nu(x) \leq \left( \int_{\Omega} w(x) \left| f(x) - \frac{a+b}{2} \right| d\nu(x) \right)^{1/q}. \quad (3.24)$$

On making use of (3.2), (3.23) and (3.24) we get the first inequality in (3.20).

The last part of (3.20) follows by the fact that

$$\left| f(x) - \frac{a+b}{2} \right| \leq \frac{1}{2}(b-a)$$

for any  $x \in \Omega$ . ■

## 4. Bounds for Special Convexity

When some convexity properties for the function  $\Phi$  are assumed, then other bounds can be derived as follows.

**Theorem 7.** *Let  $w : \Omega \rightarrow \mathbb{R}$  be a  $\nu$ -measurable function with  $w(x) \geq 0$  for  $\nu$ -a.e.  $x \in \Omega$  and  $\int_{\Omega} w(x) d\nu(x) = 1$  and  $f : \Omega \rightarrow [a, b]$  be a  $\nu$ -measurable function with  $\Phi \circ f \in L_w(\Omega, \nu)$ . Assume also that  $\Phi : [a, b] \rightarrow \mathbb{R}$  is a continuous function on  $[a, b]$ .*

(i) *If  $|\Phi|$  is concave on  $[a, b]$ , then*

$$R_G(\Phi, f; w) \leq |\Phi(E(f; w))|, \quad (4.1)$$

(ii) *If  $|\Phi|$  is convex on  $[a, b]$ , then*

$$R_G(\Phi, f; w) \leq \frac{1}{b-a} [(b - E(f; w)) |\Phi(a)| + (E(f; w) - a) \Phi(b)]. \quad (4.2)$$

*Demostración.* (i) If  $|\Phi|$  is concave on  $[a, b]$ , then by Jensen's inequality we have

$$\int_{\Omega} w(x) |(\Phi \circ f)(x)| d\nu(x) \leq \left| \Phi \left( \int_{\Omega} w(x) f(x) d\nu(x) \right) \right|. \quad (4.3)$$

From (3.2) for  $\gamma = 0$  we also have

$$R_G(\Phi, f; w) \leq \int_{\Omega} w(x) |(\Phi \circ f)(x)| d\nu(x). \quad (4.4)$$

This is an inequality of interest in itself.

On utilizing (4.3) and (4.4) we get (4.1).

(ii) Since  $|\Phi|$  is convex on  $[a, b]$ , then for any  $t \in [a, b]$  we have

$$|\Phi(t)| = \left| \Phi \left( \frac{(b-t)a + b(t-a)}{b-a} \right) \right| \leq \frac{(b-t)|\Phi(a)| + (t-a)\Phi(b)}{b-a}.$$

This implies that

$$|(\Phi \circ f)(x)| \leq \frac{(b-f(x))|\Phi(a)| + (f(x)-a)\Phi(b)}{b-a} \quad (4.5)$$

for any  $x \in \Omega$ .

If we multiply (4.5) by  $w(x)$  and integrate, then we get

$$\begin{aligned} & \int_{\Omega} w(x) |(\Phi \circ f)(x)| d\nu(x) \\ & \leq \frac{1}{b-a} \left[ \left( b \int_{\Omega} w(x) d\nu(x) - \int_{\Omega} w(x) f(x) d\nu(x) \right) |\Phi(a)| \right. \\ & \quad \left. + \left( \int_{\Omega} w(x) f(x) d\nu(x) - a \int_{\Omega} w(x) d\nu(x) \right) \Phi(b) \right], \end{aligned}$$

which, together with (4.4), produces the desired result (4.2). ■

In order to state other results we need the following definitions:

**Definition 1** ([19]). We say that a function  $f : I \rightarrow \mathbb{R}$  belongs to the class  $P(I)$  if it is nonnegative and for all  $x, y \in I$  and  $t \in [0, 1]$  we have

$$f(tx + (1 - t)y) \leq f(x) + f(y).$$

It is important to note that  $P(I)$  contains all nonnegative monotone, convex and *quasi convex functions*, i.e. functions satisfying

$$f(tx + (1 - t)y) \leq \max\{f(x), f(y)\}$$

for all  $x, y \in I$  and  $t \in [0, 1]$ .

