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Inequalities of Hermite-Hadamard Type for K-Bounded Modulus Convex Complex Functions

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Abstract. Let $D \subset \mathbb{C}$ be a convex domain of complex numbers and $K > 0$. We say that the function $f : D \subset \mathbb{C} \to \mathbb{C}$ is called K-bounded modulus convex, for the given $K > 0$, if it satisfies the condition

$$|(1-\lambda) f(x) + \lambda f(y) - f((1-\lambda) x + \lambda y)| \leq \frac{1}{2} K \lambda (1-\lambda) |x-y|^2$$

for any $x, y \in D$ and $\lambda \in [0,1]$. In this paper we establish some new Hermite-Hadamard type inequalities for the complex integral on γ, a smooth path from $\mathbb{C}$, and K-bounded modulus convex functions. Some examples for integrals on segments and circular paths are also given.

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1 Introduction

Let $(X; \|\cdot\|_X)$ and $(Y; \|\cdot\|_Y)$ be two normed linear spaces over the complex number field $\mathbb{C}$. Let C be a convex set in X. In the recent paper [1] we introduced the following class of functions:

Definition 1. *A mapping $f : C \subset X \to Y$ is called K-bounded norm convex, for some given $K > 0$, if it satisfies the condition*

$$\|(1-\lambda) f(x) + \lambda f(y) - f((1-\lambda) x + \lambda y)\|_Y \leq \frac{1}{2} K \lambda (1-\lambda) \|x-y\|_X^2 \qquad (1)$$

for any $x, y \in C$ and $\lambda \in [0,1]$. For simplicity, we denote this by $f \in \mathcal{BN}_K(C)$.

We have from (1) for $\lambda = \frac{1}{2}$ the Jensen's inequality

$$\left\| \frac{f(x) + f(y)}{2} - f\left(\frac{x+y}{2}\right) \right\|_Y \leq \frac{1}{8} K \|x-y\|_X^2$$

for any $x, y \in C$.

We observe that $\mathcal{BN}_K(C)$ is a convex subset in the linear space of all functions defined on C and with values in Y.

In the same paper [1], we obtained the following result which provides a large class of examples of such functions.

Theorem 1. *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed linear spaces, C an open convex subset of X and $f : C \to Y$ a twice-differentiable mapping on C. Then for any x, $y \in C$ and $\lambda \in [0,1]$ we have*

$$\|(1-\lambda) f(x) + \lambda f(y) - f((1-\lambda) x + \lambda y)\|_Y \leq \frac{1}{2} K \lambda (1-\lambda) \|y - x\|_X^2, \tag{2}$$

where

$$K_{f''} := \sup_{z \in C} \left\| f''(z) \right\|_{\mathcal{L}(X^2;Y)} \tag{3}$$

is assumed to be finite, namely $f \in \mathcal{BN}_{K_{f''}}(C)$.

We have the following inequalities of Hermite-Hadamard type [1]:

Theorem 2. *Let $(X; \|\cdot\|_X)$ and $(Y; \|\cdot\|_Y)$ be two normed linear spaces over the complex number field $\mathbb{C}$ with Y complete. Assume that the mapping $f : C \subset X \to Y$ is continuous on the convex set C in the norm topology. If $f \in \mathcal{BN}_K(C)$ for some $K > 0$, then we have*

$$\left\| \frac{f(x) + f(y)}{2} - \int_0^1 f((1-\lambda) x + \lambda y) \, d\lambda \right\|_Y \leq \frac{1}{12} K \|x - y\|_X^2 \tag{4}$$

and

$$\left\| \int_0^1 f((1-\lambda) x + \lambda y) \, d\lambda - f\left(\frac{x+y}{2}\right) \right\|_Y \leq \frac{1}{24} K \|x - y\|_X^2 \tag{5}$$

for any x, $y \in C$.

The constants $\frac{1}{12}$ and $\frac{1}{24}$ are best possible.

For a monograph devoted to Hermite-Hadamard type inequalities see [3] and the recent survey paper [2].

