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SOME MULTIPLE INTEGRAL INEQUALITIES VIA THE DIVERGENCE THEOREM

SILVESTRU SEVER DRAGOMIR

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Abstract. In this paper, by the use of the divergence theorem, we establish some inequalities for functions defined on closed and bounded subsets of the Euclidean space $\mathbb{R}^n$, $n \geqslant 2$.

1. Introduction

Let ∂D be a simple, closed counterclockwise curve bounding a region D and f defined on an open set containing D and having continuous partial derivatives on D. In the recent paper [4], by the use of *Green's identity*, we have shown among others that

$$\begin{aligned}&\left|\int\int_D f(x,y)\,dxdy - \frac{1}{2}\oint_{\partial D}\left[(\beta-y)f(x,y)\,dx + (x-\alpha)f(x,y)\,dy\right]\right|\\ &\leqslant \frac{1}{2}\int\int_D\left[|\alpha-x|\left|\frac{\partial f(x,y)}{\partial x}\right| + |\beta-y|\left|\frac{\partial f(x,y)}{\partial y}\right|\right]dxdy =: M(\alpha,\beta;f) \qquad (1.1)\end{aligned}$$

for all $\alpha,\ \beta \in \mathbb{C}$ and

$$M(\alpha,\beta;f) \leqslant \begin{cases} \left\|\frac{\partial f}{\partial x}\right\|_{D,\infty}\iint_D|\alpha-x|\,dxdy + \left\|\frac{\partial f}{\partial y}\right\|_{D,\infty}\iint_D|\beta-y|\,dxdy; \\ \\ \left\|\frac{\partial f}{\partial x}\right\|_{D,p}\left(\iint_D|\alpha-x|^q\,dxdy\right)^{1/q} + \left\|\frac{\partial f}{\partial y}\right\|_{D,p}\left(\iint_D|\beta-y|^q\,dxdy\right)^{1/q} \\ \text{where } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \sup_{(x,y)\in D}|\alpha-x|\left\|\frac{\partial f}{\partial x}\right\|_{D,1} + \sup_{(x,y)\in D}|\beta-y|\left\|\frac{\partial f}{\partial y}\right\|_{B,1}, \end{cases} \qquad (1.2)$$

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where $\|\cdot\|_{D,p}$ are the usual Lebesgue norms, we recall that

$$\|g\|_{D,p} := \begin{cases} \left(\iint_D |g(x,y)|^p \, dxdy\right)^{1/p}, \ p \geqslant 1; \\ \\ \sup_{(x,y)\in D} |g(x,y)|, \ p = \infty. \end{cases}$$

Applications for rectangles and disks were also provided in [4]. For some recent double integral inequalities see [1], [2] and [3].

We also considered similar inequalities for 3-dimensional bodies as follows, see [5]. Let B be a solid in the three dimensional space $\mathbb{R}^3$ bounded by an orientable closed surface ∂B. If $f : B \to \mathbb{C}$ is a continuously differentiable function defined on a open set containing B, then by making use of the *Gauss-Ostrogradsky identity*, we have obtained the following inequality

$$\begin{aligned} &\Bigg| \iiint_B f(x,y,z)\, dxdydz - \frac{1}{3}\Bigg[\int\int_{\partial B} (x-\alpha) f(x,y,z)\, dy \wedge dz \\ &+ \int\int_{\partial B} (y-\beta) f(x,y,z)\, dz \wedge dx + \int\int_{\partial B} (z-\gamma) f(x,y,z)\, dx \wedge dy \Bigg] \Bigg| \\ &\leqslant \frac{1}{3} \iiint_B \left[|\alpha - x| \left| \frac{\partial f(x,y,z)}{\partial x} \right| + |\beta - y| \left| \frac{\partial f(x,y,z)}{\partial y} \right| + |\gamma - z| \left| \frac{\partial f(x,y,z)}{\partial z} \right| \right] dxdydz \\ &=: M(\alpha,\beta,\gamma;f) \end{aligned} \tag{1.3}$$