For some results on  $P$ -functions see [19] and [28] while for quasi convex functions, the reader can consult [18].

**Definition 2** ([3]). Let  $s$  be a real number,  $s \in (0, 1]$ . A function  $f : [0, \infty) \rightarrow [0, \infty)$  is said to be *s-convex (in the second sense) or Breckner s-convex* if

$$f(tx + (1 - t)y) \leq t^s f(x) + (1 - t)^s f(y)$$

for all  $x, y \in [0, \infty)$  and  $t \in [0, 1]$ .

For some properties of this class of functions see [1], [2], [3], [4], [16], [17], [25], [27] and [29].

**Theorem 8.** Let  $w : \Omega \rightarrow \mathbb{R}$  be a  $\nu$ -measurable function with  $w(x) \geq 0$  for  $\nu$ -a.e.  $x \in \Omega$  and  $\int_{\Omega} w(x) d\nu(x) = 1$  and  $f : \Omega \rightarrow [a, b]$  be a  $\nu$ -measurable function with  $\Phi \circ f \in L_w(\Omega, \nu)$ . Assume also that  $\Phi : [a, b] \rightarrow \mathbb{R}$  is a continuous function on  $[a, b]$ .

(i) If  $|\Phi|$  belongs to the class  $P$  on  $[a, b]$ , then

$$R_G(\Phi, f; w) \leq |\Phi(a)| + \Phi(b); \tag{4.6}$$

(ii) If  $|\Phi|$  is quasi convex on  $[a, b]$ , then

$$R_G(\Phi, f; w) \leq \max\{|\Phi(a)|, \Phi(b)\}; \tag{4.7}$$

(iii) If  $|\Phi|$  is Breckner  $s$ -convex on  $[a, b]$ , then

$$\begin{aligned} R_G(\Phi, f; w) &\leq \frac{1}{(b-a)^s} \left[ |\Phi(a)| \int_{\Omega} w(x) (b-f(x))^s d\nu(x) \right. \\ &\quad \left. + \Phi(b) \int_{\Omega} w(x) (f(x)-a)^s d\nu(x) \right] \\ &\leq \frac{1}{(b-a)^s} \left[ |\Phi(a)| (b-E(f; w))^s d\nu(x) \right. \\ &\quad \left. + \Phi(b) (E(f; w)-a)^s d\nu(x) \right]. \end{aligned} \tag{4.8}$$



*Demostración.* (i) Since  $|\Phi|$  belongs to the class  $\mathcal{P}$  on  $[a, b]$ , then for any  $t \in [a, b]$  we have

$$|\Phi(t)| = \left| \Phi \left( \frac{(b-t)a + b(t-a)}{b-a} \right) \right| \leq |\Phi(a)| + \Phi|(b)|.$$

This implies that

$$|(\Phi \circ f)(x)| \leq |\Phi(a)| + \Phi|(b)| \quad (4.9)$$

for any  $x \in \Omega$ .

If we multiply (4.9) by  $w(x)$  and integrate, then we get

$$\int_{\Omega} w(x) |(\Phi \circ f)(x)| \, d\nu(x) \leq |\Phi(a)| + \Phi|(b)|, \quad (4.10)$$

which, together with (4.4), produces the desired result (4.6).

(ii) Goes in a similar way.

(iii) By Breckner  $s$ -convexity we have

$$|\Phi(t)| = \left| \Phi \left( \frac{(b-t)a + b(t-a)}{b-a} \right) \right| \leq \left( \frac{b-t}{b-a} \right)^s |\Phi(a)| + \left( \frac{t-a}{b-a} \right)^s \Phi|(b)|$$

for any  $t \in [a, b]$ .

This implies that

$$|(\Phi \circ f)(x)| \leq \frac{1}{(b-a)^s} [(b-f(x))^s |\Phi(a)| + (f(x)-a)^s \Phi|(b)|] \quad (4.11)$$

for any  $x \in \Omega$ .

