Inequalities for Double Integrals of Schur Convex Functions on Symmetric and Convex Domains

MATEMATIČKI VESNIK
МАТЕМАТИЧКИ ВЕСНИК
73, 1 (2021), 63–74
March 2021

research paper
оригинални научни рад

INEQUALITIES FOR DOUBLE INTEGRALS OF SCHUR CONVEX FUNCTIONS ON SYMMETRIC AND CONVEX DOMAINS

Silvestru Sever Dragomir

Abstract. In this paper, by making use of Green's identity for double integrals, we establish some integral inequalities for Schur convex functions defined on domains $D \subset \mathbb{R}^2$ that are symmetric, convex and have nonempty interiors. Examples for squares and disks are also provided.

1. Introduction

For any $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, let $x_{[1]} \geq \ldots \geq x_{[n]}$ denote the components of x in decreasing order, and let $x_{\downarrow} = \left(x_{[1]}, \ldots, x_{[n]}\right)$ denote the decreasing rearrangement of x. For x, $y \in \mathbb{R}^n$, $x \prec y$ if, by definition,

$$\begin{cases} \sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]}, \ k = 1, \ldots, n-1; \\ \sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]}. \end{cases}$$

When $x \prec y$, x is said to be *majorized* by y (y *majorizes* x). This notation and terminology was introduced by Hardy, Littlewood and Pólya in 1934.

Functions that preserve the ordering of majorization are said to be Schur-convex. Perhaps "Schur-increasing" would be more appropriate, but the term "Schur-convex" is by now well entrenched in the literature, [3, p.80].

A real-valued function ϕ defined on a set $\mathcal{A} \subset \mathbb{R}^n$ is said to be *Schur-convex* on $\mathcal{A}$ if

$$x \prec y \text{ on } \mathcal{A} \Rightarrow \phi(x) \leq \phi(y). \tag{1}$$

If, in addition, $\phi(x) < \phi(y)$ whenever $x \prec y$ but x is not a permutation of y, then ϕ is said to be *strictly Schur-convex* on $\mathcal{A}$. If $\mathcal{A} = \mathbb{R}^n$, then ϕ is simply said to be Schur-convex or strictly Schur-convex.

2020 Mathematics Subject Classification: 26D15
Keywords and phrases: Schur convex functions; double integral inequalities.

For fundamental properties of Schur convexity see the monograph [3] and the references therein. For some recent results, see [1, 2, 4–6].

The following result is known in the literature as *Schur-Ostrowski theorem* [3, p.84]:

THEOREM 1.1. *Let $I \subset \mathbb{R}$ be an open interval and let $\phi : I^n \to \mathbb{R}$ be continuously differentiable. Necessary and sufficient conditions for ϕ to be Schur-convex on I^n are*

$$\phi \text{ is symmetric on } I^n, \tag{2}$$

and for all $i \neq j$, with $i, j \in \{1, \ldots, n\}$,

$$(z_i - z_j)\left[\frac{\partial\phi(z)}{\partial x_i} - \frac{\partial\phi(z)}{\partial x_j}\right] \geq 0 \text{ for all } z \in I^n, \tag{3}$$

where $\frac{\partial\phi}{\partial x_k}$ denotes the partial derivative of ϕ with respect to its k-th argument.

With the aid of (2), condition (3) can be replaced by the condition

$$(z_1 - z_2)\left[\frac{\partial\phi(z)}{\partial x_1} - \frac{\partial\phi(z)}{\partial x_2}\right] \geq 0 \text{ for all } z \in I^n. \tag{4}$$

This simplified condition is sometimes more convenient to verify.

The above condition is not sufficiently general for all applications because the domain of ϕ may not be a Cartesian product.

Let $\mathcal{A} \subset \mathbb{R}^n$ be a set with the following properties:

(i) $\mathcal{A}$ is *symmetric* in the sense that $x \in \mathcal{A} \Rightarrow x\Pi \in \mathcal{A}$ for all permutations Π of the coordinates.