Let $D \subset \mathbb{C}$ be a convex domain of complex numbers and $K > 0$. Following Definition 1, we say that the function $f : D \subset \mathbb{C} \to \mathbb{C}$ is called K-bounded modulus convex, for the given $K > 0$, if it satisfies the condition

$$|(1-\lambda) f(x) + \lambda f(y) - f((1-\lambda) x + \lambda y)| \leq \frac{1}{2} K \lambda (1-\lambda) |x - y|^2 \tag{6}$$

for any x, $y \in D$ and $\lambda \in [0,1]$. For simplicity, we denote this by $f \in \mathcal{BM}_K(D)$.

All the above results can be translated for complex functions defined on convex subsets $D \subset \mathbb{C}$.

In the following, in order to obtain several inequalities for the complex integral, we need the following facts.

Suppose γ is a smooth path from $\mathbb{C}$ parametrized by $z(t)$, $t \in [a,b]$ and f is a complex function which is continuous on γ. Put $z(a) = u$ and $z(b) = w$ with u, $w \in \mathbb{C}$. We define the integral of f on $\gamma_{u,w} = \gamma$ as

$$\int_\gamma f(z) \, dz = \int_{\gamma_{u,w}} f(z) \, dz := \int_a^b f(z(t)) \, z'(t) \, dt.$$

We observe that the actual choice of parametrization of γ does not matter.

This definition immediately extends to paths that are piecewise smooth. Suppose γ is parametrized by $z(t)$, $t \in [a,b]$, which is differentiable on the intervals $[a,c]$ and $[c,b]$, then assuming that f is continuous on γ we define

$$\int_{\gamma_{u,w}} f(z)\,dz := \int_{\gamma_{u,v}} f(z)\,dz + \int_{\gamma_{v,w}} f(z)\,dz$$

where $v := z(s)$ for some $s \in (a,b)$. This can be extended for a finite number of intervals.

We also define the integral with respect to arc-length

$$\int_{\gamma_{u,w}} f(z)\,|dz| := \int_a^b f(z(t))\left|z'(t)\right| dt$$

and the length of the curve γ is then

$$\ell(\gamma) = \int_{\gamma_{u,w}} |dz| = \int_a^b \left|z'(t)\right| dt.$$

Let f and g be holomorphic in D, an open domain and suppose $\gamma \subset D$ is a piecewise smooth path from $z(a) = u$ to $z(b) = w$. Then we have the *integration by parts formula*

$$\int_{\gamma_{u,w}} f(z)\,g'(z)\,dz = f(w)\,g(w) - f(u)\,g(u) - \int_{\gamma_{u,w}} f'(z)\,g(z)\,dz. \tag{7}$$

We recall also the *triangle inequality* for the complex integral, namely

$$\left|\int_\gamma f(z)\,dz\right| \le \int_\gamma |f(z)|\,|dz| \le \|f\|_{\gamma,\infty}\,\ell(\gamma) \tag{8}$$

where $\|f\|_{\gamma,\infty} := \sup_{z\in\gamma} |f(z)|$.

We also define the p-norm with $p \ge 1$ by

$$\|f\|_{\gamma,p} := \left(\int_\gamma |f(z)|^p\,|dz|\right)^{1/p}.$$

For $p = 1$ we have

$$\|f\|_{\gamma,1} := \int_\gamma |f(z)|\,|dz|.$$

If p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then by Hölder's inequality we have

$$\|f\|_{\gamma,1} \le [\ell(\gamma)]^{1/q}\,\|f\|_{\gamma,p}.$$

Motivated by the above results, in this paper we establish some new Hermite-Hadamard type inequalities for the complex integral on γ, a smooth path from $\mathbb{C}$, and K-bounded modulus convex functions. Some examples for integrals on segments and circular paths are also given.

