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Research Article

# Ostrowski's Type Inequalities for the Complex Integral on Paths

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**ABSTRACT.** In this paper, we extend the Ostrowski inequality to the integral with respect to arc-length by providing upper bounds for the quantity

$$\left| f(v) \ell(\gamma) - \int_{\gamma} f(z) |dz| \right|$$

under the assumptions that  $\gamma$  is a smooth path parametrized by  $z(t)$ ,  $t \in [a, b]$  with the length  $\ell(\gamma)$ ,  $u = z(a)$ ,  $v = z(b)$  with  $x \in (a, b)$  and  $w = z(b)$  while  $f$  is holomorphic in  $G$ , an open domain and  $\gamma \subset G$ . An application for circular paths is also given.

**Keywords:** Complex integral, continuous functions, holomorphic functions, Ostrowski inequality.

**2020 Mathematics Subject Classification:** 26D15, 26D10, 30A10, 30A86.

## 1. INTRODUCTION

In 1938, A. Ostrowski [8], proved the following inequality concerning the distance between the integral mean  $\frac{1}{b-a} \int_a^b f(t) dt$  and the value  $f(x)$ ,  $x \in [a, b]$ .

**Theorem 1.1** (Ostrowski, 1938 [8]). *Let  $f : [a, b] \rightarrow \mathbb{R}$  be continuous on  $[a, b]$  and differentiable on  $(a, b)$  such that  $f' : (a, b) \rightarrow \mathbb{R}$  is bounded on  $(a, b)$ , i.e.,  $\|f'\|_{\infty} := \sup_{t \in (a, b)} |f'(t)| < \infty$ . Then,*

$$(1.1) \quad \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left[ \frac{1}{4} + \left( \frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] \|f'\|_{\infty} (b-a)$$

for all  $x \in [a, b]$  and the constant  $\frac{1}{4}$  is the best possible.

In [6], S. S. Dragomir and S. Wang, by the use of the Montgomery integral identity [7, p. 565],

$$(1.2) \quad f(x) - \frac{1}{b-a} \int_a^b f(t) dt = \frac{1}{b-a} \int_a^b p(x, t) f'(t) dt, \quad x \in [a, b],$$

where  $p : [a, b]^2 \rightarrow \mathbb{R}$  is given by

$$p(x, t) := \begin{cases} t - a & \text{if } t \in [a, x], \\ t - b & \text{if } t \in (x, b], \end{cases}$$

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gave a simple proof of Ostrowski’s inequality and applied it for special means (identric mean, logarithmic mean, etc.) and to the problem of estimating the error bound in approximating the Riemann integral  $\int_a^b f(t) dt$  by one arbitrary Riemann sum (see [6, Section 3]). For extensions of Ostrowski’s inequality in terms of the  $p$ -norms of the derivative, see [1], [2] and [3]. For a recent survey on Ostrowski’s inequality, see [4].

In order to extend this result for the complex integral, we need some preparations as follows. Suppose  $\gamma$  is a smooth path parametrized by  $z(t)$ ,  $t \in [a, b]$  and  $f$  is a complex function which is continuous on  $\gamma$ . Put  $z(a) = u$  and  $z(b) = w$  with  $u, w \in \mathbb{C}$ . We define the integral of  $f$  on  $\gamma_{u,w} = \gamma$  as

$$\int_{\gamma} f(z) dz = \int_{\gamma_{u,w}} f(z) dz := \int_a^b f(z(t)) z'(t) dt.$$

We observe that the actual choice of parametrization of  $\gamma$  does not matter. This definition immediately extends to paths that are piecewise smooth. Suppose  $\gamma$  is parametrized by  $z(t)$ ,  $t \in [a, b]$ , which is differentiable on the intervals  $[a, c]$  and  $[c, b]$ , then assuming that  $f$  is continuous on  $\gamma$  we define

$$\int_{\gamma_{u,w}} f(z) dz := \int_{\gamma_{u,v}} f(z) dz + \int_{\gamma_{v,w}} f(z) dz,$$

where  $v := z(c)$ . This can be extended for a finite number of intervals. We also define the integral with respect to arc-length

$$\int_{\gamma_{u,w}} f(z) |dz| := \int_a^b f(z(t)) |z'(t)| dt$$

and the length of the curve  $\gamma$  is then

$$\ell(\gamma) = \int_{\gamma_{u,w}} |dz| = \int_a^b |z'(t)| dt.$$

Let  $f$  and  $g$  be holomorphic in  $G$ , an open domain and suppose  $\gamma \subset G$  is a piecewise smooth path from  $z(a) = u$  to  $z(b) = w$ . Then, we have the *integration by parts formula*

$$(1.3) \quad \int_{\gamma_{u,w}} f(z) g'(z) dz = f(w) g(w) - f(u) g(u) - \int_{\gamma_{u,w}} f'(z) g(z) dz.$$