(ii) $\mathcal{A}$ is convex and has a nonempty interior.

We have the following result, [3, p. 85].

THEOREM 1.2. *If ϕ is continuously differentiable on the interior of $\mathcal{A}$ and continuous on $\mathcal{A}$, then necessary and sufficient conditions for ϕ to be Schur-convex on $\mathcal{A}$ are*

$$\phi \text{ is symmetric on } \mathcal{A} \tag{5}$$

and

$$(z_1 - z_2)\left[\frac{\partial\phi(z)}{\partial x_1} - \frac{\partial\phi(z)}{\partial x_2}\right] \geq 0 \text{ for all } z \in \mathcal{A}. \tag{6}$$

It is well known that any symmetric convex function defined on a symmetric convex set $\mathcal{A}$ is Schur convex, [3, p. 97]. If the function $\phi : \mathcal{A} \to \mathbb{R}$ is symmetric and quasi-convex, namely $\phi(\alpha u + (1-\alpha)v) \leq \max\{\phi(u), \phi(v)\}$ for all $\alpha \in [0, 1]$ and u, $v \in \mathcal{A}$, a symmetric convex set, then ϕ is Schur convex on $\mathcal{A}$ [3, p.98].

In this paper we establish some integral inequalities for Schur convex functions defined on domains $D \subset \mathbb{R}^2$ that are symmetric, convex and have nonempty interiors. Examples for squares and disks are also provided.

2. Main results

For a function $f : D \to \mathbb{C}$ having partial derivatives on the domain $D \subset \mathbb{R}^2$ we define $\Lambda_{\partial f,D} : D \to \mathbb{C}$ as

$$\Lambda_{\partial f,D}(x,y) := (x-y)\left(\frac{\partial f(x,y)}{\partial x} - \frac{\partial f(x,y)}{\partial y}\right).$$

Let ∂D be a simple, closed counterclockwise curve in the xy-plane, bounding a region D. Let L and M be scalar functions defined at least on an open set containing D. Assume L and M have continuous first partial derivatives. Then the following equality is well known as the Green theorem:

$$\iint_D \left(\frac{\partial M(x,y)}{\partial x} - \frac{\partial L(x,y)}{\partial y}\right) dx\,dy = \oint_{\partial D} (L(x,y)\,dx + M(x,y)\,dy). \tag{7}$$

By applying this equality for real and imaginary parts, we can also state it for complex valued functions P and Q. Moreover, if the curve ∂D is described by the function $r(t) = (x(t), y(t))$, $t \in [a,b]$, with x, y differentiable on (a,b) then we can calculate the path integral as

$$\oint_{\partial D} (L(x,y)\,dx + M(x,y)\,dy) = \int_a^b [L(x(t), y(t))\,x'(t) + M(x(t), y(t))\,y'(t)]\,dt.$$

We have the following identity of interest.

Lemma 2.1. *Let ∂D be a simple, closed counterclockwise curve in the xy-plane, bounding a region D. Assume that the function $f : D \to \mathbb{C}$ has continuous partial derivatives on the domain D. Then*

$$\frac{1}{2}\oint_{\partial D} [(x-y) f(x,y)\,dx + (x-y) f(x,y)\,dy] - \iint_D f(x,y)\,dx\,dy$$
$$= \frac{1}{2}\iint_D \Lambda_{\partial f,D}(x,y)\,dx\,dy. \tag{8}$$

Proof. Consider the functions$M(x,y) := (x-y) f(x,y)$ and $L(x,y) := (x-y) f(x,y)$ for $(x,y) \in D$.

We have

$$\frac{\partial}{\partial x}[(x-y) f(x,y)] = f(x,y) + (x-y)\frac{\partial f(x,y)}{\partial x}$$

and

$$\frac{\partial}{\partial y}[(y-x) f(x,y)] = f(x,y) + (y-x)\frac{\partial f(x,y)}{\partial y}$$

for $(x,y) \in D$.