2 Integral Inequalities

We have:

Theorem 3. *Let $D \subset \mathbb{C}$ be a convex domain of complex numbers and $K > 0$. Assume that f is holomorphic on D and $f \in \mathcal{BM}_K(D)$. If $\gamma \subset D$ parametrized by $z(t)$, $t \in [a,b]$ is a piecewise smooth path from $z(a) = u$ to $z(b) = w$ and $v \in D$, then*

$$\left| \int_\gamma f(z)\,dz - \left[f(v) + f'(v) \left(\frac{w+u}{2} - v \right) \right] (w-u) \right| \le \frac{1}{2} K \int_\gamma |z-v|^2 |dz| . \tag{9}$$

In particular, we have for $v = \frac{w+u}{2}$ that

$$\left| \int_\gamma f(z)\,dz - f\left(\frac{w+u}{2} \right) (w-u) \right| \le \frac{1}{2} K \int_\gamma \left| z - \frac{w+u}{2} \right|^2 |dz| . \tag{10}$$

Proof. Let x, $y \in D$. Since $f \in \mathcal{BM}_K(D)$, then we have

$$|f((1-\lambda)x + \lambda y) - f(x) + \lambda [f(x) - f(y)]| \le \frac{1}{2} K \lambda (1-\lambda) |x-y|^2$$

that implies that

$$\left| \frac{f(x + \lambda(y-x)) - f(x)}{\lambda} + f(x) - f(y) \right| \le \frac{1}{2} K (1-\lambda) |x-y|^2$$

for $\lambda \in (0,1)$.

Since f is holomorphic on D, then by letting $\lambda \to 0+$, we get

$$\left| f'(x)(y-x) + f(x) - f(y) \right| \le \frac{1}{2} K |x-y|^2$$

that is equivalent to

$$\left| f(y) - f(x) - f'(x)(y-x) \right| \le \frac{1}{2} K |y-x|^2 \tag{11}$$

for all x, $y \in D$.

We have

$$\begin{aligned} &\int_\gamma \left[f(z) - f(v) - f'(v)(z-v) \right] dz \qquad (12) \\ &= \int_\gamma f(z)\,dz - f(v) \int_\gamma dz - f'(v) \left(\int_\gamma z\,dz - v \int_\gamma dz \right) \\ &= \int_\gamma f(z)\,dz - f(v)(w-u) - f'(v) \left[\frac{1}{2} \left(w^2 - u^2 \right) - v(w-u) \right] \\ &= \int_\gamma f(z)\,dz - \left[f(v) + f'(v) \left(\frac{w+u}{2} - v \right) \right] (w-u) \end{aligned}$$

for any $v \in D$.

By using (11) we get

$$\begin{aligned} & \left| \int_\gamma f(z)\, dz - \left[f(v) + f'(v) \left(\frac{w+u}{2} - v \right) \right] (w-u) \right| \\ & \leq \int_\gamma \left| f(z) - f(v) - f'(v)(z-v) \right| |dz| \leq \frac{1}{2} K \int_\gamma |z-v|^2 |dz| \end{aligned}$$

for any $v \in D$, which proves the inequality (9). □

If the path γ is a segment $[u, w] \subset G$ connecting two distinct points u and w in G then we write $\int_\gamma f(z)\, dz$ as $\int_u^w f(z)\, dz$.

Corollary 4. *With the assumptions of Theorem 3, suppose $[u, w] \subset D$ is a segment connecting two distinct points u and w in D and $v \in [u, w]$. Then for $v = (1-s) u + sw$ with $s \in [0, 1]$, we have*

$$\begin{aligned} & \left| \int_u^w f(z)\, dz - f((1-s) u + sw)(w-u) \right. \\ & \qquad \left. - f'((1-s) u + sw) \left(\frac{1}{2} - s \right) (w-u)^2 \right| \\ & \qquad \qquad \leq \frac{1}{6} K |w-u|^3 \left[(1-s)^3 + s^3 \right]. \end{aligned} \tag{13}$$

In particular, we have, see also (5),

$$\left| \int_u^w f(z)\, dz - f\left(\frac{w+u}{2} \right) (w-u) \right| \leq \frac{1}{24} K |w-u|^3. \tag{14}$$

