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Q–NORM INEQUALITIES FOR SEQUENCES OF HILBERT SPACE OPERATORS

S. S. DRAGOMIR, M. S. MOSLEHIAN AND J. SÁNDOR

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Abstract. We give some inequalities related to a large class of operator norms, the so-called Q-norms, for a (not necessary commutative) family of bounded linear operators acting on a Hilbert space that are related to the classical Schwarz inequality. Applications for vector inequalities are also provided.

1. Introduction and preliminaries

Let $(\mathscr{H};\langle\cdot,\cdot\rangle)$ be a real or complex Hilbert space, $\mathbb{B}(\mathscr{H})$ the Banach algebra of all bounded linear operators acting on $\mathscr{H}$ and I denote the identity operator on $\mathscr{H}$.

In many estimates one needs to use upper bounds for the norm of a sum of products $A_1,\dots,A_n,B_1,\dots,B_n \in \mathbb{B}(\mathscr{H})$, where separate information for norms of operators are provided. The special case in which $B_i=\alpha_i I$ $(\alpha_i\in\mathbb{C}, 1\leqslant i\leqslant n)$ and the norms in the question are the operator norm has already investigated in [4]. The celebrated Hölder's discrete inequality stating

$$\left|\sum_{i=1}^n \alpha_i\beta_i\right| \leqslant \left(\sum_{i=1}^n |\alpha_i|^p\right)^{1/p}\left(\sum_{i=1}^n |\beta_i|^q\right)^{1/q} \tag{1}$$

$$\left(1<p,\quad 1<q,\quad \frac{1}{p}+\frac{1}{q}=1,\quad \alpha_i,\beta_i\in\mathbb{C}\quad (1\leqslant i\leqslant n)\right)$$

and its variance in the special case $p=q=2$ (the so-called Cauchy–Bunyakovsky–Schwarz (CBS) discrete inequality) are of special interest and of larger utility.

To each symmetric gauge functions defined on the sequences of real numbers there corresponds a unitarily invariant norm $|||\cdot|||$ defined on a two-sided ideal $\mathscr{C}_{|||\cdot|||}$ of $\mathbb{B}(\mathscr{H})$ enjoying the invariant property $|||UAV|||=|||A|||$ for all $A\in\mathscr{C}_{|||\cdot|||}$ and all unitary operators $U,V\in\mathbb{B}(\mathscr{H})$. It is known that $\mathscr{C}_{|||\cdot|||}$ is a Banach space under the norm $|||\cdot|||$. If $B\in\mathscr{C}_{|||\cdot|||}$, then $|||B^*|||=|||B|||$ and $|||AB|||\leqslant\|A\|\,|||B|||$ for all $A\in\mathbb{B}(\mathscr{H})$, from which we easily conclude that

$$|||ABC|||\leqslant\|A\|\,|||B|||\,\|C\| \qquad (A,C\in\mathbb{B}(\mathscr{H})).$$

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Some well-known examples of unitarily invariant norms are the Schatten p-norms $\|B\|_p := \operatorname{tr}(|B|^p)^{1/p}$ for $1 \leqslant p < \infty$, operator norm $\|\cdot\|$, and the Ky-Fan norms; cf. [1, 6]. A unitarily invariant norm $\|\cdot\|_Q$ is called a Q-norm if there is a unitarily invariant norm $\|\cdot\|_{\widehat{Q}}$ such that $\|BB^*\|_{\widehat{Q}} = \|B\|_Q^2$.

The aim of the present paper is to establish some upper bounds of interest for the quantity $\|\sum_{i=1}^n A_iB_i\|_Q$ under certain assumptions on A_i's and B_i's. Our results are significant since we do not use any commutative assumption on the families of operators and as well we provide inequalities for a class of norms which are larger than operator norms (compare the results with those of [4]). Applications for vector inequalities are also given.

We need the following lemmas concerning real numbers (the second lemma is obvious):

LEMMA 1.1. *(Young's inequality; see* [5, p. 30 or p. 49]*)* *If* $p > 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$*, then*

$$ab \leqslant \frac{a^p}{p} + \frac{b^q}{q}$$

for all $a, b > 0$.

LEMMA 1.2. *If* $a, b > 0$*, then*

$$ab \leqslant \frac{1}{4}(a+b)^2 \leqslant \frac{1}{2}(a^2+b^2).$$

LEMMA 1.3. *(Daykin–Eliezer; see* [2]*)* *If* $a_k > 1$, $b_k > 1$ *and* $\frac{1}{p} + \frac{1}{q} < 1$*, then*

$$\left(\sum_{k=1}^n a_k b_k\right)^{\frac{1}{p}+\frac{1}{q}} \leqslant \left(\sum_{k=1}^n a_k^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^n b_k^q\right)^{\frac{1}{q}}.$$

2. Some General Results

The following result containing 9 different inequalities may be stated:

THEOREM 2.1. *Let* $A_1, \ldots, A_n \in \mathbb{B}(\mathscr{H})$ *and* $B_1, \ldots, B_n \in \mathscr{C}_Q$*. Then*

$$\left\|\sum_{i=1}^n A_iB_i\right\|_Q^2 \leqslant \begin{cases} A \\ B \\ C \end{cases} \tag{2}$$

where

$$A := \begin{cases} \max\limits_{1\leqslant k\leqslant n} \|A_k\|^2 \sum\limits_{i,j=1}^{n} \left\|B_iB_j^*\right\|_{\widehat{Q}}, \\ \max\limits_{1\leqslant k\leqslant n} \|A_k\| \left(\sum\limits_{i=1}^{n} \|A_i\|^r\right)^{\frac{1}{r}} \left(\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n}\left\|B_iB_j^*\right\|_{\widehat{Q}}\right)^s\right)^{\frac{1}{s}}, \\ \qquad\qquad \text{where } r>1,\ \frac{1}{r}+\frac{1}{s}=1, \\ \max\limits_{1\leqslant k\leqslant n} \|A_k\| \sum\limits_{i=1}^{n} \|A_i\| \max\limits_{1\leqslant i\leqslant n} \left(\sum\limits_{j=1}^{n}\left\|B_iB_j^*\right\|_{\widehat{Q}}\right), \end{cases}$$

$$B := \begin{cases} \max\limits_{1\leqslant i\leqslant n} \|A_i\| \left(\sum\limits_{k=1}^{n} \|A_k\|^p\right)^{\frac{1}{p}} \sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n}\left\|B_iB_j^*\right\|_{\widehat{Q}}^q\right)^{\frac{1}{q}}, \\ \left(\sum\limits_{i=1}^{n} \|A_i\|^t\right)^{\frac{1}{t}} \left(\sum\limits_{k=1}^{n} \|A_k\|^p\right)^{\frac{1}{p}} \left[\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n}\left\|B_iB_j^*\right\|_{\widehat{Q}}^q\right)^{\frac{u}{q}}\right]^{\frac{1}{u}}, \\ \qquad\qquad \text{where } t>1,\ \frac{1}{t}+\frac{1}{u}=1, \\ \sum\limits_{i=1}^{n} \|A_i\| \left(\sum\limits_{k=1}^{n} \|A_k\|^p\right)^{\frac{1}{p}} \max\limits_{1\leqslant i\leqslant n}\left\{\left(\sum\limits_{j=1}^{n}\left\|B_iB_j^*\right\|_{\widehat{Q}}^q\right)^{\frac{1}{q}}\right\}, \end{cases} \tag{3}$$

for $p>1,\ \dfrac{1}{p}+\dfrac{1}{q}=1$*; and*

$$C := \begin{cases} \max\limits_{1\leqslant i\leqslant n} \|A_i\| \sum\limits_{k=1}^{n} \|A_k\| \sum\limits_{i=1}^{n} \max\limits_{1\leqslant j\leqslant n}\left\{\left\|B_iB_j^*\right\|_{\widehat{Q}}\right\}, \\ \left(\sum\limits_{i=1}^{n} \|A_i\|^m\right)^{\frac{1}{m}} \sum\limits_{k=1}^{n} \|A_k\| \left[\sum\limits_{i=1}^{n}\left(\max\limits_{1\leqslant j\leqslant n}\left\{\left\|B_iB_j^*\right\|_{\widehat{Q}}\right\}\right)^l\right]^{\frac{1}{l}}, \\ \qquad\qquad \text{where } m,l>1,\ \frac{1}{m}+\frac{1}{l}=1, \\ \left(\sum\limits_{k=1}^{n} \|A_k\|\right)^2 \max\limits_{1\leqslant i,j\leqslant n}\left\{\left\|B_iB_j^*\right\|_{\widehat{Q}}\right\}. \end{cases}$$

Proof. In the operator partial order of $\mathbb{B}(\mathscr{H})$, we have

$$\begin{aligned} 0 &\leqslant \left(\sum_{i=1}^{n} B_iA_i\right)^* \left(\sum_{i=1}^{n} B_iA_i\right) \\ &= \sum_{i=1}^{n} A_i^*B_i^* \sum_{j=1}^{n} B_jA_j = \sum_{i,j=1}^{n} A_i^*B_i^*B_jA_j. \end{aligned} \tag{4}$$

Taking the norm in (4) and noticing that $\|BB^*\|_{\widehat{Q}} = \|B\|_Q^2$ for any $B \in \mathscr{C}_Q$, we have

$$\begin{aligned}\left\|\sum_{i=1}^n A_iB_i\right\|_Q^2 &= \left\|\sum_{i=1}^n\sum_{j=1}^n A_i^*B_i^*B_jA_j\right\|_{\widehat{Q}} \\ &\leqslant \sum_{i=1}^n\sum_{j=1}^n \|A_i\|\,\|A_j\|\,\|B_iB_j^*\|_{\widehat{Q}} \\ &= \sum_{i=1}^n \|A_i\|\left(\sum_{j=1}^n \|A_j\|\,\|B_iB_j^*\|_{\widehat{Q}}\right) =: M.\end{aligned}$$

Utilizing Hölder's discrete inequality we have that

$$\sum_{j=1}^n \|A_j\|\,\|B_iB_j^*\|_{\widehat{Q}} \leqslant \begin{cases} \max\limits_{1\leqslant k\leqslant n}\|A_k\| \sum\limits_{j=1}^n \left\|B_iB_j^*\right\|_{\widehat{Q}}, \\ \left(\sum\limits_{k=1}^n \|A_k\|^p\right)^{\frac{1}{p}} \left(\sum\limits_{j=1}^n \left\|B_iB_j^*\right\|_{\widehat{Q}}^q\right)^{\frac{1}{q}}, \\ \qquad\qquad \text{where } p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \sum\limits_{k=1}^n \|A_k\| \max\limits_{1\leqslant j\leqslant n}\left\|B_iB_j^*\right\|_{\widehat{Q}}, \end{cases}$$

for any $i \in \{1,\ldots,n\}$.