If we add these two equalities, then we get

$$\frac{\partial M(x,y)}{\partial x} - \frac{\partial L(x,y)}{\partial y} = 2f(x,y) + \Lambda_{\partial f,D}(x,y) \tag{9}$$

for $(x,y) \in D$.

If we integrate this equality on D, then we obtain

$$\iint_D \left(\frac{\partial M(x,y)}{\partial x} - \frac{\partial L(x,y)}{\partial y} \right) dx\,dy$$
$$= 2 \iint_D f(x,y)\,dx\,dy + \iint_D \Lambda_{\partial f,D}(x,y)\,dx\,dy. \tag{10}$$

From Green's identity we also have

$$\iint_D \left(\frac{\partial M(x,y)}{\partial x} - \frac{\partial L(x,y)}{\partial y} \right) dx\,dy = \oint_{\partial D} (L(x,y)\,dx + M(x,y)\,dy)$$
$$= \oint_{\partial D} [(x-y) f(x,y)\,dx + (x-y) f(x,y)\,dy]. \tag{11}$$

By employing (10) and (11) we deduce the desired equality (8). □

Corollary 2.2. *With the assumptions of Lemma 2.1 and if the curve ∂D is described by the function $r(t) = (x(t), y(t))$, $t \in [a,b]$, with x, y differentiable on (a,b), then*

$$\frac{1}{2} \int_a^b (x(t) - y(t)) f(x(t), y(t)) (x'(t) + y'(t))\,dt - \iint_D f(x,y)\,dx\,dy$$
$$= \frac{1}{2} \iint_D \Lambda_{\partial f,D}(x,y)\,dx\,dy. \tag{12}$$

We have the following result for Schur convex functions defined on symmetric convex domains of $\mathbb{R}^2$.

Theorem 2.3. *Let $D \subset \mathbb{R}^2$ be symmetric, convex and has a nonempty interior. If ϕ is continuously differentiable on the interior of D, continuous and Schur convex on D and ∂D is a simple, closed counterclockwise curve in the xy-plane bounding D, then*

$$\iint_D \phi(x,y)\,dx\,dy \le \frac{1}{2} \oint_{\partial D} [(x-y)\phi(x,y)\,dx + (x-y)\phi(x,y)\,dy]. \tag{13}$$

If ϕ is Schur concave on D, then the sign of inequality reverses in (13).

The proof follows by Lemma 2.1 and Theorem 1.1.

Corollary 2.4. *Let $D \subset \mathbb{R}^2$ be symmetric, convex and has a nonempty interior. If ϕ is continuously differentiable on the interior of D, continuous and convex or quasi-convex on D and ∂D is a simple, closed counterclockwise curve in the xy-plane bounding D, then the inequality* (13) *is valid.*

Remark 2.5. With the assumptions of Theorem 2.3 and if the curve ∂D is described by the function $r(t) = (x(t), y(t))$, $t \in [a,b]$, with x, y differentiable on (a,b), then

$$\iint_D \phi(x,y)\,dx\,dy \le \frac{1}{2} \int_a^b (x(t) - y(t)) \phi(x(t), y(t)) (x'(t) + y'(t))\,dt. \tag{14}$$

Let $a < b$. Put $A = (a, a)$, $B = (b, a)$, $C = (b, b)$, $D = (a, b) \in \mathbb{R}^2$ the vertices of the square $ABCD = [a, b]^2$. Consider the counterclockwise segments

$$AB : \begin{cases} x = (1-t)\, a + tb, & t \in [0,1] \\ y = a \end{cases}$$

$$BC : \begin{cases} x = b \\ y = (1-t)\, a + tb, & t \in [0,1] \end{cases}$$

$$CD : \begin{cases} x = (1-t)\, b + ta \\ y = b, \quad t \in [0,1] \end{cases}$$

and

$$DA : \begin{cases} x = a \\ y = (1-t)\, b + ta, & t \in [0,1]. \end{cases}$$

Therefore $\partial (ABCD) = AB \cup BC \cup CD \cup DA$.