Proof. It follows by Theorem 3 by observing that

$$\begin{aligned} \int_u^w |z-v|^2 |dz| &= |w-u| \int_0^1 |(1-t) u + tw - (1-s) u - sw|^2\, dt \\ &= |w-u| \int_0^1 |(1-t) u + tw - (1-s) u - sw|^2\, dt \\ &= |w-u|^3 \int_0^1 (t-s)^2\, dt = \frac{1}{3} |w-u|^3 \left[(1-s)^3 + s^3 \right] \end{aligned}$$

for $s \in [0, 1]$. □

Theorem 5. *Let $D \subset \mathbb{C}$ be a convex domain of complex numbers and $K > 0$. Assume that f is holomorphic on D and $f \in \mathcal{BM}_K(D)$. If $\gamma \subset D$ parametrized by $z(t)$, $t \in [a, b]$ is a piecewise smooth path from $z(a) = u$ to $z(b) = w$ and $v \in D$, then*

$$\left| \frac{1}{2} \left[f(w)(w-v) + f(u)(v-u) + f(v)(w-u) \right] - \int_\gamma f(z)\, dz \right|$$

$$\leq \frac{1}{4}K\int_{\gamma}|z-v|^{2}\,|dz|\,. \quad (15)$$

In particular, we have for $v=\frac{w+u}{2}$ *that*

$$\left|\frac{1}{2}\left[\frac{f(w)+f(u)}{2}+f\left(\frac{w+u}{2}\right)\right](w-u)-\int_{\gamma}f(z)\,dz\right| \leq \frac{1}{4}K\int_{\gamma}\left|z-\frac{w+u}{2}\right|^{2}|dz|\,. \quad (16)$$

Proof. By using (11) we get

$$\int_{\gamma}\left|f(v)-f(z)-f'(z)(v-z)\right||dz|\leq\frac{1}{2}K\int_{\gamma}|v-z|^{2}\,|dz| \quad (17)$$

for $v\in D$.

By the complex integral properties, we have

$$\left|\int_{\gamma}\left[f(v)-f(z)-f'(z)(v-z)\right]dz\right| \leq\int_{\gamma}\left|f(v)-f(z)-f'(z)(v-z)\right||dz| \quad (18)$$

for $v\in D$.

Using integration by parts, we get

$$\begin{aligned}
&\int_{\gamma}\left[f(v)-f(z)-f'(z)(v-z)\right]dz\\
&=f(v)\int_{\gamma}dz-\int_{\gamma}f(z)\,dz-\int_{\gamma}f'(z)(v-z)\,dz\\
&=f(v)(w-u)-\int_{\gamma}f(z)\,dz-\left[f(z)(v-z)|_{u}^{w}+\int_{\gamma}f(z)\,dz\right]\\
&=f(v)(w-u)-\int_{\gamma}f(z)\,dz-f(w)(v-w)+f(u)(v-u)-\int_{\gamma}f(z)\,dz\\
&=f(w)(w-v)+f(u)(v-u)+f(v)(w-u)-2\int_{\gamma}f(z)\,dz,
\end{aligned}$$

which implies that

$$\frac{1}{2}\left[f(w)(w-v)+f(u)(v-u)+f(v)(w-u)\right]-\int_{\gamma}f(z)\,dz =\frac{1}{2}\int_{\gamma}\left[f(v)-f(z)-f'(z)(v-z)\right]dz \quad (19)$$

for $v\in D$.

By utilising (17)-(19) we get the desired result (15). □

We have:

Corollary 6. *With the assumptions of Theorem 3, suppose* $[u,w]\subset D$ *is a segment connecting two distinct points* u *and* w *in* D *and* $v\in[u,w]$. *Then for* $v=(1-s)u+sw$ *with* $s\in[0,1]$, *we have*

$$\left|\frac{1}{2}\left[(1-s)f(w)+sf(u)+f((1-s)u+sw)\right](w-u)-\int_u^w f(z)\,dz\right| \leq \frac{1}{12}K|w-u|^3\left[(1-s)^3+s^3\right]. \quad (20)$$