This provides the following inequalities:

$$M \leqslant \begin{cases} \max\limits_{1\leqslant k\leqslant n}\|A_k\| \sum\limits_{i=1}^n \|A_i\| \left(\sum\limits_{j=1}^n \left\|B_iB_j^*\right\|_{\widehat{Q}}\right) =: M_1, \\ \left(\sum\limits_{k=1}^n \|A_k\|^p\right)^{\frac{1}{p}} \sum\limits_{i=1}^n \|A_i\| \left(\sum\limits_{j=1}^n \left\|B_iB_j^*\right\|_{\widehat{Q}}^q\right)^{\frac{1}{q}} := M_p, \\ \qquad\qquad \text{where } p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \sum\limits_{k=1}^n \|A_k\| \sum\limits_{i=1}^n \|A_i\| \left(\max\limits_{1\leqslant j\leqslant n}\left\|B_iB_j^*\right\|_{\widehat{Q}}\right) := M_\infty. \end{cases}$$

Utilizing Hölder's inequality for $r,s>1,\ \frac{1}{r}+\frac{1}{s}=1,$ we have:

$$\sum_{i=1}^{n}\|A_i\|\left(\sum_{j=1}^{n}\left\|B_iB_j^*\right\|_{\widehat{Q}}\right)\leqslant\begin{cases}\max\limits_{1\leqslant i\leqslant n}\|A_i\|\sum\limits_{i,j=1}^{n}\left\|B_iB_j^*\right\|_{\widehat{Q}},\\[2ex] \left(\sum\limits_{i=1}^{n}\|A_i\|^r\right)^{\frac{1}{r}}\left[\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n}\left\|B_iB_j^*\right\|_{\widehat{Q}}\right)^s\right]^{\frac{1}{s}},\\ \qquad\qquad\text{where } r>1,\ \frac{1}{r}+\frac{1}{s}=1,\\[2ex] \sum\limits_{i=1}^{n}\|A_i\|\max\limits_{1\leqslant i\leqslant n}\left(\sum\limits_{j=1}^{n}\left\|B_iB_j^*\right\|_{\widehat{Q}}\right),\end{cases}$$

and thus we can state that

$$M_1\leqslant\begin{cases}\max\limits_{1\leqslant k\leqslant n}\|A_k\|^2\sum\limits_{i,j=1}^{n}\left\|B_iB_j^*\right\|_{\widehat{Q}},\\[2ex] \max\limits_{1\leqslant k\leqslant n}\|A_k\|\left(\sum\limits_{i=1}^{n}\|A_i\|^r\right)^{\frac{1}{r}}\left(\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n}\left\|B_iB_j^*\right\|_{\widehat{Q}}\right)^s\right)^{\frac{1}{s}},\\ \qquad\qquad\text{where } r>1,\ \frac{1}{r}+\frac{1}{s}=1,\\[2ex] \max\limits_{1\leqslant k\leqslant n}\|A_k\|\sum\limits_{i=1}^{n}\|A_i\|\max\limits_{1\leqslant i\leqslant n}\left(\sum\limits_{j=1}^{n}\left\|B_iB_j^*\right\|_{\widehat{Q}}\right),\end{cases}$$

and the first part of the theorem is proved.

By Hölder's inequality we can also have that (for $p>1,\ \frac{1}{p}+\frac{1}{q}=1$)

$$M_p\leqslant\left(\sum_{k=1}^{n}\|A_k\|^p\right)^{\frac{1}{p}}\times\begin{cases}\max\limits_{1\leqslant i\leqslant n}\|A_i\|\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n}\left\|B_iB_j^*\right\|_{\widehat{Q}}^q\right)^{\frac{1}{q}},\\[2ex] \left(\sum\limits_{i=1}^{n}\|A_i\|^t\right)^{\frac{1}{t}}\left[\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n}\left\|B_iB_j^*\right\|_{\widehat{Q}}^q\right)^{\frac{u}{q}}\right]^{\frac{1}{u}},\\ \qquad\qquad\text{where } t>1,\ \frac{1}{t}+\frac{1}{u}=1,\\[2ex] \sum\limits_{i=1}^{n}\|A_i\|\max\limits_{1\leqslant i\leqslant n}\left\{\left(\sum\limits_{j=1}^{n}\left\|B_iB_j^*\right\|_{\widehat{Q}}^q\right)^{\frac{1}{q}}\right\},\end{cases}$$

and the second part of (2) is proved.