We recall also the *triangle inequality* for the complex integral, namely

$$(1.4) \quad \left| \int_{\gamma} f(z) dz \right| \leq \int_{\gamma} |f(z)| |dz| \leq \|f\|_{\gamma, \infty} \ell(\gamma),$$

where  $\|f\|_{\gamma, \infty} := \sup_{z \in \gamma} |f(z)|$ .

We also define the  $p$ -norm with  $p \geq 1$  by

$$\|f\|_{\gamma, p} := \left( \int_{\gamma} |f(z)|^p |dz| \right)^{1/p}.$$

For  $p = 1$ , we have

$$\|f\|_{\gamma, 1} := \int_{\gamma} |f(z)| |dz|.$$

If  $p, q > 1$  with  $\frac{1}{p} + \frac{1}{q} = 1$ , then by Hölder’s inequality, we have

$$\|f\|_{\gamma, 1} \leq [\ell(\gamma)]^{1/q} \|f\|_{\gamma, p}.$$

In the recent paper [5], we obtained the following result for functions of complex variable:

**Theorem 1.2.** *Let  $f$  be holomorphic in  $G$ , an open domain and suppose  $\gamma \subset G$  is a smooth path from  $z(a) = u$  to  $z(b) = w$ . If  $v = z(x)$  with  $x \in (a, b)$ , then  $\gamma_{u,w} = \gamma_{u,v} \cup \gamma_{v,w}$ ,*

$$(1.5) \quad \left| f(v)(w-u) - \int_{\gamma} f(z) dz \right| \\ \leq \|f'\|_{\gamma_{u,v};\infty} \int_{\gamma_{u,v}} |z-u| |dz| + \|f'\|_{\gamma_{v,w};\infty} \int_{\gamma_{v,w}} |z-w| |dz| \\ \leq \left[ \int_{\gamma_{u,v}} |z-u| |dz| + \int_{\gamma_{v,w}} |z-w| |dz| \right] \|f'\|_{\gamma_{u,w};\infty}$$

and

$$(1.6) \quad \left| f(v)(w-u) - \int_{\gamma} f(z) dz \right| \\ \leq \max_{z \in \gamma_{u,v}} |z-u| \|f'\|_{\gamma_{u,v};1} + \max_{z \in \gamma_{v,w}} |z-w| \|f'\|_{\gamma_{v,w};1} \\ \leq \max \left\{ \max_{z \in \gamma_{u,v}} |z-u|, \max_{z \in \gamma_{v,w}} |z-w| \right\} \|f'\|_{\gamma_{u,w};1}.$$

If  $p, q > 1$  with  $\frac{1}{p} + \frac{1}{q} = 1$ , then

$$(1.7) \quad \left| f(v)(w-u) - \int_{\gamma} f(z) dz \right| \\ \leq \left( \int_{\gamma_{u,v}} |z-u|^q |dz| \right)^{1/q} \|f'\|_{\gamma_{u,v};p} + \left( \int_{\gamma_{v,w}} |z-w|^q |dz| \right)^{1/q} \|f'\|_{\gamma_{v,w};p} \\ \leq \left( \int_{\gamma_{u,v}} |z-u|^q |dz| + \int_{\gamma_{v,w}} |z-w|^q |dz| \right)^{1/q} \|f'\|_{\gamma_{u,w};p}.$$

Motivated by the above results, in this paper, we extend the Ostrowski inequality to the complex integral, by providing upper bounds for the quantity

$$\left| f(v)\ell(\gamma) - \int_{\gamma} f(z) |dz| \right|$$

under the assumptions that  $\gamma$  is a smooth path parametrized by  $z(t)$ ,  $t \in [a, b]$ , with the length  $\ell(\gamma)$ ,  $u = z(a)$ ,  $v = z(x)$  with  $x \in (a, b)$  and  $w = z(b)$  while  $f$  is holomorphic in  $G$ , an open domain and  $\gamma \subset G$ . An application for circular paths is also given.