In particular, we have for $v=\frac{w+u}{2}$ *that*

$$\left|\frac{1}{2}\left[\frac{f(w)+f(u)}{2}+f\left(\frac{w+u}{2}\right)\right](w-u)-\int_u^w f(z)\,dz\right| \leq \frac{1}{48}K|w-u|^3. \quad (21)$$

We observe that, if f is holomorphic on D and $K=\sup_{z\in D}|f''(z)|$ is finite, then by (9) and (10) we have

$$\left|\int_\gamma f(z)\,dz-\left[f(v)+f'(v)\left(\frac{w+u}{2}-v\right)\right](w-u)\right| \leq \frac{1}{2}\sup_{z\in D}\left|f''(z)\right|\int_\gamma|z-v|^2|dz| \quad (22)$$

for all $v\in D$. In particular,

$$\left|\int_\gamma f(z)\,dz-f\left(\frac{w+u}{2}\right)(w-u)\right| \leq \frac{1}{2}\sup_{z\in D}\left|f''(z)\right|\int_\gamma\left|z-\frac{w+u}{2}\right|^2|dz|. \quad (23)$$

From (15) and (16) we get

$$\left|\frac{1}{2}\left[f(w)(w-v)+f(u)(v-u)+f(v)(w-u)\right]-\int_\gamma f(z)\,dz\right| \leq \frac{1}{4}\sup_{z\in D}\left|f''(z)\right|\int_\gamma|z-v|^2|dz|. \quad (24)$$

for all $v\in D$. In particular,

$$\left|\frac{1}{2}\left[\frac{f(w)+f(u)}{2}+f\left(\frac{w+u}{2}\right)\right](w-u)-\int_\gamma f(z)\,dz\right| \leq \frac{1}{4}\sup_{z\in D}\left|f''(z)\right|\int_\gamma\left|z-\frac{w+u}{2}\right|^2|dz|. \quad (25)$$

The inequalities (22)-(25) provide many examples of interest as follows.

If we consider the function $f(z) = \exp z$, $z \in \mathbb{C}$ and $\gamma \subset \mathbb{C}$ parametrized by $z(t)$, $t \in [a, b]$ is a piecewise smooth path from $z(a) = u$ to $z(b) = w$ then by (22)-(25) we have by the inequalities

$$\left|\exp w - \exp u - \left(1 + \frac{w+u}{2} - v\right)(w-u)\exp v\right| \leq \frac{1}{2}\sup_{z\in D}|\exp z| \int_\gamma |z-v|^2 |dz| \quad (26)$$

for all $v \in \mathbb{C}$. In particular,

$$\left|\exp w - \exp u - \exp\left(\frac{w+u}{2}\right)(w-u)\right| \leq \frac{1}{2}\sup_{z\in D}|\exp z| \int_\gamma \left|z - \frac{w+u}{2}\right|^2 |dz|. \quad (27)$$

We also have

$$\left|\frac{1}{2}\left[(w-v)\exp w + (v-u)\exp u + (w-u)\exp v\right] - \exp w + \exp u\right| \leq \frac{1}{4}\sup_{z\in D}|\exp z| \int_\gamma |z-v|^2 |dz|. \quad (28)$$

for all $v \in \mathbb{C}$. In particular,

$$\left|\frac{1}{2}\left[\frac{\exp w + \exp u}{2} + \exp\left(\frac{w+u}{2}\right)\right](w-u) - \exp w + \exp u\right| \leq \frac{1}{4}\sup_{z\in D}|\exp z| \int_\gamma \left|z - \frac{w+u}{2}\right|^2 |dz|. \quad (29)$$

Consider the function $F(z) = \operatorname{Log}(z)$ where $\operatorname{Log}(z) = \ln|z| + i\operatorname{Arg}(z)$ and $\operatorname{Arg}(z)$ is such that $-\pi < \operatorname{Arg}(z) \leq \pi$. Log is called the *"principal branch"* of the complex logarithmic function. F is analytic on all of $\mathbb{C}\backslash\{x+iy : x \leq 0,\ y = 0\}$ and $F'(z) = \frac{1}{z}$ on this set.