for all α, β, γ complex numbers. Moreover, we have the bounds

$$\begin{aligned} &M(\alpha,\beta,\gamma;f) \\ &\leqslant \frac{1}{3} \begin{cases} \left\|\frac{\partial f}{\partial x}\right\|_{B,\infty} \iiint_B |\alpha - x|\, dxdydz + \left\|\frac{\partial f}{\partial y}\right\|_{B,\infty} \iiint_B |\beta - y|\, dxdydz \\ + \left\|\frac{\partial f}{\partial z}\right\|_{B,\infty} \iiint_B |\gamma - z|\, dxdydz; \\ \\ \left\|\frac{\partial f}{\partial x}\right\|_{B,p} \left(\iiint_B |\alpha - x|^q\, dxdydz\right)^{1/q} + \left\|\frac{\partial f}{\partial y}\right\|_{B,p} \left(\iiint_B |\beta - y|^q\, dxdydz\right)^{1/q} \\ + \left\|\frac{\partial f}{\partial z}\right\|_{B,p} \left(\iiint_B |\gamma - z|\, dxdydz\right)^{1/q}, \ p,q > 1, \ \frac{1}{p} + \frac{1}{q} = 1; \\ \\ \sup_{(x,y,z)\in B} |\alpha - x| \left\|\frac{\partial f}{\partial x}\right\|_{B,1} + \sup_{(x,y,z)\in B} |\beta - y| \left\|\frac{\partial f}{\partial y}\right\|_{B,1} \\ + \sup_{(x,y,z)\in B} |\gamma - z| \left\|\frac{\partial f}{\partial z}\right\|_{B,1}. \end{cases} \end{aligned} \tag{1.4}$$

Applications for 3-dimensional balls were also given in [5]. For some triple integral inequalities see [6] and [9].

Motivated by the above results, in this paper we establish several similar inequalities for multiple integrals for functions defined on bonded subsets of $\mathbb{R}^n$ $(n \geqslant 2)$ with smooth (or piecewise smooth) boundary ∂B. To achieve this goal we make use of the well known divergence theorem for multiple integrals as summarized below.

2. Some preliminary facts

Let B be a bounded open subset of $\mathbb{R}^n$ $(n \geqslant 2)$ with smooth (or piecewise smooth) boundary ∂B. Let $F = (F_1, \ldots, F_n)$ be a smooth vector field defined in $\mathbb{R}^n$, or at least in $B \cup \partial B$. Let $\mathbf{n}$ be the unit outward-pointing normal of ∂B. Then the *Divergence Theorem* states, see for instance [8]:

$$\int_B \operatorname{div} F dV = \int_{\partial B} F \cdot n dA, \tag{2.1}$$

where

$$\operatorname{div} F = \nabla \cdot F = \sum_{k=1}^{n} \frac{\partial F_k}{\partial x_k},$$

dV is the element of volume in $\mathbb{R}^n$ and dA is the element of surface area on ∂B.

If $\mathbf{n} = (\mathbf{n}_1, \ldots, \mathbf{n}_n)$, $x = (x_1, \ldots, x_n) \in B$ and use the notation dx for dV we can write (2.1) more explicitly as

$$\sum_{k=1}^{n} \int_B \frac{\partial F_k(x)}{\partial x_k} dx = \sum_{k=1}^{n} \int_{\partial B} F_k(x) n_k(x) dA. \tag{2.2}$$

By taking the real and imaginary part, we can extend the above equality for complex valued functions F_k, $k \in \{1, \ldots, n\}$ defined on B.

If $n = 2$, the normal is obtained by rotating the tangent vector through 90° (in the correct direction so that it points out). The quantity tds can be written (dx_1, dx_2) along the surface, so that

$$ndA := nds = (dx_2, -dx_1)$$

Here t is the tangent vector along the boundary curve and ds is the element of arc-length.

From (2.2) we get for $B \subset \mathbb{R}^2$ that

$$\int_B \frac{\partial F_1(x_1,x_2)}{\partial x_1} dx_1 dx_2 + \int_B \frac{\partial F_2(x_1,x_2)}{\partial x_2} dx_1 dx_2 = \int_{\partial B} F_1(x_1,x_2) dx_2 - \int_{\partial B} F_2(x_1,x_2) dx_1, \tag{2.3}$$

which is *Green's theorem* in plane.