Finally, we may state that

$$M_\infty \leqslant \sum_{k=1}^{n} \|A_k\| \times \begin{cases} \max\limits_{1\leqslant i\leqslant n} \|A_i\| \sum\limits_{i=1}^{n} \max\limits_{1\leqslant j\leqslant n} \left\{ \left\| B_i B_j^* \right\|_{\widehat{Q}} \right\}, \\ \left(\sum\limits_{i=1}^{n} \|A_i\|^m \right)^{\frac{1}{m}} \left[\sum\limits_{i=1}^{n} \left(\max\limits_{1\leqslant j\leqslant n} \left\{ \left\| B_i B_j^* \right\|_{\widehat{Q}} \right\} \right)^l \right]^{\frac{1}{l}}, \\ \qquad \text{where } m,l>1,\ \frac{1}{m}+\frac{1}{l}=1, \\ \sum\limits_{i=1}^{n} \|A_i\| \max\limits_{1\leqslant i,j\leqslant n} \left\{ \left\| B_i B_j^* \right\|_{\widehat{Q}} \right\}, \end{cases}$$

giving the last part of (2).

REMARK 2.2. One can obtain various particular inequalities as well. For instance, by using the usual Cauchy–Bunyakovsky–Schwarz inequality and norm properties of the space $\mathscr{C}_Q$ we get

$$\begin{aligned} \left\| \sum_{i=1}^{n} A_i B_i \right\|_Q^2 &\leqslant \left(\sum_{i=1}^{n} \|A_i B_i\|_Q \right)^2 \\ &\leqslant \left(\sum_{i=1}^{n} \|A_i\| \, \|B_i\|_Q \right)^{\frac{1}{2}} \\ &\leqslant \sum_{i=1}^{n} \|A_i\|^2 \sum_{i=1}^{n} \|B_i\|_Q^2 . \end{aligned} \tag{5}$$

If we consider again the Cauchy–Bunyakovsky–Schwarz inequality

$$\left\| \sum_{i=1}^{n} A_i B_i \right\|_Q^2 \leqslant \sum_{i=1}^{n} \|A_i\|_Q^2 \sum_{i=1}^{n} \|B_i\|_Q^2 \tag{6}$$

then we can conclude that (5) is a refinement of the (CBS) inequality (6) whenever the operator norm $\|\cdot\|$ is majorized by $\|\cdot\|_Q$.

COROLLARY 2.3. *Let $A_1,\ldots,A_n \in \mathbb{B}(\mathscr{H})$ and $B_1,\ldots,B_n \in \mathscr{C}_Q$ so that $B_i B_j^* = 0$ with $i \neq j$. Then*

$$\left\| \sum_{i=1}^{n} A_i B_i \right\|_Q^2 \leqslant \begin{cases} \tilde{A} \\ \tilde{B} \\ \tilde{C} \end{cases} \tag{7}$$

where

$$\tilde{A} := \begin{cases} \max\limits_{1\leqslant k\leqslant n} \|A_k\|^2 \sum\limits_{i=1}^{n} \|B_i\|_Q^2, \\ \max\limits_{1\leqslant k\leqslant n} \|A_k\| \left(\sum\limits_{i=1}^{n} \|A_i\|^r\right)^{\frac{1}{r}} \left(\sum\limits_{i=1}^{n} \|B_i\|_Q^{2s}\right)^{\frac{1}{s}}, \\ \qquad \text{where } r>1,\ \frac{1}{r}+\frac{1}{s}=1, \\ \max\limits_{1\leqslant k\leqslant n} \|A_k\| \sum\limits_{i=1}^{n} \|A_i\| \max\limits_{1\leqslant i\leqslant n} \left\{\|B_i\|_Q^2\right\}, \end{cases}$$

$$\tilde{B} := \begin{cases} \max\limits_{1\leqslant i\leqslant n} \|A_i\| \left(\sum\limits_{k=1}^{n} \|A_k\|^p\right)^{\frac{1}{p}} \sum\limits_{i=1}^{n} \|B_i\|_Q^2, \\ \left(\sum\limits_{i=1}^{n} \|A_i\|^t\right)^{\frac{1}{t}} \left(\sum\limits_{k=1}^{n} \|A_k\|^p\right)^{\frac{1}{p}} \left[\sum\limits_{i=1}^{n} \|B_i\|_Q^{2u}\right]^{\frac{1}{u}}, \\ \qquad \text{where } t>1,\ \frac{1}{t}+\frac{1}{u}=1, \\ \sum\limits_{i=1}^{n} \|A_i\| \left(\sum\limits_{k=1}^{n} \|A_k\|^p\right)^{\frac{1}{p}} \max\limits_{1\leqslant i\leqslant n} \left\{\|B_i\|_Q^2\right\}, \end{cases}$$

where $p>1$ *and*

$$\tilde{C} := \begin{cases} \max\limits_{1\leqslant i\leqslant n} \|A_i\| \sum\limits_{k=1}^{n} \|A_k\| \sum\limits_{i=1}^{n} \|B_i\|_Q^2, \\ \left(\sum\limits_{i=1}^{n} \|A_i\|^m\right)^{\frac{1}{m}} \sum\limits_{k=1}^{n} \|A_k\| \left(\sum\limits_{i=1}^{n} \|B_i\|_Q^{2l}\right)^{\frac{1}{l}}, \\ \qquad \text{where } m,l>1,\ \frac{1}{m}+\frac{1}{l}=1, \\ \left(\sum\limits_{k=1}^{n} \|A_k\|\right)^2 \max\limits_{1\leqslant i,j\leqslant n} \left\{\|B_i\|_Q^2\right\}. \end{cases}$$