## 2. OSTROWSKI TYPE RESULTS

We have the following result for functions of complex variable:

**Theorem 2.3.** Let  $f$  be holomorphic in  $G$ , an open domain and suppose  $\gamma \subset G$  is a smooth path from  $z(a) = u$  to  $z(b) = w$ . If  $v = z(x)$  with  $x \in (a, b)$ , then  $\gamma_{u,w} = \gamma_{u,v} \cup \gamma_{v,w}$  and

$$(2.8) \quad \left| f(v) \ell(\gamma) - \int_{\gamma} f(z) |dz| \right| \leq \ell(\gamma_{u,v}) \|f'\|_{\gamma_{u,v};1} + \ell(\gamma_{v,w}) \|f'\|_{\gamma_{v,w};1}$$

$$\leq \begin{cases} \frac{1}{2} [\ell(\gamma_{u,w}) + |\ell(\gamma_{u,v}) - \ell(\gamma_{v,w})|] \|f'\|_{\gamma_{u,w};1}, \\ [\ell^p(\gamma_{u,v}) + \ell^p(\gamma_{v,w})]^{1/p} \left( \|f'\|_{\gamma_{u,v};1}^q + \|f'\|_{\gamma_{v,w};1}^q \right)^{1/q} \\ p, q > 1 \text{ and } \frac{1}{p} + \frac{1}{q} = 1, \\ \frac{1}{2} \ell(\gamma_{u,w}) \left[ \|f'\|_{\gamma_{u,w};1} + \left| \|f'\|_{\gamma_{u,v};1} - \|f'\|_{\gamma_{v,w};1} \right| \right]. \end{cases}$$

*Proof.* Using the integration by parts formula, we have

$$\begin{aligned} \int_{\gamma_{u,v}} f(z) |dz| &= \int_a^x f(z(t)) |z'(t)| dt \\ &= \int_a^x f(z(t)) d \left( \int_a^t |z'(s)| ds \right) \\ &= f(z(t)) \int_a^t |z'(s)| ds \Big|_a^x - \int_a^x \frac{df(z(t))}{dt} \left( \int_a^t |z'(s)| ds \right) dt \\ &= f(z(x)) \int_a^x |z'(s)| ds - \int_a^x f'(z(t)) \left( \int_a^t |z'(s)| ds \right) z'(t) dt \\ &= f(v) \ell(\gamma_{u,v}) - \int_a^x f'(z(t)) \left( \int_a^t |z'(s)| ds \right) z'(t) dt \end{aligned}$$

and

$$\begin{aligned} \int_{\gamma_{v,w}} f(z) |dz| &= \int_x^b f(z(t)) |z'(t)| dt \\ &= - \int_x^b f(z(t)) d \left( \int_t^b |z'(s)| ds \right) \\ &= - f(z(t)) \int_t^b |z'(s)| ds \Big|_x^b + \int_x^b \frac{df(z(t))}{dt} \left( \int_t^b |z'(s)| ds \right) dt \\ &= f(z(x)) \int_x^b |z'(s)| ds + \int_x^b f'(z(t)) \left( \int_t^b |z'(s)| ds \right) z'(t) dt \\ &= f(v) \ell(\gamma_{v,w}) + \int_x^b f'(z(t)) \left( \int_t^b |z'(s)| ds \right) z'(t) dt. \end{aligned}$$

If we add these two equalities, we get

$$\begin{aligned} & \int_{\gamma_{u,v}} f(z) |dz| + \int_{\gamma_{v,w}} f(z) |dz| \\ &= f(v) \ell(\gamma_{u,v}) + f(v) \ell(\gamma_{v,w}) - \int_a^x f'(z(t)) \left( \int_a^t |z'(s)| ds \right) z'(t) dt \\ &+ \int_x^b f'(z(t)) \left( \int_t^b |z'(s)| ds \right) z'(t) dt, \end{aligned}$$

which gives the following equality of interest

$$\begin{aligned} (2.9) \quad & f(v) \ell(\gamma_{u,w}) - \int_{\gamma_{u,w}} f(z) |dz| \\ &= \int_a^x f'(z(t)) \left( \int_a^t |z'(s)| ds \right) z'(t) dt - \int_x^b f'(z(t)) \left( \int_t^b |z'(s)| ds \right) z'(t) dt. \end{aligned}$$