If we multiply (4.11) by  $w(x)$  and integrate, then we get

$$\begin{aligned} \int_{\Omega} w(x) |(\Phi \circ f)(x)| \, d\nu(x) &\leq \frac{1}{(b-a)^s} \left[ |\Phi(a)| \int_{\Omega} w(x) (b-f(x))^s \, d\nu(x) \right. \\ &\quad \left. + \Phi|(b)| \int_{\Omega} w(x) (f(x)-a)^s \, d\nu(x) \right], \end{aligned} \quad (4.12)$$

which, together with (4.4), produces the first part of (4.8).

The last part follows by Jensen's integral inequality for concave functions, namely

$$\int_{\Omega} w(x) (b-f(x))^s \, d\nu(x) \leq \left( b - \int_{\Omega} w(x) f(x) \, d\nu(x) \right)^s$$

and

$$\int_{\Omega} w(x) (f(x)-a)^s \, d\nu(x) \leq \left( \int_{\Omega} w(x) f(x) \, d\nu(x) - a \right)^s,$$

where  $s \in (0, 1)$ . ■

## 5. Some Examples

Let  $f : \Omega \rightarrow [0, \infty)$  be a  $\nu$ -measurable function and  $w : \Omega \rightarrow \mathbb{R}$  a  $\nu$ -measurable function with  $w(x) \geq 0$  for  $\nu$ -a.e.  $x \in \Omega$  and  $\int_{\Omega} w(x) d\nu(x) = 1$ . We define, for the function  $\Phi(t) = t^p$ ,  $p > 0$ , the *generalized  $(p, f)$ -mean difference*  $R_G(p, f; w)$  by

$$R_G(p, f; w) := \frac{1}{2} \int_{\Omega} \int_{\Omega} w(x) w(y) |f^p(x) - f^p(y)| d\nu(x) d\nu(y) \quad (5.1)$$

and the *generalized  $(p, f)$ -mean deviation*  $M_D(p, f; w)$  by

$$M_D(p, f; w) := \int_{\Omega} w(x) |f^p(x) - E(p, f; w)| d\nu(x), \quad (5.2)$$

where

$$E(p, f; w) := \int_{\Omega} f^p(y) w(y) d\nu(y) \quad (5.3)$$

is the *generalized  $(p, f)$ -expectation*.

If  $f : \Omega \rightarrow [a, b] \subset [0, \infty)$  is a  $\nu$ -measurable function, then by (3.1) we have

$$R_G(p, f; w) \leq \frac{1}{2} (b^p - a^p). \quad (5.4)$$

By (3.7) we have

$$R_G(p, f; w) \leq p \delta_p(a, b) R_G(f; w), \quad (5.5)$$

where

$$\delta_p(a, b) := \begin{cases} b^{p-1} & \text{if } p \geq 1, \\ a^{p-1} & \text{if } p \in (0, 1) \end{cases}$$

and

$$R_G(p, f; w) \leq \frac{p}{2^{1/\alpha}} \left[ \frac{b^{\alpha(p-1)+1} - a^{\alpha(p-1)+1}}{\alpha(p-1)+1} \right]^{1/\alpha} R_G^{1/\beta}(f; w), \quad (5.6)$$

where  $\alpha > 1$ ,  $\frac{1}{\alpha} + \frac{1}{\beta} = 1$ .

From (3.20) we also have

$$\begin{aligned} & R_G(p, f; w) \\ & \leq \begin{cases} \delta_p(a, b) M(f; w), \\ p \left( \frac{b^{\alpha(p-1)+1} - a^{\alpha(p-1)+1}}{\alpha(p-1)+1} \right)^{1/\alpha} M^{1/\beta}(f; w) & \text{if } \alpha > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1 \end{cases} \\ & \leq \begin{cases} \frac{1}{2} (b - a) \delta_p(a, b), \\ \frac{1}{2^{1/\beta}} (b - a)^{1/\beta} p \left( \frac{b^{\alpha(p-1)+1} - a^{\alpha(p-1)+1}}{\alpha(p-1)+1} \right)^{1/\alpha} & \text{if } \alpha > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1, \end{cases} \end{aligned} \quad (5.7)$$

where  $M(f; w)$  is defined by (3.21).