If we consider $f : D \to \mathbb{C}$, $f(z) = \frac{1}{z}$ where $D \subset \mathbb{C}\backslash\{x+iy : x \leq 0,\ y = 0\}$, then F is a primitive of f on D and if $\gamma \subset D$ parametrized by $z(t)$, $t \in [a, b]$ is a piecewise smooth path from $z(a) = u$ to $z(b) = w$, then

$$\int_\gamma f(z)\,dz = \operatorname{Log}(w) - \operatorname{Log}(u).$$

For $D \subset \mathbb{C}\backslash\{x+iy : x \leq 0,\ y = 0\}$, define $d := \inf_{z\in D}|z|$ and assume that $d \in (0, \infty)$. By the inequalities (22)-(25) we then have

$$\left|\operatorname{Log}(w)-\operatorname{Log}(u)-\left[\frac{1}{v}-\frac{1}{v^{2}}\left(\frac{w+u}{2}-v\right)\right](w-u)\right| \leq \frac{1}{d^{3}}\int_{\gamma}|z-v|^{2}|dz| \quad (30)$$

for all $v\in D$. In particular,

$$\left|\operatorname{Log}(w)-\operatorname{Log}(u)-\left(\frac{w+u}{2}\right)^{-1}(w-u)\right| \leq \frac{1}{d^{3}}\int_{\gamma}\left|z-\frac{w+u}{2}\right|^{2}|dz|. \quad (31)$$

We also have

$$\left|\frac{1}{2}\left(\frac{w-v}{w}+\frac{v-u}{u}+\frac{w-u}{v}\right)-\operatorname{Log}(w)+\operatorname{Log}(u)\right| \leq \frac{1}{2d^{3}}\int_{\gamma}|z-v|^{2}|dz|. \quad (32)$$

for all $v\in D$. In particular,

$$\left|\frac{1}{2}\left[\frac{u+w}{2wu}+\left(\frac{w+u}{2}\right)^{-1}\right](w-u) -\operatorname{Log}(w)+\operatorname{Log}(u)\right| \leq \frac{1}{2d^{3}}\int_{\gamma}\left|z-\frac{w+u}{2}\right|^{2}|dz|. \quad (33)$$

3 Examples for Circular Paths

Let $[a,b]\subseteq[0,2\pi]$ and the circular path $\gamma_{[a,b],R}$ be centered at 0 and with radius $R>0$

$$z(t)=R\exp(it)=R(\cos t+i\sin t),\ t\in[a,b].$$

If $[a,b]=[0,\pi]$ then we get a half circle while for $[a,b]=[0,2\pi]$ we get the full circle.

Since

$$\begin{aligned}\left|e^{is}-e^{it}\right|^{2}&=\left|e^{is}\right|^{2}-2\operatorname{Re}\left(e^{i(s-t)}\right)+\left|e^{it}\right|^{2}\\&=2-2\cos(s-t)=4\sin^{2}\left(\frac{s-t}{2}\right)\end{aligned}$$

for any t, $s\in\mathbb{R}$, then

$$\left|e^{is}-e^{it}\right|^{r}=2^{r}\left|\sin\left(\frac{s-t}{2}\right)\right|^{r} \quad (34)$$

for any t, $s\in\mathbb{R}$ and $r>0$. In particular,

$$\left|e^{is}-e^{it}\right|=2\left|\sin\left(\frac{s-t}{2}\right)\right|$$

for any $t,\ s\in\mathbb{R}$.