For any function f defined on $ABCD$, we have

$$\begin{aligned}
&\oint_{AB} [(x-y) f(x,y)\, dx + (x-y) f(x,y)\, dy] \\
&\quad = (b-a) \int_0^1 ((1-t)\, a + tb - a)\, f((1-t)\, a + tb, a)\, dt \\
&\quad = (b-a)^2 \int_0^1 t f((1-t)\, a + tb, a)\, dt,
\end{aligned}$$

$$\begin{aligned}
&\oint_{BC} [(x-y) f(x,y)\, dx + (x-y) f(x,y)\, dy] \\
&\quad = (b-a) \int_0^1 (b - (1-t)\, a - tb)\, f(b, (1-t)\, a + tb)\, dt \\
&\quad = (b-a)^2 \int_0^1 (1-t)\, f(b, (1-t)\, a + tb)\, dt,
\end{aligned}$$

$$\begin{aligned}
&\oint_{CD} [(x-y) f(x,y)\, dx + (x-y) f(x,y)\, dy] \\
&\quad = (a-b) \int_0^1 ((1-t)\, b + ta - b)\, f((1-t)\, b + ta, b)\, dt \\
&\quad = (a-b)^2 \int_0^1 t f((1-t)\, b + ta, b)\, dt \\
&\quad = (a-b)^2 \int_0^1 (1-t)\, f((1-t)\, a + tb, b)\, dt \text{ (by change of variable).}
\end{aligned}$$

and

$$\oint_{DA} [(x-y) f(x,y)\, dx + (x-y) f(x,y)\, dy]$$

$$= (a-b)\int_0^1 (a-(1-t)\,b-ta)\, f\,(a,(1-t)\,b+ta)\,dt$$
$$= (a-b)^2\int_0^1 (1-t)\, f\,(a,(1-t)\,b+ta)\,dt$$
$$= (a-b)^2\int_0^1 tf\,(a,(1-t)\,a+tb)\,dt \text{ (by change of variable).}$$

Therefore

$$\oint_{\partial(ABCD)} [(x-y)\, f\,(x,y)\,dx + (x-y)\, f\,(x,y)\,dy] \tag{15}$$
$$= (b-a)^2\int_0^1 tf\,((1-t)\,a+tb,a)\,dt + (b-a)^2\int_0^1 (1-t)\, f\,(b,(1-t)\,a+tb)\,dt$$
$$+ (b-a)^2\int_0^1 (1-t)\, f\,((1-t)\,a+tb,b)\,dt + (b-a)^2\int_0^1 tf\,(a,(1-t)\,a+tb)\,dt$$
$$= (b-a)^2\int_0^1 t\,[f\,((1-t)\,a+tb,a) + f\,(a,(1-t)\,a+tb)]\,dt$$
$$+ (b-a)^2\int_0^1 (1-t)\,[f\,(b,(1-t)\,a+tb) + f\,((1-t)\,a+tb,b)]\,dt.$$

Since the vast majority of examples of Schur convex functions are defined on the Cartesian product of intervals, we can state the following result of interest.