If $n = 3$ and if ∂B is described as a level-set of a function of 3 variables i.e. $\partial B = \left\{x_1, x_2, x_3 \in \mathbb{R}^3 \mid G(x_1,x_2,x_3) = 0\right\}$, then a vector pointing in the direction of $\mathbf{n}$ is $\operatorname{grad} G$. We shall use the case where $G(x_1,x_2,x_3) = x_3 - g(x_1,x_2)$, $(x_1,x_2) \in D$, a domain in $\mathbb{R}^2$ for some differentiable function g on D and B corresponds to the inequality $x_3 < g(x_1,x_2)$, namely

$$B = \left\{(x_1,x_2,x_3) \in \mathbb{R}^3 \mid x_3 < g(x_1,x_2)\right\}.$$

Then

$$\mathbf{n} = \frac{(-g_{x_1}, -g_{x_2}, 1)}{\left(1 + g_{x_1}^2 + g_{x_2}^2\right)^{1/2}}, \; dA = \left(1 + g_{x_1}^2 + g_{x_2}^2\right)^{1/2} dx_1 dx_2$$

and

$$\mathbf{n}dA = (-g_{x_1}, -g_{x_2}, 1)\, dx_1 dx_2.$$

From (2.2) we get

$$\begin{aligned}
&\int_B \left(\frac{\partial F_1(x_1,x_2,x_3)}{\partial x_1} + \frac{\partial F_2(x_1,x_2,x_3)}{\partial x_2} + \frac{\partial F_3(x_1,x_2,x_3)}{\partial x_3} \right) dx_1 dx_2 dx_3 \\
&= - \int_D F_1(x_1,x_2,g(x_1,x_2))\, g_{x_1}(x_1,x_2)\, dx_1 dx_2 - \int_D F_1(x_1,x_2,g(x_1,x_2))\, g_{x_2}(x_1,x_2) dx_1 dx_2 \\
&\quad + \int_D F_3(x_1,x_2,g(x_1,x_2))\, dx_1 dx_2 \qquad (2.4)
\end{aligned}$$

which is the *Gauss-Ostrogradsky theorem* in space.

3. Identities of interest

We have the following identity of interest:

THEOREM 1. *Let B be a bounded open subset of $\mathbb{R}^n$ $(n \geqslant 2)$ with smooth (or piecewise smooth) boundary ∂B. Let f be a continuously differentiable function defined in $\mathbb{R}^n$, or at least in $B \cup \partial B$ and with complex values. If α_k, $\beta_k \in \mathbb{C}$ for $k \in \{1,...,n\}$ with $\sum_{k=1}^n \alpha_k = 1$, then*

$$\int_B f(x)\, dx = \sum_{k=1}^n \int_B (\beta_k - \alpha_k x_k) \frac{\partial f(x)}{\partial x_k} dx + \sum_{k=1}^n \int_{\partial B} (\alpha_k x_k - \beta_k) f(x)\, n_k(x)\, dA. \qquad (3.1)$$

We also have

$$\int_B f(x)\, dx = \frac{1}{n} \sum_{k=1}^n \int_B (\gamma_k - x_k) \frac{\partial f(x)}{\partial x_k} dx + \frac{1}{n} \sum_{k=1}^n \int_{\partial B} (x_k - \gamma_k) f(x)\, n_k(x)\, dA \qquad (3.2)$$

for all $\gamma_k \in \mathbb{C}$, where $k \in \{1,...,n\}$.

Proof. Let $x = (x_1,...,x_n) \in B$. We consider

$$F_k(x) = (\alpha_k x_k - \beta_k) f(x), \; k \in \{1,...,n\}$$

and take the partial derivatives $\frac{\partial F_k(x)}{\partial x_k}$ to get

$$\frac{\partial F_k(x)}{\partial x_k} = \alpha_k f(x) + (\alpha_k x_k - \beta_k) \frac{\partial f(x)}{\partial x_k}, \; k \in \{1,...,n\}.$$

If we sum this equality over k from 1 to n we get

$$\sum_{k=1}^n \frac{\partial F_k(x)}{\partial x_k} = \sum_{k=1}^n \alpha_k f(x) + \sum_{k=1}^n (\alpha_k x_k - \beta_k) \frac{\partial f(x)}{\partial x_k} = f(x) + \sum_{k=1}^n (\alpha_k x_k - \beta_k) \frac{\partial f(x)}{\partial x_k} \qquad (3.3)$$

for all $x = (x_1, ..., x_n) \in B$.