For $s=a$ and $s=b$ we have

$$\left|e^{ia}-e^{it}\right|=2\left|\sin\left(\frac{a-t}{2}\right)\right| \text{ and } \left|e^{ib}-e^{it}\right|=2\left|\sin\left(\frac{b-t}{2}\right)\right|.$$

If $u=R\exp(ia)$ and $w=R\exp(ib)$ then

$$\begin{aligned}w-u&=R[\exp(ib)-\exp(ia)]=R[\cos b+i\sin b-\cos a-i\sin a]\\&=R[\cos b-\cos a+i(\sin b-\sin a)].\end{aligned}$$

Since

$$\cos b-\cos a=-2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{b-a}{2}\right)$$

and

$$\sin b-\sin a=2\sin\left(\frac{b-a}{2}\right)\cos\left(\frac{a+b}{2}\right),$$

hence

$$\begin{aligned}w-u&=R\left[-2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{b-a}{2}\right)+2i\sin\left(\frac{b-a}{2}\right)\cos\left(\frac{a+b}{2}\right)\right]\\&=2R\sin\left(\frac{b-a}{2}\right)\left[-\sin\left(\frac{a+b}{2}\right)+i\cos\left(\frac{a+b}{2}\right)\right]\\&=2Ri\sin\left(\frac{b-a}{2}\right)\left[\cos\left(\frac{a+b}{2}\right)+i\sin\left(\frac{a+b}{2}\right)\right]\\&=2Ri\sin\left(\frac{b-a}{2}\right)\exp\left[\left(\frac{a+b}{2}\right)i\right].\end{aligned}$$

We also have

$$z'(t)=Ri\exp(it) \text{ and } \left|z'(t)\right|=R$$

for $t\in[a,b]$.

In what follows we assume that f is defined on a domain containing the circular path $\gamma_{[a,b],R}$ and that f is holomorphic on that domain.

Consider the circular path $\gamma_{[a,b],R}$ and assume that $v=R\exp(is)\in\gamma_{[a,b],R}$ with $s\in[a,b]$. Then by using the inequality (9) we get

$$\begin{aligned}&\Bigg|Ri\int_a^b f(R\exp(it))\exp(it)\,dt\\&-\left[f(R\exp(is))+f'(R\exp(is))\left(\frac{R\exp(ib)+R\exp(ia)}{2}-R\exp(is)\right)\right]\\&\qquad\times 2Ri\sin\left(\frac{b-a}{2}\right)\exp\left[\left(\frac{a+b}{2}\right)i\right]\Bigg|\\&\le\frac{1}{2}\sup_{t\in[a,b]}\left|f''(R\exp(it))\right|R\int_a^b\left|R\exp(it)-R\exp(is)\right|^2dt\end{aligned}$$

$$= \frac{1}{2} \sup_{t\in[a,b]} \left| f''(R\exp(it)) \right| R^3 \int_a^b 4\sin^2\left(\frac{s-t}{2}\right) dt$$
$$= 2 \sup_{t\in[a,b]} \left| f''(R\exp(it)) \right| R^3 \int_a^b \sin^2\left(\frac{s-t}{2}\right) dt,$$

which is equivalent to

$$\begin{aligned}\Bigg| \int_a^b & f(R\exp(it))\exp(it)\,dt \\ & - 2R\left[f(R\exp(is)) + f'(R\exp(is))\left(\frac{\exp(ib)+\exp(ia)}{2} - \exp(is)\right)\right] \\ & \times \sin\left(\frac{b-a}{2}\right)\exp\left[\left(\frac{a+b}{2}\right)i\right]\Bigg| \\ & \le 2\sup_{t\in[a,b]}\left| f''(R\exp(it))\right| R^2 \int_a^b \sin^2\left(\frac{s-t}{2}\right) dt \end{aligned} \tag{35}$$

for $s \in [a,b]$.

Since

$$\sin^2\left(\frac{s-t}{2}\right) = \frac{1-\cos(s-t)}{2},$$

hence

$$\begin{aligned}&\int_a^b \sin^2\left(\frac{s-t}{2}\right) dt \\ &= \int_a^b \frac{1-\cos(s-t)}{2} dt = \frac{1}{2}\left[b-a-\sin(b-s)-\sin(s-a)\right] \\ &= \frac{1}{2}\left[b-a-2\sin\left(\frac{b-a}{2}\right)\cos\left(\frac{a+b}{2}-s\right)\right] \\ &= \frac{b-a}{2} - \sin\left(\frac{b-a}{2}\right)\cos\left(\frac{a+b}{2}-s\right)\end{aligned}$$

for $s \in [a,b]$.