Corollary 2.6. *If ϕ is continuously differentiable on the interior of $D = [a,b]^2$, continuous on D and Schur convex, then*

$$\frac{1}{(b-a)^2}\int_a^b\int_a^b \phi\,(x,y)\,dx\,dy \le \int_0^1 t\phi\,((1-t)\,a+tb,a)\,dt + \int_0^1 (1-t)\,\phi\,((1-t)\,a+tb,b)\,dt. \tag{16}$$

Proof. From (13) we get

$$\frac{1}{(b-a)^2}\int_a^b\int_a^b \phi\,(x,y)\,dx\,dy \tag{17}$$
$$\le \int_0^1 t\left[\frac{\phi\,((1-t)\,a+tb,a)+\phi\,(a,(1-t)\,a+tb)}{2}\right]dt + \int_0^1 (1-t)\left[\frac{\phi\,((1-t)\,a+tb,b)+\phi\,(b,(1-t)\,a+tb)}{2}\right]dt.$$

Since ϕ is symmetric on $D = [a,b]^2$, hence

$$\phi\,((1-t)\,a+tb,a) = \phi\,(a,(1-t)\,a+tb)$$

and

$$\phi\,((1-t)\,a+tb,b) = \phi\,(b,(1-t)\,a+tb)$$

for all $t \in [0,1]$ and by (17) we get (16). □

Remark 2.7. By making the change of variable $x = (1-t)\,a + tb$, $t \in [0,1]$, then $dx = (b-a)\,dt$, $t = \frac{x-a}{b-a}$ and by (16) we get

$$\frac{1}{(b-a)^2}\int_a^b\int_a^b \phi(x,y)\,dx\,dy \le \frac{1}{b-a}\int_a^b \frac{x-a}{b-a}\phi(x,a)\,dx + \frac{1}{b-a}\int_a^b \frac{b-x}{b-a}\phi(x,b)\,dx, \tag{18}$$

or, equivalently,

$$\int_a^b\int_a^b \phi(x,y)\,dx\,dy \le \int_a^b (x-a)\,\phi(x,a)\,dx + \int_a^b (b-x)\,\phi(x,b)\,dx. \tag{19}$$

3. Lower and upper Schur convexity

Start with the following extensions of Schur convex functions:

Definition 3.1. Let D be symmetric, convex and has a nonempty interior in $\mathbb{R}^2$ and a symmetric function $f : D \to \mathbb{R}$ having continuous partial derivatives on $D \subset \mathbb{R}^2$.

(i) For $m \in \mathbb{R}$, f is called m-lower Schur convex on D if

$$m\,(x-y)^2 \le \Lambda_{\partial f,D}(x,y) \text{ for all } (x,y) \in D. \tag{20}$$

(ii) For $M \in \mathbb{R}$, f is called M-upper Schur convex on D if

$$\Lambda_{\partial f,D}(x,y) \le M\,(x-y)^2 \text{ for all } (x,y) \in D. \tag{21}$$

(iii) For m, $M \in \mathbb{R}$ with $m < M$, f is called (m,M)-Schur convex on D if

$$m\,(x-y)^2 \le \Lambda_{\partial f,D}(x,y) \le M\,(x-y)^2 \text{ for all } (x,y) \in D. \tag{22}$$

We have the following simple but useful result.

Proposition 3.2. *Let D be symmetric, convex and has a nonempty interior in $\mathbb{R}^2$ and a symmetric function $f : D \to \mathbb{R}$ having continuous partial derivatives on $D \subset \mathbb{R}^2$.*

(i) For $m \in \mathbb{R}$, f is m-lower Schur convex on D iff $f_m : D \to \mathbb{R}$,

$$f_m(x,y) := f(x,y) - \frac{1}{2}m\left(x^2+y^2\right)$$

is Schur convex on D.

(ii) For $M \in \mathbb{R}$, f is M-upper Schur convex on D iff $f_M : D \to \mathbb{R}$,

$$f_M(x,y) := \frac{1}{2}M\left(x^2+y^2\right) - f(x,y)$$

is Schur convex on D.

(iii) For m, $M \in \mathbb{R}$ with $m < M$, f is (m,M)-Schur convex on D iff f_m and f_M are Schur convex on D.