As in the proof of the Theorem 2.1 one has

$$\left\|\sum_{i=1}^{n} A_i B_i\right\|_Q^2 \leqslant \sum_{i=1}^{n} \|A_i\| \left(\sum_{j=1}^{n} \|A_j\|\, \|B_i B_j^*\|_{\widehat{Q}}\right) =: M$$

Using Hölder's generalized inequality from Lemma 1.3, and assuming that $\|A_j\| > 1,\ \|B_i B_j^*\|_{\widehat{Q}} > 1,$ we get

$$\left(\sum_{j=1}^{n} \|A_j\|\, \|B_i B_j^*\|_{\widehat{Q}}\right) \leqslant \left(\sum_{k=1}^{n} \|A_k\|^p\right)^{\frac{q}{p+q}} \left(\sum_{j=1}^{n} \|B_i B_j^*\|_{\widehat{Q}}^q\right)^{\frac{P}{p+q}}.$$

Thus the following result is true:

THEOREM 2.4. *Let* $A_1,\dots,A_n\in\mathbb{B}(\mathscr{H})$ *and* $B_1,\dots,B_n\in\mathscr{C}_Q$. *Assume that* $\|A_j\|>1$ *and* $\|B_iB_j^*\|_{\widehat{Q}}>1$ *for all* $i,j\in\{1,2,\dots,n\}$. *Then*

$$\left\|\sum_{i=1}^n A_iB_i\right\|_Q^2\leqslant\left(\sum_{k=1}^n\|A_k\|^p\right)^{\frac{q}{p+q}}\sum_{i=1}^n\|A_i\|\left(\sum_{j=1}^n\|B_iB_j^*\|_Q^q\right)^{\frac{p}{p+q}},$$

where $\dfrac{1}{p}+\dfrac{1}{q}<1$.

3. Other Results

A different approach is embodied in the following theorem:

THEOREM 3.1. *If* $A_1,\dots,A_n\in\mathbb{B}(\mathscr{H})$ *and* $B_1,\dots,B_n\in\mathscr{C}_Q$, *then*

$$\left\|\sum_{i=1}^n A_iB_i\right\|_Q^2\leqslant\sum_{i=1}^n\|A_i\|^2\sum_{j=1}^n\left\|B_iB_j^*\right\|_{\widehat{Q}} \tag{8}$$

$$\leqslant\begin{cases}\sum\limits_{i=1}^n\|A_i\|^2\max\limits_{1\leqslant i\leqslant n}\left[\sum\limits_{j=1}^n\left\|B_iB_j^*\right\|_{\widehat{Q}}\right],\\[2ex] \left(\sum\limits_{i=1}^n\|A_i\|^{2p}\right)^{\frac{1}{p}}\left[\sum\limits_{i=1}^n\left(\sum\limits_{j=1}^n\left\|B_iB_j^*\right\|_{\widehat{Q}}\right)^q\right]^{\frac{1}{q}},\\ \qquad\qquad\text{where } p>1,\ \frac{1}{p}+\frac{1}{q}=1,\\[2ex] \max\limits_{1\leqslant i\leqslant n}\|A_i\|^2\sum\limits_{i,j=1}^n\left\|B_iB_j^*\right\|_{\widehat{Q}}.\end{cases}$$

Proof. From the proof of Theorem 2.1 we have that

$$\left\|\sum_{i=1}^n A_iB_i\right\|_Q^2\leqslant\sum_{i=1}^n\sum_{j=1}^n\|A_i\|\left\|A_j\right\|\left\|B_iB_j^*\right\|_{\widehat{Q}}.$$

Using the simple observation that

$$\|A_i\|\left\|A_j\right\|\leqslant\frac{1}{2}\left(\|A_i\|^2+\left\|A_j\right\|^2\right),\quad i,j\in\{1,\dots,n\},$$

we get

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{n}\|A_i\|\,\|A_j\|\,\|B_iB_j^*\|_{\widehat{Q}} &\leqslant \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left[\|A_i\|^2+\|A_j\|^2\right]\|B_iB_j^*\|_{\widehat{Q}}\\ &=\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left[\|A_i\|^2\|B_iB_j^*\|_{\widehat{Q}}+\|A_j\|^2\|B_jB_i^*\|_{\widehat{Q}}\right]\\ &=\sum_{i=1}^{n}\|A_i\|^2\sum_{j=1}^{n}\|B_iB_j^*\|_{\widehat{Q}},\end{aligned}$$

which proves the first inequality in (8).

The second part follows by Hölder's inequality and the details are omitted.

REMARK 3.2. If in (8) we choose $A_1 = \ldots = A_n = I$, then we get

$$\left\|\sum_{i=1}^{n}B_i\right\|_Q \leqslant \left(\sum_{i,j=1}^{n}\|B_iB_j^*\|_{\widehat{Q}}\right)^{1/2} \leqslant \left(\sum_{i,j=1}^{n}\|B_i\|_{\widehat{Q}}\|B_j^*\|_{\widehat{Q}}\right)^{1/2} = \left\|\sum_{i=1}^{n}B_i\right\|_{\widehat{Q}},$$

which is a refinement for the generalized triangle inequality for $\|\cdot\|_{\widehat{Q}}$, whenever $\|\cdot\|_Q$ is majorized by $\|\cdot\|_{\widehat{Q}}$.