Now, if we take the integral in the equality (3.3) over $(x_1, ..., x_n) \in B$ we get

$$\int_B \left(\sum_{k=1}^{n} \frac{\partial F_k(x)}{\partial x_k} \right) dx = \int_B f(x)\,dx + \sum_{k=1}^{n} \int_B \left[(\alpha_k x_k - \beta_k) \frac{\partial f(x)}{\partial x_k} \right] dx. \tag{3.4}$$

By the Divergence Theorem (2.2) we also have

$$\int_B \left(\sum_{k=1}^{n} \frac{\partial F_k(x)}{\partial x_k} \right) dx = \sum_{k=1}^{n} \int_{\partial B} (\alpha_k x_k - \beta_k) f(x) n_k(x)\, dA \tag{3.5}$$

and by making use of (3.4) and (3.5) we get

$$\int_B f(x)\,dx + \sum_{k=1}^{n} \int_B \left[(\alpha_k x_k - \beta_k) \frac{\partial f(x)}{\partial x_k} \right] dx = \sum_{k=1}^{n} \int_{\partial B} (\alpha_k x_k - \beta_k) f(x) n_k(x)\, dA$$

which gives the desired representation (3.1).

The identity (3.2) follows by (3.1) for $\alpha_k = \frac{1}{n}$ and $\beta_k = \frac{1}{n}\gamma_k,\ k \in \{1, ..., n\}$.

For the body B we consider the coordinates for the *centre of gravity*

$$G\left(\overline{x_{B,1}}, ..., \overline{x_{B,n}}\right)$$

defined by

$$\overline{x_{B,k}} := \frac{1}{V(B)} \int_B x_k dx,\ k \in \{1, ..., n\},$$

where

$$V(B) := \int_B x dx$$

is the volume of B.

COROLLARY 1. *With the assumptions of Theorem 1 we have*

$$\int_B f(x)\,dx = \sum_{k=1}^{n} \int_B \alpha_k \left(\overline{x_{B,k}} - x_k \right) \frac{\partial f(x)}{\partial x_k} dx + \sum_{k=1}^{n} \int_{\partial B} \alpha_k \left(x_k - \overline{x_{B,k}} \right) f(x) n_k(x)\, dA \tag{3.6}$$

and, in particular,

$$\int_B f(x)\,dx = \frac{1}{n} \sum_{k=1}^{n} \int_B \left(\overline{x_{B,k}} - x_k \right) \frac{\partial f(x)}{\partial x_k} dx + \frac{1}{n} \sum_{k=1}^{n} \int_{\partial B} \left(x_k - \overline{x_{B,k}} \right) f(x) n_k(x)\, dA. \tag{3.7}$$

The proof follows by (3.1) on taking $\beta_k = \alpha_k \overline{x_{B,k}},\ k \in \{1, ..., n\}$.

For a function f as in Theorem 1 above, we define the points

$$x_{B,\partial f,k} := \frac{\int_B x_k \frac{\partial f(x)}{\partial x_k} dx}{\int_B \frac{\partial f(x)}{\partial x_k} dx},\ k \in \{1, ..., n\},$$

provided that all denominators are not zero.

COROLLARY 2. *With the assumptions of Theorem 1 we have*

$$\int_B f(x)\,dx = \sum_{k=1}^{n} \int_{\partial B} \alpha_k \left(x_k - x_{B,\partial f,k}\right) f(x)\, n_k(x)\, dA \tag{3.8}$$

and, in particular,

$$\int_B f(x)\,dx = \frac{1}{n} \sum_{k=1}^{n} \int_{\partial B} \left(x_k - x_{B,\partial f,k}\right) f(x)\, n_k(x)\, dA. \tag{3.9}$$

The proof follows by (3.1) on taking $\beta_k = \alpha_k x_{B,\partial f,k},\ k \in \{1,...,n\}$ and observing that

$$\sum_{k=1}^{n} \int_B (\beta_k - \alpha_k x_k) \frac{\partial f(x)}{\partial x_k} dx = \sum_{k=1}^{n} \alpha_k \int_B \left(x_{B,\partial f,k} - x_k\right) \frac{\partial f(x)}{\partial x_k} dx = 0.$$

For a function f as in Theorem 1 above, we define the points

$$x_{\partial B,f,k} := \frac{\int_{\partial B} x_k f(x)\, n_k(x)\, dA}{\int_{\partial B} f(x)\, n_k(x)\, dA},\ k \in \{1,...,n\}$$

provided that all denominators are not zero.