If  $p \in (0, 1)$ , then the function  $|\Phi(t)| = t^p$  is concave on  $[a, b] \subset [0, \infty)$  and by (4.1) we have

$$R_G(p, f; w) \leq E^p(f; w). \quad (5.8)$$

For  $p \geq 1$  the function  $|\Phi(t)| = t^p$  is convex on  $[a, b] \subset [0, \infty)$  and by (4.2) we have

$$R_G(p, f; w) \leq \frac{1}{b-a} [(b - E(f; w))^p + (E(f; w) - a)^p]. \quad (5.9)$$

Let  $f: \Omega \rightarrow [0, \infty)$  be a  $\nu$ -measurable function and  $w: \Omega \rightarrow \mathbb{R}$  a  $\nu$ -measurable function with  $w(x) \geq 0$  for  $\nu$ -a.e.  $x \in \Omega$  and  $\int_{\Omega} w(x) d\nu(x) = 1$ . We define, for the function  $\Phi(t) = \ln t$ , the *generalized  $(\ln, f)$ -mean difference*  $R_G(\ln, f; w)$  by

$$R_G(\ln, f; w) := \frac{1}{2} \int_{\Omega} \int_{\Omega} w(x) w(y) |\ln f(x) - \ln f(y)| d\nu(x) d\nu(y) \quad (5.10)$$

and the *generalized  $(p, f)$ -mean deviation*  $M_D(\ln, f; w)$  by

$$M_D(\ln, f; w) := \int_{\Omega} w(x) |\ln f(x) - E(\ln, f; w)| d\nu(x), \quad (5.11)$$

where

$$E(\ln, f; w) := \int_{\Omega} w(y) \ln f(y) d\nu(y) \quad (5.12)$$

is the *generalized  $(\ln, f)$ -expectation*.

If  $f: \Omega \rightarrow [a, b] \subset [0, \infty)$  is a  $\nu$ -measurable function, then by (3.1) we have

$$R_G(\ln, f; w) \leq \frac{1}{2} (\ln b - \ln a). \quad (5.13)$$

By (3.7) we have

$$R_G(\ln, f; w) \leq \begin{cases} \frac{1}{a} R_G(f; w), \\ \frac{1}{2^{1/p}} \left( \frac{b^{p-1} - a^{p-1}}{(p-1)b^{p-1}a^{p-1}} \right)^{1/p} R_G^{1/q}(f; w) \text{ if } p > 1, \frac{1}{p} + \frac{1}{q} = 1. \end{cases} \quad (5.14)$$

By (3.20) we have

$$R_G(\ln, f; w) \leq \begin{cases} \frac{1}{a} M(f; w), \\ \left( \frac{b^{p-1} - a^{p-1}}{(p-1)b^{p-1}a^{p-1}} \right)^{1/p} M^{1/q}(f; w) \text{ if } p > 1, \frac{1}{p} + \frac{1}{q} = 1 \end{cases} \quad (5.15)$$

$$\leq \begin{cases} \frac{1}{2} \left( \frac{b}{a} - 1 \right), \\ \frac{1}{2^{1/q}} (b - a)^{1/q} \left( \frac{b^{p-1} - a^{p-1}}{(p-1)b^{p-1}a^{p-1}} \right)^{1/p} \text{ if } p > 1, \frac{1}{p} + \frac{1}{q} = 1. \end{cases}$$

Now, observe that the function  $|\Phi(t)| = |\ln t|$  is convex on  $(0, 1)$  and concave on  $[1, \infty)$ . If  $f : \Omega \rightarrow [a, b] \subset (0, 1)$  is a  $\nu$ -measurable function, then by (4.2) we have

$$R_G(\ln, f; w) \leq \frac{1}{b-a} [(b - E(f; w)) |\ln a| + (E(f; w) - a) |\ln b|] \quad (5.16)$$

and if  $f : \Omega \rightarrow [a, b] \subset [1, \infty)$ , then by (4.1) we have

$$R_G(\ln, f; w) \leq \ln(E(f; w)). \quad (5.17)$$

The interested reader may state similar bounds for functions  $\Phi$  such as  $\Phi(t) = \exp t$ ,  $t \in \mathbb{R}$  or  $\Phi(t) = t \ln t$ ,  $t > 0$ . We omit the details.

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