The following corollary may be stated:

COROLLARY 3.3. *If $A_1,\ldots,A_n \in \mathbb{B}(\mathscr{H})$ and $B_1,\ldots,B_n \in \mathscr{C}_Q$ are such that $B_iB_j^* = 0$ for $i \neq j,\ i,j \in \{1,\ldots,n\}$, then*

$$\left\|\sum_{i=1}^{n}A_iB_i\right\|_Q^2 \leqslant \sum_{i=1}^{n}\|A_i\|^2\|B_i\|_Q^2 \tag{9}$$

$$\leqslant \begin{cases} \sum_{i=1}^{n}\|A_i\|^2 \max_{1\leqslant i\leqslant n}\|B_i\|_Q^2, \\ \left(\sum_{i=1}^{n}\|A_i\|^{2p}\right)^{\frac{1}{p}}\left[\sum_{j=1}^{n}\|B_i\|_Q^{2q}\right]^{\frac{1}{q}}, \\ \qquad \text{where } p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \max_{1\leqslant i\leqslant n}\|A_i\|^2\sum_{i=1}^{n}\|B_i\|_Q^2. \end{cases}$$

THEOREM 3.4. *If $A_1,\ldots,A_n \in \mathbb{B}(\mathscr{H})$ and $B_1,\ldots,B_n \in \mathscr{C}_Q$, then*

$$\left\|\sum_{i=1}^{n} A_iB_i\right\|_Q^2 \leqslant \begin{cases} \max\limits_{1\leqslant i\leqslant n} \|A_i\|^2 \sum\limits_{i,j=1}^{n} \left\|B_iB_j^*\right\|_{\widehat{Q}}, \\ \left(\sum\limits_{i=1}^{n} \|A_i\|^p\right)^{\frac{2}{p}} \left(\sum\limits_{i,j=1}^{n} \left\|B_iB_j^*\right\|_{\widehat{Q}}^q\right)^{\frac{1}{q}}, \\ \qquad \text{where } p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \left(\sum\limits_{i=1}^{n} \|A_i\|\right)^2 \max\limits_{1\leqslant i,j\leqslant n} \left\{\left\|B_iB_j^*\right\|_{\widehat{Q}}\right\}. \end{cases} \tag{10}$$

Proof. We know that

$$\left\|\sum_{i=1}^{n} A_iB_i\right\|_Q^2 \leqslant \sum_{i=1}^{n}\sum_{j=1}^{n} \|A_i\| \left\|A_j\right\| \left\|B_iB_j^*\right\|_{\widehat{Q}} =: P.$$

Firstly, we obviously have that

$$P \leqslant \max_{1\leqslant i,j\leqslant n} \left\{\|A_i\| \left\|A_j\right\|\right\} \sum_{i,j=1}^{n} \left\|B_iB_j^*\right\|_{\widehat{Q}} = \max_{1\leqslant i\leqslant n} \|A_i\|^2 \sum_{i,j=1}^{n} \left\|B_iB_j^*\right\|_{\widehat{Q}}.$$

Secondly, by the Hölder inequality for double sums, we obtain

$$\begin{aligned} P &\leqslant \left[\sum_{i,j=1}^{n} \left(\|A_i\| \left\|A_j\right\|\right)^p\right]^{\frac{1}{p}} \left(\sum_{i,j=1}^{n} \left\|B_iB_j^*\right\|_{\widehat{Q}}^q\right)^{\frac{1}{q}} \\ &= \left(\sum_{i=1}^{n} \|A_i\|^p \sum_{j=1}^{n} \left\|A_j\right\|^p\right)^{\frac{1}{p}} \left(\sum_{i,j=1}^{n} \left\|B_iB_j^*\right\|_{\widehat{Q}}^q\right)^{\frac{1}{q}} \\ &= \left(\sum_{i=1}^{n} \|A_i\|^p\right)^{\frac{2}{p}} \left(\sum_{i,j=1}^{n} \left\|B_iB_j^*\right\|_{\widehat{Q}}^q\right)^{\frac{1}{q}}, \end{aligned}$$

where $p>1,\ \dfrac{1}{p}+\dfrac{1}{q}=1$.

Finally, we have

$$\begin{aligned} P &\leqslant \max_{1\leqslant i,j\leqslant n} \left\{\left\|B_iB_j^*\right\|_{\widehat{Q}}\right\} \sum_{i,j=1}^{n} \|A_i\| \left\|A_j\right\| \\ &= \left(\sum_{i=1}^{n} \|A_i\|\right)^2 \max_{1\leqslant i,j\leqslant n} \left\{\left\|B_iB_j^*\right\|_{\widehat{Q}}\right\} \end{aligned}$$

and the theorem is proved.