COROLLARY 3. *With the assumptions of Theorem 1 we have*

$$\int_B f(x)\,dx = \sum_{k=1}^{n} \int_B \alpha_k \left(x_{\partial B,f,k} - x_k\right) \frac{\partial f(x)}{\partial x_k} dx \tag{3.10}$$

and, in particular,

$$\int_B f(x)\,dx = \frac{1}{n} \sum_{k=1}^{n} \int_B \left(x_{\partial B,f,k} - x_k\right) \frac{\partial f(x)}{\partial x_k} dx. \tag{3.11}$$

The proof follows by (3.1) on taking $\beta_k = \alpha_k x_{\partial B,f,k},\ k \in \{1,...,n\}$ and observing that

$$\sum_{k=1}^{n} \int_{\partial B} (\alpha_k x_k - \beta_k) f(x)\, n_k(x)\, dA = 0.$$

4. Some integral inequalities

We have the following result generalizing the inequalities from the introduction:

THEOREM 2. *Let B be a bounded open subset of $\mathbb{R}^n$ $(n \geqslant 2)$ with smooth (or piecewise smooth) boundary ∂B. Let f be a continuously differentiable function defined in $\mathbb{R}^n$, or at least in $B \cup \partial B$ and with complex values. If $\alpha_k,\ \beta_k \in \mathbb{C}$ for*

$k \in \{1,...,n\}$ *with* $\sum_{k=1}^{n} \alpha_k = 1$, *then*

$$\left| \int_B f(x)\,dx - \sum_{k=1}^{n} \int_{\partial B} (\alpha_k x_k - \beta_k) f(x) n_k(x)\,dA \right|$$
$$\leqslant \sum_{k=1}^{n} \int_B |\beta_k - \alpha_k x_k| \left| \frac{\partial f(x)}{\partial x_k} \right| dx \leqslant \begin{cases} \sum_{k=1}^{n} \int_B |\beta_k - \alpha_k x_k|\,dx \left\| \frac{\partial f(x)}{\partial x_k} \right\|_{B,\infty} \\ \sum_{k=1}^{n} \left(\int_B |\beta_k - \alpha_k x_k|^q\,dx \right)^{1/q} \left\| \frac{\partial f(x)}{\partial x_k} \right\|_{B,p} \\ \text{where } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \sum_{k=1}^{n} \sup_{x\in B} |\beta_k - \alpha_k x_k| \left\| \frac{\partial f(x)}{\partial x_k} \right\|_{B,1} \end{cases} \tag{4.1}$$

We also have

$$\left| \int_B f(x)\,dx - \frac{1}{n} \sum_{k=1}^{n} \int_{\partial B} (x_k - \gamma_k) f(x) n_k(x)\,dA \right|$$
$$\leqslant \frac{1}{n} \sum_{k=1}^{n} \int_B |\gamma_k - x_k| \left| \frac{\partial f(x)}{\partial x_k} \right| dx \leqslant \frac{1}{n} \begin{cases} \sum_{k=1}^{n} \int_B |\gamma_k - x_k|\,dx \left\| \frac{\partial f(x)}{\partial x_k} \right\|_{B,\infty} \\ \sum_{k=1}^{n} \left(\int_B^q |\gamma_k - x_k|^q\,dx \right)^{1/q} \left\| \frac{\partial f(x)}{\partial x_k} \right\|_{B,p} \\ \text{where } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \sum_{k=1}^{n} \sup_{x\in B} |\gamma_k - x_k| \left\| \frac{\partial f(x)}{\partial x_k} \right\|_{B,1} \end{cases} \tag{4.2}$$

for all $\gamma_k \in \mathbb{C}$, *where* $k \in \{1,...,n\}$.