Proof. (i) Observe that

$$\begin{aligned}\Lambda_{\partial f_m,D}(x,y) &= (x-y)\left(\frac{\partial f_m(x,y)}{\partial x}-\frac{\partial f_m(x,y)}{\partial y}\right)\\ &= (x-y)\left(\frac{\partial f(x,y)}{\partial x}-mx-\frac{\partial f(x,y)}{\partial y}+my\right)\\ &= (x-y)\left(\frac{\partial f(x,y)}{\partial x}-\frac{\partial f(x,y)}{\partial y}-m(x-y)\right)\\ &= \Lambda_{\partial f,D}(x,y)-m(x-y)^2,\end{aligned}$$

for all $(x,y)\in D$, which proves the statement.

The statements (ii) and (iii) follow in a similar way. □

THEOREM 3.3. *Let ∂D be a simple, closed counterclockwise curve in the xy-plane, bounding a domain $D\subset\mathbb{R}^2$ that is symmetric, convex and has a nonempty interior.*
(i) Assume that the function $f:D\to\mathbb{R}$ is m-lower Schur convex, then

$$\begin{aligned}&\frac{1}{2}m\iint_D (x-y)^2\,dx\,dy \qquad (23)\\ &\le \frac{1}{2}\oint_{\partial D}[(x-y)f(x,y)\,dx+(x-y)f(x,y)\,dy]-\iint_D f(x,y)\,dx\,dy.\end{aligned}$$

(ii) Assume that the function $f:D\to\mathbb{R}$ is M-upper Schur convex, then

$$\begin{aligned}&\frac{1}{2}\oint_{\partial D}[(x-y)f(x,y)\,dx+(x-y)f(x,y)\,dy]-\iint_D f(x,y)\,dx\,dy \qquad (24)\\ &\le \frac{1}{2}M\iint_D (x-y)^2\,dx\,dy.\end{aligned}$$

(iii) Assume that the function $f:D\to\mathbb{R}$ is (m,M)-Schur convex, then

$$\begin{aligned}&\frac{1}{2}m\iint_D (x-y)^2\,dx\,dy \qquad (25)\\ &\le \frac{1}{2}\oint_{\partial D}[(x-y)f(x,y)\,dx+(x-y)f(x,y)\,dy]-\iint_D f(x,y)\,dx\,dy\\ &\le \frac{1}{2}M\iint_D (x-y)^2\,dx\,dy.\end{aligned}$$

Proof. (i) Since $f_m(x,y):=f(x,y)-\frac{1}{2}m\left(x^2+y^2\right)$ is Schur convex on D, then by (13) we get

$$\iint_D f_m(x,y)\,dx\,dy\le\frac{1}{2}\oint_{\partial D}[(x-y)f_m(x,y)\,dx+(x-y)f_m(x,y)\,dy],$$

namely
$$\iint_D\left[f(x,y)-\frac{1}{2}m\left(x^2+y^2\right)\right]dx\,dy \qquad (26)$$

$$\leq \frac{1}{2}\oint\limits_{\partial D}\left\{(x-y)\left[f(x,y)-\frac{1}{2}m\left(x^2+y^2\right)\right]dx\right.$$
$$\left.+(x-y)\left[f(x,y)-\frac{1}{2}m\left(x^2+y^2\right)\right]dy\right\}.$$

Since
$$\iint_D\left[f(x,y)-\frac{1}{2}m\left(x^2+y^2\right)\right]dx\,dy=\iint_D f(x,y)\,dx\,dy$$
$$-\frac{1}{2}m\iint_D\left(x^2+y^2\right)dx\,dy$$

and
$$\frac{1}{2}\oint\limits_{\partial D}\left\{(x-y)\left[f(x,y)-\frac{1}{2}m\left(x^2+y^2\right)\right]dx\right.$$
$$\left.+(x-y)\left[f(x,y)-\frac{1}{2}m\left(x^2+y^2\right)\right]dy\right\}$$
$$=\frac{1}{2}\oint\limits_{\partial D}\left[(x-y)f(x,y)\,dx+(x-y)f(x,y)\,dy\right]$$
$$-\frac{1}{4}m\oint\limits_{\partial D}\left[\left(x^2+y^2\right)dx+\left(x^2+y^2\right)dy\right],$$