By taking the modulus in (2.9) and using the properties of modulus, we get

$$\begin{aligned} (2.10) \quad & \left| f(v) \ell(\gamma_{u,w}) - \int_{\gamma_{u,w}} f(z) |dz| \right| \\ & \leq \left| \int_a^x f'(z(t)) \left( \int_a^t |z'(s)| ds \right) z'(t) dt \right| \\ & + \left| \int_x^b f'(z(t)) \left( \int_t^b |z'(s)| ds \right) z'(t) dt \right| \\ & \leq \int_a^x |f'(z(t))| |z'(t)| \left( \int_a^t |z'(s)| ds \right) dt \\ & + \int_x^b |f'(z(t))| |z'(t)| \left( \int_t^b |z'(s)| ds \right) dt =: B(x) \end{aligned}$$

for  $x \in [a, b]$ . We have

$$\int_a^t |z'(s)| ds \leq \int_a^x |z'(s)| ds \text{ for } t \in [a, x]$$

and

$$\int_t^b |z'(s)| ds \leq \int_x^b |z'(s)| ds \text{ for } t \in [x, b],$$

then

$$\begin{aligned} B(x) & \leq \int_a^x |z'(s)| ds \int_a^x |f'(z(t))| |z'(t)| dt \\ & + \int_x^b |z'(s)| ds \int_x^b |f'(z(t))| |z'(t)| dt \\ & = \ell(\gamma_{u,v}) \int_{\gamma_{u,v}} |f'(z)| |dz| + \ell(\gamma_{v,w}) \int_{\gamma_{v,w}} |f'(z)| |dz| \end{aligned}$$

and by (2.10), we get the first inequality in (2.8). The second part follows by Hölder’s inequalities

$$mn + cd \leq \begin{cases} \max \{m, c\} (n + d) \\ (m^p + c^p)^{1/p} (n^q + d^q)^{1/q}, \text{ for } p, q > 1, \frac{1}{p} + \frac{1}{q} = 1, \end{cases}$$

where  $m, n, c, d \geq 0$ . □

**Corollary 2.1.** *With the assumption of Theorem 2.3 and if there exists  $m \in \gamma$  such that  $\ell(\gamma_{u,m}) = \ell(\gamma_{m,w})$ , then*

$$(2.11) \quad \left| f(m) \ell(\gamma) - \int_{\gamma} f(z) |dz| \right| \leq \frac{1}{2} \ell(\gamma) \|f'\|_{\gamma_{u,w};1}$$

and if  $s \in \gamma$  such that  $\int_{\gamma_{u,s}} |f'(z)| |dz| = \int_{\gamma_{s,w}} |f'(z)| |dz|$ , then

$$(2.12) \quad \left| f(s) \ell(\gamma) - \int_{\gamma} f(z) |dz| \right| \leq \frac{1}{2} \ell(\gamma) \|f'\|_{\gamma_{u,w};1}.$$

We have also the following result for  $p$ -norms:

**Theorem 2.4.** *With the assumption of Theorem 2.3, we have for  $p, q > 1$  with  $\frac{1}{p} + \frac{1}{q} = 1$  that*

$$(2.13) \quad \begin{aligned} & \left| f(v) \ell(\gamma) - \int_{\gamma} f(z) |dz| \right| \\ & \leq \frac{1}{q+1} \left[ \ell^{1+1/q}(\gamma_{u,v}) \|f'\|_{\gamma_{u,v};p} + \ell^{1+1/q}(\gamma_{v,w}) \|f'\|_{\gamma_{v,w};p} \right] \\ & \leq \frac{1}{q+1} \begin{cases} \frac{1}{2^{1+1/q}} [\ell(\gamma_{u,w}) + |\ell(\gamma_{u,v}) - \ell(\gamma_{v,w})|]^{1+1/q} \\ \quad \times [\|f'\|_{\gamma_{u,v};p} + \|f'\|_{\gamma_{v,w};p}], \\ \left[ \ell^{q+1}(\gamma_{u,v}) + \ell^{q+1}(\gamma_{v,w}) \right]^{1/q} \|f'\|_{\gamma_{u,w};p}, \\ \frac{1}{2^{1/p}} \left[ \|f'\|_{\gamma_{u,w};p}^p + \left| \|f'\|_{\gamma_{u,v};p}^p - \|f'\|_{\gamma_{v,w};p}^p \right| \right]^{1/p} \\ \quad \times \left[ \ell^{1+1/q}(\gamma_{u,v}) + \ell^{1+1/q}(\gamma_{v,w}) \right]. \end{cases} \end{aligned}$$