COROLLARY 3.5. *If* $A_1,\dots,A_n \in \mathbb{B}(\mathscr{H})$ *and* $B_1,\dots,B_n \in \mathscr{C}_Q$ *are such that* $B_iB_j^* = 0$ *for* $i,j \in \{1,\dots,n\}$ *with* $i \neq j$, *then*

$$\left\|\sum_{i=1}^{n} A_iB_i\right\|_Q^2 \leqslant \begin{cases} \max\limits_{1\leqslant i\leqslant n} \|A_i\|^2 \sum\limits_{i=1}^{n} \|B_i\|_Q^2, \\ \left(\sum\limits_{i=1}^{n} \|A_i\|^p\right)^{\frac{2}{p}} \left(\sum\limits_{i=1}^{n} \|B_i\|_Q^{2q}\right)^{\frac{1}{q}}, \\ \quad \text{where } p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \left(\sum\limits_{i=1}^{n} \|A_i\|\right)^2 \max\limits_{1\leqslant i\leqslant n} \left\{\|B_i\|_Q^2\right\}. \end{cases} \tag{11}$$

THEOREM 3.6. *If* $\dfrac{1}{p}+\dfrac{1}{q} < 1$ *and* $\|A_i\| > 1$ *and* $\|B_iB_j^*\| > 1$ *for all* $1 \leqslant i,j \leqslant n$ *then*

$$\left\|\sum_{i=1}^{n} A_iB_i\right\|_Q^2 \leqslant \left(\sum_{i=1}^{n} \|A_i\|^p\right)^{\frac{2q}{p+q}} \left(\sum_{i,j=1}^{n} \|B_iB_j^*\|_{\widehat{Q}}^q\right)^{\frac{p}{p+q}}.$$

Proof. Using Lemma 1.3 for double sums, we have, as in the proof of Theorem 3.4, that

$$p = \sum_{i=1}^{n}\sum_{j=1}^{n} \|A_i\|\,\|A_j\|\,\|B_iB_j^*\|_{\widehat{Q}} \leqslant \left(\sum_{i,j=1}^{n} (\|A_i\|\,\|A_j\|)^p\right)^{\frac{q}{p+q}} \left(\sum_{i,j=1}^{n} \|B_iB_j^*\|_{\widehat{Q}}^q\right)^{\frac{p}{p+q}},$$

if we assume that $\|A_i\| > 1$ and $\|B_iB_j^*\| > 1$ for all $1 \leqslant i,j \leqslant n$.

Thus, we get

$$p \leqslant \left(\sum_{i=1}^{n} \|A_i\|^p\right)^{\frac{2q}{p+q}} \left(\sum_{i,j=1}^{n} \|B_iB_j^*\|_{\widehat{Q}}^q\right)^{\frac{p}{p+q}},$$

if $\dfrac{1}{p}+\dfrac{1}{q} < 1$.

Finally, the following result may be stated as well:

THEOREM 3.7. *If* $\dfrac{1}{p}+\dfrac{1}{q} = 1$, $A_1,\dots,A_n \in \mathbb{B}(\mathscr{H})$ *and* $B_1,\dots,B_n \in \mathscr{C}_Q$, *then*

$$\left\|\sum_{i=1}^{n} A_iB_i\right\|_Q^2 \leqslant \left(\frac{1}{p}\sum_{i=1}^{n} \|A_i\|^p + \frac{1}{q}\sum_{i=1}^{n} \|A_i\|^q\right) \max_{1\leqslant i\leqslant n} \left[\sum_{j=1}^{n} \|B_iB_j^*\|_{\widehat{Q}}\right].$$

Proof. As in the proof of Theorem 3.1 we have

$$\left\|\sum_{i=}^{n} A_i B_i\right\|_Q^2 \leqslant \sum_{i=1}^{n}\sum_{j=1}^{n} \|A_i\|\,\|A_j\|\,\|B_i B_j^*\|_{\widehat{Q}}.$$

Now, using Lemma 1.1, we can write

$$\|A_i\|\,\|A_j\| \leqslant \frac{1}{p}\|A_i\|^p + \frac{1}{q}\|A_j\|^q.$$

Thus, we have

$$\begin{aligned}
\sum_{i=1}^{n}\sum_{j=1}^{n} \|A_i\|\,\|A_j\|\,\|B_i B_j^*\|_{\widehat{Q}} &\leqslant \sum_{i=1}^{n}\sum_{j=1}^{n}\left[\frac{1}{p}\|A_i\|^p + \frac{1}{q}\|A_j\|^q\right]\|B_i B_j^*\|_{\widehat{Q}} \\
&= \frac{1}{p}\sum_{i=1}^{n}\sum_{j=1}^{n}\|A_i\|^p\,\|B_i B_j^*\|_{\widehat{Q}} + \frac{1}{q}\sum_{i=1}^{n}\sum_{j=1}^{n}\|A_j\|^q\,\|B_i B_j^*\|_{\widehat{Q}} \\
&= \frac{1}{p}\sum_{i=1}^{n}\|A_i\|^p\sum_{j=1}^{n}\|B_i B_j^*\|_{\widehat{Q}} + \frac{1}{q}\sum_{j=1}^{n}\|A_j\|^q\sum_{i=1}^{n}\|B_i B_j^*\|_{\widehat{Q}} \\
&\leqslant \left(\frac{1}{p}\sum_{i=1}^{n}\|A_i\|^p + \frac{1}{q}\sum_{j=1}^{n}\|A_j\|^q\right)\max_{1\leqslant i\leqslant n}\left[\sum_{j=1}^{n}\|B_i B_j^*\|_{\widehat{Q}}\right].
\end{aligned}$$