hence, by (26), we get

$$\frac{1}{2}m\left\{\frac{1}{2}\oint\limits_{\partial D}\left[(x-y)\left(x^2+y^2\right)dx+(x-y)\left(x^2+y^2\right)dy\right]-\iint_D\left(x^2+y^2\right)dx\,dy\right\}$$
$$\leq\frac{1}{2}\oint\limits_{\partial D}\left[(x-y)f(x,y)\,dx+(x-y)f(x,y)\,dy\right]-\iint_D f(x,y)\,dx\,dy. \tag{27}$$

Further, if we use the identity (8) for the function $g(x,y)=x^2+y^2$ we get

$$\frac{1}{2}\oint\limits_{\partial D}\left[(x-y)\left(x^2+y^2\right)dx+(x-y)\left(x^2+y^2\right)dy\right]-\iint_D\left(x^2+y^2\right)dx\,dy$$
$$=\frac{1}{2}\iint_D 2(x-y)^2\,dx\,dy=\iint_D(x-y)^2\,dx\,dy,$$

which together with (27) gives the desired result (23).

The statements (ii) and (iii) follow in a similar way and we omit the details. □

If f is symmetric on D, for all $(x,y)\in D$, we have

$$\Lambda_{\partial f,D}(x,y)=(x-y)\left(\frac{\partial f(x,y)}{\partial x}-\frac{\partial f(x,y)}{\partial y}\right)=(x-y)\left(\frac{\partial f(x,y)}{\partial x}-\frac{\partial f(y,x)}{\partial x}\right).$$

If
$$0<k\leq\left|\frac{\frac{\partial f(x,y)}{\partial x}-\frac{\partial f(y,x)}{\partial x}}{x-y}\right|\leq K<\infty \text{ for all } (x,y)\in D \text{ with } x\neq y, \tag{28}$$

then
$$0\leq k(x-y)^2\leq\Lambda_{\partial f,D}(x,y)\leq K(x-y)^2 \text{ for all } (x,y)\in D.$$

By making use of Theorem 3.3 we can state the following result.

Corollary 3.4. *Let ∂D be a simple, closed counterclockwise curve in the xy-plane, bounding a domain $D \subset \mathbb{R}^2$ that is symmetric, convex and has a nonempty interior. If f is continuously differentiable on the interior of D, continuous and symmetric on D and the partial derivative $\frac{\partial f}{\partial x}$ satisfies the condition* (28), *then we have the inequalities*

$$\begin{aligned} 0 &\leq \frac{1}{2} k \iint_D (x-y)^2 \, dx \, dy \qquad (29) \\ &\leq \frac{1}{2} \oint_{\partial D} \left[(x-y) f(x,y) \, dx + (x-y) f(x,y) \, dy \right] - \iint_D f(x,y) \, dx \, dy \\ &\leq \frac{1}{2} K \iint_D (x-y)^2 \, dx \, dy. \end{aligned}$$

Remark 3.5. If $D = [a,b]^2$ and since

$$\int_a^b \int_a^b (x-y)^2 \, dx \, dy = \int_a^b \frac{(b-x)^3 + (x-a)^3}{3} dx = \frac{1}{6} (b-a)^4$$

hence by (29) we get

$$\begin{aligned} 0 &\leq \frac{1}{12} k (b-a)^4 \qquad (30) \\ &\leq \int_a^b (x-a) f(x,a) \, dx + \int_a^b (b-x) f(x,b) \, dx - \int_a^b \int_a^b f(x,y) \, dx \, dy \\ &\leq \frac{1}{12} K (b-a)^4 , \end{aligned}$$

provided that f is *continuously differentiable* on the interior of $[a,b]^2$, continuous and symmetric on $[a,b]^2$ and the partial derivative $\frac{\partial f}{\partial x}$ satisfies the condition (28).