*Proof.* Using the weighted Hölder integral inequality for  $p, q > 1$  with  $\frac{1}{p} + \frac{1}{q} = 1$ , we have

$$\begin{aligned} & \int_a^x |f'(z(t))| |z'(t)| \left( \int_a^t |z'(s)| ds \right) dt \\ & \leq \left( \int_a^x |f'(z(t))|^p |z'(t)| dt \right)^{1/p} \left( \int_a^x \left( \int_a^t |z'(s)| ds \right)^q |z'(t)| dt \right)^{1/q} \\ & = \left( \int_a^x |f'(z(t))|^p |z'(t)| dt \right)^{1/p} \left( \frac{\left( \int_a^x |z'(s)| ds \right)^{q+1}}{q+1} \right)^{1/q} \\ & = \frac{\left( \int_a^x |z'(s)| ds \right)^{1+1/q}}{(q+1)^{1/q}} \left( \int_a^x |f'(z(t))|^p |z'(t)| dt \right)^{1/p} \end{aligned}$$

and

$$\begin{aligned}
& \int_x^b |f'(z(t))| |z'(t)| \left( \int_t^b |z'(s)| ds \right) dt \\
& \leq \left( \int_x^b |f'(z(t))|^p |z'(t)| dt \right)^{1/p} \left( \int_x^b \left( \int_t^b |z'(s)| ds \right)^q |z'(t)| dt \right)^{1/q} \\
& = \left( \int_x^b |f'(z(t))|^p |z'(t)| dt \right)^{1/p} \left( \frac{\left( \int_x^b |z'(s)| ds \right)^{q+1}}{q+1} \right)^{1/q} \\
& = \frac{\left( \int_x^b |z'(s)| ds \right)^{1+1/q}}{(q+1)^{1/q}} \left( \int_x^b |f'(z(t))|^p |z'(t)| dt \right)^{1/p}
\end{aligned}$$

for  $x \in (a, b)$ . If we add these two inequalities, we get

$$\begin{aligned}
B(x) & \leq \frac{\left( \int_a^x |z'(s)| ds \right)^{1+1/q}}{(q+1)^{1/q}} \left( \int_a^x |f'(z(t))|^p |z'(t)| dt \right)^{1/p} \\
& + \frac{\left( \int_x^b |z'(s)| ds \right)^{1+1/q}}{(q+1)^{1/q}} \left( \int_x^b |f'(z(t))|^p |z'(t)| dt \right)^{1/p} \\
& = \frac{1}{q+1} \left[ \ell^{1+1/q}(\gamma_{u,v}) \left( \int_{\gamma_{u,v}} |f'(z)|^p |dz| \right)^{1/p} + \ell^{1+1/q}(\gamma_{v,w}) \left( \int_{\gamma_{v,w}} |f'(z)|^p |dz| \right)^{1/p} \right],
\end{aligned}$$

which proves the first inequality in (2.13). We also have

$$\begin{aligned}
& \ell^{1+1/q}(\gamma_{u,v}) \left( \int_{\gamma_{u,v}} |f'(z)|^p |dz| \right)^{1/p} + \ell^{1+1/q}(\gamma_{v,w}) \left( \int_{\gamma_{v,w}} |f'(z)|^p |dz| \right)^{1/p} \\
& \leq \max \left\{ \ell^{1+1/q}(\gamma_{u,v}), \ell^{1+1/q}(\gamma_{v,w}) \right\} \\
& \times \left[ \left( \int_{\gamma_{u,v}} |f'(z)|^p |dz| \right)^{1/p} + \left( \int_{\gamma_{v,w}} |f'(z)|^p |dz| \right)^{1/p} \right] \\
& = [\max \{ \ell(\gamma_{u,v}), \ell(\gamma_{v,w}) \}]^{1+1/q} \left[ \left( \int_{\gamma_{u,v}} |f'(z)|^p |dz| \right)^{1/p} + \left( \int_{\gamma_{v,w}} |f'(z)|^p |dz| \right)^{1/p} \right] \\
& = \frac{1}{2^{1+1/q}} [\ell(\gamma_{u,w}) + |\ell(\gamma_{u,v}) - \ell(\gamma_{v,w})|]^{1+1/q} \\
& \times \left[ \left( \int_{\gamma_{u,v}} |f'(z)|^p |dz| \right)^{1/p} + \left( \int_{\gamma_{v,w}} |f'(z)|^p |dz| \right)^{1/p} \right]
\end{aligned}$$