REMARK 3.8. Another variant may be obtained via Lemma 1.2:

$$\|A_i\|\,\|A_j\| \leqslant \frac{1}{4}\left(\|A_i\| + \|A_j\|\right)^2, \text{ so}$$

$$\begin{aligned}
\sum_{i=1}^{n}\sum_{j=1}^{n} \|A_i\|\,\|A_j\|\,\|B_i B_j^*\|_{\widehat{Q}} &\leqslant \frac{1}{4}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\|A_i\| + \|A_j\|\right)^2\|B_i B_j^*\|_{\widehat{Q}} \\
&\leqslant \sup_{1\leqslant i\leqslant n}\sum_{j=1}^{n}\|B_i B_j^*\|_{\widehat{Q}}\left(\sum_{i,j=1}^{n}\left(\|A_i\| + \|A_j\|\right)^2\right).
\end{aligned}$$

4. Applications

Throughout this section we assume $\|\cdot\|_Q$ (and hence $\|\cdot\|_{\widehat{Q}}$) to be the operator norm. If by $M(\mathbf{B},\mathbf{A})$ we denote any of the bounds provided by (2), (5), (8) or (10) for the quantity $\left\|\sum_{i=1}^{n} A_i B_i\right\|^2$, then we may state the following general fact:

Under the assumptions of Theorem 2.1, we have:

$$\left\|\sum_{i=1}^{n} A_i B_i x\right\|^2 \leqslant \|x\|^2 M(\mathbf{B},\mathbf{A}). \tag{12}$$

for any $x \in \mathscr{H}$ *and*

$$\left| \sum_{i=1}^{n} \langle A_i B_i x, y \rangle \right|^2 \leqslant \|x\|^2 \|y\|^2 M(\mathbf{B}, \mathbf{A}). \tag{13}$$

for any $x, y \in \mathscr{H}$*, respectively.*

The proof follows by the Schwarz inequality in the Hilbert space $(\mathscr{H}, \langle .,. \rangle)$.

Now, we consider the non zero vectors $y_1, \ldots, y_n \in \mathscr{H}$. Define $B_i : \mathscr{H} \to \mathscr{H}$ by $B_i = \frac{y_i \otimes y_i}{\|y_i\|}$ $(1 \leqslant i \leqslant n)$. Then

$$\left\| B_i B_j^* \right\| = \frac{\left\| \langle y_j, y_i \rangle y_i \otimes y_j \right\|}{\|y_i\| \|y_j\|} = \left| \langle y_i, y_j \rangle \right| \quad (1 \leqslant i, j \leqslant n).$$

If $(y_i)_{1 \leqslant i \leqslant n}$ is an orthogonal family on $\mathscr{H}$, then $\|B_i\| = 1$ and $B_i B_j = 0$ for $i, j \in \{1, \ldots, n\}$, $i \neq j$.

Now, utilizing, for instance, the inequalities in Theorem 3.1 we may state that:

$$\left\| \sum_{i=1}^{n} A_i \frac{\langle x, y_i \rangle}{\|y_i\|} y_i \right\|^2 \leqslant \|x\|^2 \sum_{i=1}^{n} \|A_i\|^2 \sum_{j=1}^{n} \left| \langle y_i, y_j \rangle \right| \tag{14}$$

$$\leqslant \|x\|^2 \times \begin{cases} \sum\limits_{i=1}^{n} \|A_i\|^2 \max\limits_{1 \leqslant i \leqslant n} \left[\sum\limits_{j=1}^{n} \left| \langle y_i, y_j \rangle \right| \right], \\ \left(\sum\limits_{i=1}^{n} \|A_i\|^{2p} \right)^{\frac{1}{p}} \left[\sum\limits_{i=1}^{n} \left(\sum\limits_{j=1}^{n} \left| \langle y_i, y_j \rangle \right| \right)^q \right]^{\frac{1}{q}}, \\ \qquad \text{where } p > 1,\ \frac{1}{p} + \frac{1}{q} = 1, \\ \max\limits_{1 \leqslant i \leqslant n} \|A_i\|^2 \sum\limits_{i,j=1}^{n} \left| \langle y_i, y_j \rangle \right|. \end{cases}$$

for any $x, y_1, \ldots, y_n \in \mathscr{H}$ and $A_1, \ldots, A_n \in \mathbb{B}(\mathscr{H})$.

The choice $A_i = \|y_i\| I$ $(i = 1, \cdots, n)$ will produce some interesting bounds for the norm of the Fourier series $\|\sum_{i=1}^{n} \langle x, y_i \rangle\|$. Notice the vectors y_i $(i = 1, \cdots, n)$ are not necessarily orthonormal.

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S. S. Dragomir
Research Group in Mathematical Inequalities & Applications
School of Engineering & Science
Victoria University
P. O. Box 14428
Melbourne City
Victoria 8001
Australia
e-mail: sever.dragomir@vu.edu.au

M. S. Moslehian
Department of Pure Mathematics
Ferdowsi University of Mashhad
P.O. Box 1159
Mashhad 91775
Iran
Centre of Excellence in Analysis on Algebraic Structures (CEAAS)
Ferdowsi University of Mashhad
Iran
e-mail: moslehian@ferdowsi.um.ac.ir and moslehian@ams.org

J. Sándor
Department of Mathematics
Babes-Bolyai University
Str. Kogalniceanu
400084 Cluj-Napoca
Romania
e-mail: jsandor@math.ubbcluj.ro