Therefore by (35) we get

$$\begin{aligned}\Bigg| \int_a^b & f(R\exp(it))\exp(it)\,dt \\ & - 2R\left[f(R\exp(is)) + f'(R\exp(is))\left(\frac{\exp(ib)+\exp(ia)}{2} - \exp(is)\right)\right] \\ & \times \sin\left(\frac{b-a}{2}\right)\exp\left[\left(\frac{a+b}{2}\right)i\right]\Bigg| \\ & \le 2R^2\sup_{t\in[a,b]}\left| f''(R\exp(it))\right| \left[\frac{b-a}{2} - \sin\left(\frac{b-a}{2}\right)\cos\left(\frac{a+b}{2}-s\right)\right] \end{aligned} \tag{36}$$

for $s\in[a,b]$.

In particular, for $s=\frac{a+b}{2}$, we obtain from (36) the best possible inequality

$$\begin{aligned}&\left|\int_a^b f(R\exp(it))\exp(it)\,dt\right.\\&\qquad-2R\left[f\left(R\exp\left(\frac{a+b}{2}i\right)\right)+f'\left(R\exp\left(\frac{a+b}{2}i\right)\right)\right.\\&\qquad\times\left.\left(\frac{\exp(ib)+\exp(ia)}{2}-\exp\left(\frac{a+b}{2}i\right)\right)\right]\\&\qquad\left.\times\sin\left(\frac{b-a}{2}\right)\exp\left[\left(\frac{a+b}{2}\right)i\right]\right|\\&\qquad\le 2R^2\sup_{t\in[a,b]}\left|f''(R\exp(it))\right|\left[\frac{b-a}{2}-\sin\left(\frac{b-a}{2}\right)\right].\qquad(37)\end{aligned}$$

By utilising the inequality (24) for the circular path $\gamma_{[a,b],R}$ and $v=R\exp(is)\in\gamma_{[a,b],R}$ with $s\in[a,b]$, we also get

$$\begin{aligned}&\left|f(R\exp(ib))\sin\left(\frac{b-s}{2}\right)\exp\left[\left(\frac{s+b}{2}\right)i\right]\right.\\&\qquad+f(R\exp(ia))\sin\left(\frac{s-a}{2}\right)\exp\left[\left(\frac{a+s}{2}\right)i\right]\\&\qquad+f(R\exp(is))\sin\left(\frac{b-a}{2}\right)\exp\left[\left(\frac{a+b}{2}\right)i\right]\\&\qquad\left.-\int_a^b f(R\exp(it))\exp(it)\,dt\right|\\&\le R^2\sup_{t\in[a,b]}\left|f''(R\exp(it))\right|\left[\frac{b-a}{2}-\sin\left(\frac{b-a}{2}\right)\cos\left(\frac{a+b}{2}-s\right)\right].\qquad(38)\end{aligned}$$

In particular, for $s=\frac{a+b}{2}$, we get from (38) best possible inequality

$$\begin{aligned}&\left|f(R\exp(bi))\sin\left(\frac{b-a}{4}\right)\exp\left[\left(\frac{a+3b}{4}\right)i\right]\right.\\&\qquad+f(R\exp(ia))\sin\left(\frac{b-a}{4}\right)\exp\left[\left(\frac{3a+b}{4}\right)i\right]\\&\qquad+f\left(R\exp\left(\frac{a+b}{2}i\right)\right)\sin\left(\frac{b-a}{2}\right)\exp\left[\left(\frac{a+b}{2}\right)i\right]\\&\qquad\left.-\int_a^b f(R\exp(it))\exp(it)\,dt\right|\\&\qquad\le R^2\sup_{t\in[a,b]}\left|f''(R\exp(it))\right|\left[\frac{b-a}{2}-\sin\left(\frac{b-a}{2}\right)\right].\qquad(39)\end{aligned}$$

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