and

$$\begin{aligned}
& \ell^{1+1/q}(\gamma_{u,v}) \left( \int_{\gamma_{u,v}} |f'(z)|^p |dz| \right)^{1/p} + \ell^{1+1/q}(\gamma_{v,w}) \left( \int_{\gamma_{v,w}} |f'(z)|^p |dz| \right)^{1/p} \\
& \leq \max \left\{ \left( \int_{\gamma_{u,v}} |f'(z)|^p |dz| \right)^{1/p}, \left( \int_{\gamma_{v,w}} |f'(z)|^p |dz| \right)^{1/p} \right\} \\
& \times \left[ \ell^{1+1/q}(\gamma_{u,v}) + \ell^{1+1/q}(\gamma_{v,w}) \right] \\
& = \left[ \max \left\{ \int_{\gamma_{u,v}} |f'(z)|^p |dz|, \int_{\gamma_{v,w}} |f'(z)|^p |dz| \right\} \right]^{1/p} \\
& \times \left[ \ell^{1+1/q}(\gamma_{u,v}) + \ell^{1+1/q}(\gamma_{v,w}) \right] \\
& = \frac{1}{2^{1/p}} \left[ \int_{\gamma_{u,v}} |f'(z)|^p |dz| + \left| \int_{\gamma_{u,v}} |f'(z)|^p |dz| - \int_{\gamma_{v,w}} |f'(z)|^p |dz| \right| \right]^{1/p} \\
& \times \left[ \ell^{1+1/q}(\gamma_{u,v}) + \ell^{1+1/q}(\gamma_{v,w}) \right].
\end{aligned}$$

By Hölder's discrete inequality, we have

$$\begin{aligned}
& \ell^{1+1/q}(\gamma_{u,v}) \left( \int_{\gamma_{u,v}} |f'(z)|^p |dz| \right)^{1/p} + \ell^{1+1/q}(\gamma_{v,w}) \left( \int_{\gamma_{v,w}} |f'(z)|^p |dz| \right)^{1/p} \\
& \leq \left[ \ell^{q+1}(\gamma_{u,v}) + \ell^{q+1}(\gamma_{v,w}) \right]^{1/q} \left[ \int_{\gamma_{u,v}} |f'(z)|^p |dz| + \int_{\gamma_{v,w}} |f'(z)|^p |dz| \right]^{1/p} \\
& = \left[ \ell^{q+1}(\gamma_{u,v}) + \ell^{q+1}(\gamma_{v,w}) \right]^{1/q} \left( \int_{\gamma_{u,w}} |f'(z)|^p |dz| \right)^{1/p},
\end{aligned}$$

which prove the last part of (2.13). □

We have:

**Corollary 2.2.** *With the assumption of Theorem 2.4 and if there exists  $m \in \gamma$  such that  $\ell(\gamma_{u,m}) = \ell(\gamma_{m,w})$ , then*

$$(2.14) \quad \left| f(m) \ell(\gamma) - \int_{\gamma} f(z) |dz| \right| \leq \frac{1}{2^{1+1/q}(q+1)} \ell^{1+1/q}(\gamma_{u,w}) \left[ \|f'\|_{\gamma_{u,m};p} + \|f'\|_{\gamma_{m,w};p} \right]$$

and if  $s \in \gamma$  such that  $\int_{\gamma_{u,s}} |f'(z)|^p |dz| = \int_{\gamma_{s,w}} |f'(z)|^p |dz|$ , then

$$(2.15) \quad \left| f(s) \ell(\gamma) - \int_{\gamma} f(z) |dz| \right| \leq \frac{1}{2^{1/p}(q+1)} \left[ \ell^{1+1/q}(\gamma_{u,s}) + \ell^{1+1/q}(\gamma_{s,w}) \right] \|f'\|_{\gamma_{u,w};p}.$$

Finally we have:

**Theorem 2.5.** *With the assumption of Theorem 2.3, we have*

$$(2.16) \quad \left| f(v) \ell(\gamma) - \int_{\gamma} f(z) |dz| \right| \leq \frac{1}{2} \left[ \ell^2(\gamma_{u,v}) \|f'\|_{\gamma_{u,v};\infty} + \ell^2(\gamma_{v,w}) \|f'\|_{\gamma_{v,w};\infty} \right] \\ \leq \frac{1}{2} \begin{cases} \frac{1}{4} [\ell(\gamma_{u,w}) + |\ell(\gamma_{u,v}) - \ell(\gamma_{v,w})|]^2 \\ \quad \times [\|f'\|_{\gamma_{u,v};\infty} + \|f'\|_{\gamma_{v,w};\infty}], \\ \left[ \ell^{2q}(\gamma_{u,v}) + \ell^{2q}(\gamma_{v,w}) \right]^{1/q} \left( \|f'\|_{\gamma_{u,v};\infty}^p + \|f'\|_{\gamma_{v,w};\infty}^p \right)^{1/p}, \\ \quad p, q > 1 \text{ and } \frac{1}{p} + \frac{1}{q} = 1, \\ \left[ \ell^2(\gamma_{u,v}) + \ell^2(\gamma_{v,w}) \right] \|f'\|_{\gamma_{u,w};\infty}. \end{cases}$$