Proof. By the identity (3.1) we have

$$\begin{aligned} &\left| \int_B f(x)\,dx - \sum_{k=1}^{n} \int_{\partial B} (\alpha_k x_k - \beta_k) f(x) n_k(x)\,dA \right| \\ &= \left| \sum_{k=1}^{n} \int_B (\beta_k - \alpha_k x_k) \frac{\partial f(x)}{\partial x_k} dx \right| \leqslant \sum_{k=1}^{n} \left| \int_B (\beta_k - \alpha_k x_k) \frac{\partial f(x)}{\partial x_k} dx \right| \\ &\leqslant \sum_{k=1}^{n} \int_B \left| (\beta_k - \alpha_k x_k) \frac{\partial f(x)}{\partial x_k} \right| dx, \end{aligned}$$

which proves the first inequality in (4.1).

By Hölder's integral inequality for multiple integrals we have

$$\int_B \left|(\beta_k - \alpha_k x_k)\frac{\partial f(x)}{\partial x_k}\right| dx$$

$$\leqslant \begin{cases} \sup_{x\in B}\left|\frac{\partial f(x)}{\partial x_k}\right| \int_B |\beta_k - \alpha_k x_k|\, dx \\ \left(\int_B \left|\frac{\partial f(x)}{\partial x_k}\right|^p\right)^{1/p} \left(\int_B |\beta_k - \alpha_k x_k|^q\, dx\right)^{1/q} \\ \text{where } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \sup_{x\in B}|\beta_k - \alpha_k x_k| \int_B \left|\frac{\partial f(x)}{\partial x_k}\right| dx \end{cases} = \begin{cases} \int_B |\beta_k - \alpha_k x_k|\, dx \left\|\frac{\partial f}{\partial x_k}\right\|_{B,\infty} \\ \left(\int_B |\beta_k - \alpha_k x_k|^q\, dx\right)^{1/q} \left\|\frac{\partial f}{\partial x_k}\right\|_{B,p}, \\ \text{where } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \sup_{x\in B}|\beta_k - \alpha_k x_k| \left\|\frac{\partial f}{\partial x_k}\right\|_{B,1} \end{cases}$$

which proves the last part of (4.1).

COROLLARY 4. *With the assumptions of Theorem 2 we have*

$$\left|\int_B f(x)\,dx - \frac{1}{n}\sum_{k=1}^n \int_{\partial B}\left(x_k - \overline{x_{B,k}}\right) f(x)\, n_k(x)\, dA\right|$$

$$\leqslant \frac{1}{n}\sum_{k=1}^n \int_B \left|\overline{x_{B,k}} - x_k\right| \left|\frac{\partial f(x)}{\partial x_k}\right| dx \leqslant \frac{1}{n}\begin{cases} \sum_{k=1}^n \int_B \left|\overline{x_{B,k}} - x_k\right| dx \left\|\frac{\partial f(x)}{\partial x_k}\right\|_{B,\infty} \\ \sum_{k=1}^n \left(\int_B^q \left|\overline{x_{B,k}} - x_k\right|^q dx\right)^{1/q} \left\|\frac{\partial f(x)}{\partial x_k}\right\|_{B,p} \\ \textit{where } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \sum_{k=1}^n \sup_{x\in B}\left|\overline{x_{B,k}} - x_k\right| \left\|\frac{\partial f(x)}{\partial x_k}\right\|_{B,1} \end{cases} \quad (4.3)$$

and

$$\left|\int_B f(x)\,dx\right| \leqslant \frac{1}{n}\sum_{k=1}^n \int_B \left|x_{\partial B,f,k} - x_k\right| \left|\frac{\partial f(x)}{\partial x_k}\right| dx$$

$$\leqslant \frac{1}{n}\begin{cases} \sum_{k=1}^n \int_B \left|x_{\partial B,f,k} - x_k\right| dx \left\|\frac{\partial f(x)}{\partial x_k}\right\|_{B,\infty} \\ \sum_{k=1}^n \left(\int_B^q \left|x_{\partial B,f,k} - x_k\right|^q dx\right)^{1/q} \left\|\frac{\partial f(x)}{\partial x_k}\right\|_{B,p} \\ \textit{where } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \sum_{k=1}^n \sup_{x\in B}\left|x_{\partial B,f,k} - x_k\right| \left\|\frac{\partial f(x)}{\partial x_k}\right\|_{B,1}. \end{cases} \quad (4.4)$$

We also have the dual result:

THEOREM 3. *With the assumption of Theorem 2 we have*

$$\left|\int_B f(x)\,dx-\sum_{k=1}^{n}\int_B(\beta_k-\alpha_k x_k)\frac{\partial f(x)}{\partial x_k}dx\right|$$
$$\leqslant\sum_{k=1}^{n}\int_{\partial B}|\alpha_k x_k-\beta_k|\,|n_k(x)|\,|f(x)|\,dA\leqslant\begin{cases}\|f\|_{\partial B,\infty}\sum_{k=1}^{n}\int_{\partial B}|\alpha_k x_k-\beta_k|\,|n_k(x)|\,dA;\\ \\ \|f\|_{\partial B,p}\sum_{k=1}^{n}\left(\int_{\partial B}|\alpha_k x_k-\beta_k|^q|n_k(x)|^q dA\right)^{1/q}\\ \text{where } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ \\ \|f\|_{\partial B,1}\sum_{k=1}^{n}\sup_{x\in\partial B}\left||\alpha_k x_k-\beta_k|\,|n_k(x)|\right|,\end{cases}\tag{4.5}$$

where

$$\|f\|_{\partial B,p}:=\begin{cases}\left(\int_{\partial B}|f(x)|^p\,dA\right)^{1/p},\ p\geqslant 1;\\ \\ \sup_{x\in\partial B}|f(x)|,\ p=\infty.\end{cases}$$

In particular,

$$\left|\int_B f(x)\,dx-\frac{1}{n}\sum_{k=1}^{n}\int_B(\gamma_k-x_k)\frac{\partial f(x)}{\partial x_k}dx\right|$$
$$\leqslant\frac{1}{n}\sum_{k=1}^{n}\int_{\partial B}|\gamma_k-x_k|\,|n_k(x)|\,|f(x)|\,dA\leqslant\frac{1}{n}\begin{cases}\|f\|_{\partial B,\infty}\sum_{k=1}^{n}\int_{\partial B}|\gamma_k-x_k|\,|n_k(x)|\,dA;\\ \\ \|f\|_{\partial B,p}\sum_{k=1}^{n}\left(\int_{\partial B}|\gamma_k-x_k|^q\,|n_k(x)|^q\,dA\right)^{1/q}\\ \text{where } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ \\ \|f\|_{\partial B,1}\sum_{k=1}^{n}\sup_{x\in\partial B}\left[|\gamma_k-x_k|\,|n_k(x)|\right].\end{cases}\tag{4.6}$$

Proof. From the identity (3.1) we have

$$\left|\int_B f(x)\,dx-\sum_{k=1}^{n}\int_B(\beta_k-\alpha_k x_k)\frac{\partial f(x)}{\partial x_k}dx\right|$$
$$=\left|\sum_{k=1}^{n}\int_{\partial B}(\alpha_k x_k-\beta_k)f(x)\,n_k(x)\,dA\right|\leqslant\sum_{k=1}^{n}\left|\int_{\partial B}(\alpha_k x_k-\beta_k)f(x)\,n_k(x)\,dA\right|$$
$$\leqslant\sum_{k=1}^{n}\int_{\partial B}|(\alpha_k x_k-\beta_k)f(x)\,n_k(x)|\,dA,$$

which proves the first inequality in (4.5).

By Hölder's inequality for functions defined on ∂B we have

$$\int_{\partial B}|\alpha_k x_k-\beta_k||n_k(x)||f(x)|dA \leqslant \begin{cases} \int_{\partial B}|\alpha_k x_k-\beta_k||n_k(x)|dA\,\|f\|_{\partial B,\infty}; \\ \\ \left(\int_{\partial B}|\alpha_k x_k-\beta_k|^q|n_k(x)|^q dA\right)^{1/q}\|f\|_{\partial B,p} \\ \text{where } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \sup_{x\in\partial B}||\alpha_k x_k-\beta_k||n_k(x)||\,\|f\|_{\partial B,1}, \end{cases}$$

which proves the second part of the inequality (4.5).