4. Examples for disks

We consider the closed disk $D(O,R)$ centered in $O(0,0)$ and of radius $R > 0$, parameterized by

$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta, \quad r \in [0,R], \ \theta \in [0, 2\pi], \end{cases}$$

and the circle $\mathcal{C}(O,R)$, parameterized by

$$\begin{cases} x = R \cos \theta \\ y = R \sin \theta, \quad \theta \in [0, 2\pi]. \end{cases}$$

Observe that, if $\phi : D(O,R) \to \mathbb{R}$, then

$$\oint_{\mathcal{C}(O,R)} \left[(x-y) \phi(x,y) \, dx + (x-y) \phi(x,y) \, dy \right]$$

$$
\begin{aligned}
&= -\int_0^{2\pi} R\left(R\cos\theta - R\sin\theta\right)\sin\theta\phi\left(R\cos\theta, R\sin\theta\right)d\theta \\
&\quad + \int_0^{2\pi} R\left(R\cos\theta - R\sin\theta\right)\cos\theta\phi\left(R\cos\theta, R\sin\theta\right)d\theta \\
&= R^2\int_0^{2\pi}\phi\left(R\cos\theta, R\sin\theta\right)\left(\cos\theta - \sin\theta\right)^2 d\theta.
\end{aligned}
$$

Also, we have $$\iint_{D(O,R)}\phi\left(x,y\right)dx\,dy = \int_0^R\int_0^{2\pi}\phi\left(r\cos\theta, r\sin\theta\right)r\,dr\,d\theta.$$

Using Theorem 2.3 we can state the following result.

Proposition 4.1. *If ϕ is continuously differentiable on the interior of $D\left(O,R\right)$, continuous and Schur convex on $D\left(O,R\right)$, then*

$$
\int_0^R\int_0^{2\pi}\phi\left(r\cos\theta, r\sin\theta\right)r\,dr\,d\theta \le \frac{1}{2}R^2\int_0^{2\pi}\phi\left(R\cos\theta, R\sin\theta\right)\left(\cos\theta - \sin\theta\right)^2 d\theta. \quad (31)
$$

Now, observe that

$$
\begin{aligned}
&\iint_{D(O,R)}\left(x-y\right)^2 dx\,dy = \int_0^R\int_0^{2\pi}\left(R\cos\theta - R\sin\theta\right)^2 r\,dr\,d\theta \\
&= \frac{1}{2}R^4\int_0^{2\pi}\left(\cos\theta - \sin\theta\right)^2 d\theta = \frac{1}{2}R^4\int_0^{2\pi}\left(1 - 2\sin\theta\cos\theta\right)d\theta = \pi R^4.
\end{aligned}
$$

By Corollary 3.4, the following holds.

Proposition 4.2. *If ϕ is continuously differentiable on the interior of $D\left(O,R\right)$, continuous and Schur convex on $D\left(O,R\right)$ and the derivative $\frac{\partial f}{\partial x}$ satisfies the condition* (28) *on $D\left(O,R\right)$, then*

$$
\begin{aligned}
\frac{1}{2}\pi kR^4 \le \frac{1}{2}R^2 &\int_0^{2\pi}\phi\left(R\cos\theta, R\sin\theta\right)\left(\cos\theta - \sin\theta\right)^2 d\theta \\
&- \int_0^R\int_0^{2\pi}\phi\left(r\cos\theta, r\sin\theta\right)r\,dr\,d\theta \le \frac{1}{2}\pi KR^4.
\end{aligned}
$$

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(received 18.07.2019; in revised form 06.05.2020; available online 07.12.2020)

Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia

DST-NRF Centre of Excellence in the Mathematical and Statistical Sciences, School of Computer Science & Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa

E-mail: sever.dragomir@vu.edu.au