*Proof.* We have

$$\int_a^x |f'(z(t))| |z'(t)| \left( \int_a^t |z'(s)| ds \right) dt \\ \leq \max_{t \in [a,x]} |f'(z(t))| \int_a^x \left( \int_a^t |z'(s)| ds \right) |z'(t)| dt \\ = \frac{1}{2} \max_{t \in [a,x]} |f'(z(t))| \left( \int_a^x |z'(s)| ds \right)^2 = \frac{1}{2} \ell^2(\gamma_{u,v}) \|f'\|_{\gamma_{u,v};\infty}$$

and

$$\int_x^b |f'(z(t))| |z'(t)| \left( \int_t^b |z'(s)| ds \right) dt \\ \leq \max_{t \in [x,b]} |f'(z(t))| \int_x^b \left( \int_t^b |z'(s)| ds \right) |z'(t)| dt \\ = \frac{1}{2} \max_{t \in [x,b]} |f'(z(t))| \left( \int_x^b |z'(s)| ds \right)^2 = \frac{1}{2} \ell^2(\gamma_{v,w}) \|f'\|_{\gamma_{v,w};\infty},$$

which by addition produce

$$B(x) \leq \frac{1}{2} \left[ \ell^2(\gamma_{u,v}) \|f'\|_{\gamma_{u,v};\infty} + \ell^2(\gamma_{v,w}) \|f'\|_{\gamma_{v,w};\infty} \right]$$

and by utilising the inequality (2.10), we get the first part of (2.16).

The second part is obvious and we omit the details.  $\square$

**Corollary 2.3.** *With the assumption of Theorem 2.3 and if there exists  $m \in \gamma$  such that  $\ell(\gamma_{u,m}) = \ell(\gamma_{m,w})$ , then*

$$\left| f(m) \ell(\gamma) - \int_{\gamma} f(z) |dz| \right| \leq \frac{1}{8} \left[ \|f'\|_{\gamma_{u,m};\infty} + \|f'\|_{\gamma_{m,w};\infty} \right] \ell^2(\gamma_{u,w}) \\ \leq \frac{1}{4} \|f'\|_{\gamma_{u,w};\infty} \ell^2(\gamma_{u,w}).$$

## 3. EXAMPLES FOR CIRCULAR PATHS

Let  $[a, b] \subseteq [0, 2\pi]$  and the circular path  $\gamma_{[a,b],R}$  centered in 0 and with radius  $R > 0$

$$z(t) = R \exp(it) = R(\cos t + i \sin t), \quad t \in [a, b].$$

If  $[a, b] = [0, \pi]$ , then we get a half circle, while for  $[a, b] = [0, 2\pi]$ , we get the full circle. We have

$$z'(t) = Ri \exp(it), \quad t \in [a, b]$$

and  $|z'(t)| = R$  for  $t \in [a, b]$  giving that

$$\ell(\gamma_{[a,b],R}) = \int_a^b |z'(t)| dt = R(b-a).$$

If  $x \in [a, b]$  and  $v = R \exp(ix)$ , then by (2.8), we have

$$\begin{aligned} & \left| R(b-a)f(R \exp(ix)) - R \int_a^b f(R \exp(it)) dt \right| \\ & \leq R(x-a)R \int_a^x |f'(R \exp(it))| dt + R(b-x)R \int_x^b |f'(R \exp(it))| dt, \end{aligned}$$

that is equivalent to

$$(3.17) \quad \begin{aligned} & \left| (b-a)f(R \exp(ix)) - \int_a^b f(R \exp(it)) dt \right| \\ & \leq R(x-a) \int_a^x |f'(R \exp(it))| dt + R(b-x) \int_x^b |f'(R \exp(it))| dt \end{aligned}$$

for  $x \in [a, b]$ . In particular, if we take  $x = \frac{a+b}{2}$  in (3.17), then we get

$$(3.18) \quad \left| (b-a)f\left(R \exp\left(\frac{a+b}{2}i\right)\right) - \int_a^b f(R \exp(it)) dt \right| \leq \frac{1}{2}R(b-a) \int_a^b |f'(R \exp(it))| dt.$$