We also have:

COROLLARY 5. *With the assumptions of Theorem 2 we have*

$$\begin{aligned} &\left|\int_B f(x)\,dx-\frac{1}{n}\sum_{k=1}^n\int_B\left(\overline{x_{B,k}}-x_k\right)\frac{\partial f(x)}{\partial x_k}dx\right| \\ &\leqslant\frac{1}{n}\sum_{k=1}^n\int_{\partial B}\left|\overline{x_{B,k}}-x_k\right||n_k(x)||f(x)|dA \\ &\leqslant\frac{1}{n}\begin{cases}\|f\|_{\partial B,\infty}\sum_{k=1}^n\int_{\partial B}\left|\overline{x_{B,k}}-x_k\right||n_k(x)|dA; \\ \\ \|f\|_{\partial B,p}\sum_{k=1}^n\left(\int_{\partial B}\left|\overline{x_{B,k}}-x_k\right|^q|n_k(x)|^q dA\right)^{1/q} \\ \textit{where } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \|f\|_{\partial B,1}\sum_{k=1}^n\sup_{x\in\partial B}\left[\left|\overline{x_{B,k}}-x_k\right||n_k(x)|\right]\end{cases}\end{aligned} \tag{4.7}$$

and

$$\begin{aligned} \left|\int_B f(x)\,dx\right| &\leqslant\frac{1}{n}\sum_{k=1}^n\int_{\partial B}\left|x_{B,\partial f,k}-x_k\right||n_k(x)||f(x)|dA \\ &\leqslant\frac{1}{n}\begin{cases}\|f\|_{\partial B,\infty}\sum_{k=1}^n\int_{\partial B}\left|x_{B,\partial f,k}-x_k\right||n_k(x)|dA; \\ \\ \|f\|_{\partial B,p}\sum_{k=1}^n\left(\int_{\partial B}\left|x_{B,\partial f,k}-x_k\right|^q|n_k(x)|^q dA\right)^{1/q} \\ \textit{where } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \|f\|_{\partial B,1}\sum_{k=1}^n\sup_{x\in\partial B}\left[\left|x_{B,\partial f,k}-x_k\right||n_k(x)|\right].\end{cases}\end{aligned} \tag{4.8}$$

If we take $n=2$ in Theorem 3, then we get other results from [4], while for $n=3$ we recapture some results from [5].

REFERENCES

[1] H. BUDAK AND M. Z. SARIKAYA, *Generalized weighted Čebysev and Ostrowski type inequalities for double integrals*, TWMS J. Appl. Eng. Math. **7** (2017), no. 2, 272–281.

[2] H. BUDAK AND M. Z. SARIKAYA, *On weighted Grüss type inequalities for double integrals*, Commun. Fac. Sci. Univ. Ank. Sér. A1 Math. Stat., **66** (2017), no. 2, 53–61.
[3] S. S. DRAGOMIR, *Double integral inequalities of Hermite-Hadamard type for h-convex functions on linear spaces*, Analysis (Berlin) **37** (2017), no. 1, 13–22.
[4] S. S. DRAGOMIR, *Some inequalities for double and path integrals on general domains via Green's identity*, Preprint RGMIA Res. Rep. Coll. **22** (2019), Art. 55, 17 pp. Online http://rgmia.org/papers/v22/v22a55.pdf.
[5] S. S. DRAGOMIR, *Some triple integral inequalities for functions defined on 3-dimensional bodies via Gauss-Ostrogradsky identity*, Preprint RGMIA Res. Rep. Coll. **22** (2019), Art. 60, 21 pp. Online http://rgmia.org/papers/v22/v22a60.pdf.
[6] B. G. PACHPATTE, *New inequalities of Ostrowski and Grüss type for triple integrals*, Tamkang J. Math. **40** (2009), no. 2, 117–127.
[7] D. B. PACHPATTE, *Some Ostrowski type inequalities for double integrals on time scales*, Acta Appl. Math. **161** (2019), 1–11.
[8] M. SINGER, *The divergence theorem*, Online https://www.maths.ed.ac.uk/~jmf/Teaching/Lectures/divthm.pdf.
[9] W. T. SULAIMAN, *Integral inequalities concerning triple integrals*, J. Concr. Appl. Math. **8** (2010), no. 4, 585–593.

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Silvestru Sever Dragomir
Mathematics, College of Engineering & Science
Victoria University
PO Box 14428, Melbourne City, MC 8001, Australia
DST-NRF Centre of Excellence in the Mathematical and Statistical Sciences
School of Computer Science & Applied Mathematics,
University of the Witwatersrand
Private Bag 3, Johannesburg 2050, South Africa
e-mail: sever.dragomir@vu.edu.au