If  $m \in [a, b]$  is such that

$$\int_a^m |f'(R \exp(it))| dt = \int_m^b |f'(R \exp(it))| dt,$$

then from (3.17), we get

$$(3.19) \quad \begin{aligned} & \left| (b-a)f(R \exp(mi)) - \int_a^b f(R \exp(it)) dt \right| \\ & \leq \frac{1}{2}R(b-a) \int_a^b |f'(R \exp(it))| dt. \end{aligned}$$

By making use of (2.13), we get

$$\begin{aligned} & \left| R(b-a)f(R \exp(ix)) - R \int_a^b f(R \exp(it)) dt \right| \\ & \leq \frac{1}{q+1} \left[ R^{1+1/q}(x-a)^{1+1/q} R^{1/p} \left( \int_a^x |f'(R \exp(it))|^p dt \right)^{1/p} \right. \\ & \quad \left. + R^{1+1/q}(b-x)^{1+1/q} R^{1/p} \left( \int_x^b |f'(R \exp(it))|^p dt \right)^{1/p} \right], \end{aligned}$$

that is equivalent to

$$(3.20) \quad \left| (b-a) f(R \exp(ix)) - \int_a^b f(R \exp(it)) dt \right| \\ \leq \frac{1}{q+1} R \left[ (x-a)^{1+1/q} \left( \int_a^x |f'(R \exp(it))|^p dt \right)^{1/p} \right. \\ \left. + (b-x)^{1+1/q} \left( \int_x^b |f'(R \exp(it))|^p dt \right)^{1/p} \right]$$

for  $x \in [a, b]$ . If we take  $x = \frac{a+b}{2}$  in (3.20), then we get

$$(3.21) \quad \left| (b-a) f\left(R \exp\left(\frac{a+b}{2}i\right)\right) - \int_a^b f(R \exp(it)) dt \right| \\ \leq \frac{1}{(q+1)2^{1+1/q}} R (b-a)^{1+1/q} \left[ \left( \int_a^{\frac{a+b}{2}} |f'(R \exp(it))|^p dt \right)^{1/p} \right. \\ \left. + \left( \int_{\frac{a+b}{2}}^b |f'(R \exp(it))|^p dt \right)^{1/p} \right].$$

If  $c \in [a, b]$  is such that

$$\int_a^c |f'(R \exp(it))|^p dt = \int_c^b |f'(R \exp(it))|^p dt,$$

then by (3.20), we get

$$(3.22) \quad \left| (b-a) f(R \exp(ic)) - \int_a^b f(R \exp(it)) dt \right| \\ \leq \frac{1}{(q+1)2^{1/p}} R \left[ (c-a)^{1+1/q} + (b-c)^{1+1/q} \right] \left( \int_a^b |f'(R \exp(it))|^p dt \right)^{1/p}.$$

Further, if we use (2.16), then we have

$$\left| R(b-a) f(R \exp(ix)) - R \int_a^b f(R \exp(it)) dt \right| \\ \leq \frac{1}{2} R^2 \left[ (x-a)^2 \sup_{t \in [a, x]} |f'(R \exp(it))| + (b-x)^2 \sup_{t \in [x, b]} |f'(R \exp(it))| \right] \\ \leq \frac{1}{2} R^2 \left[ (x-a)^2 + (b-x)^2 \right] \sup_{t \in [a, b]} |f'(R \exp(it))|,$$

that is equivalent to

$$\begin{aligned}
 (3.23) \quad & \left| (b-a) f(R \exp(xi)) - \int_a^b f(R \exp(it)) dt \right| \\
 & \leq \frac{1}{2} R \left[ (x-a)^2 \sup_{t \in [a,x]} |f'(R \exp(it))| + (b-x)^2 \sup_{t \in [x,b]} |f'(R \exp(it))| \right] \\
 & \leq R \left[ \frac{1}{4} (b-a)^2 + \left( x - \frac{a+b}{2} \right)^2 \right] \sup_{t \in [a,b]} |f'(R \exp(it))|
 \end{aligned}$$

for  $x \in [a, b]$ . In particular, we have

$$\begin{aligned}
 (3.24) \quad & \left| (b-a) f\left(R \exp\left(\frac{a+b}{2}i\right)\right) - \int_a^b f(R \exp(it)) dt \right| \\
 & \leq \frac{1}{8} R (b-a)^2 \left[ \sup_{t \in [a, \frac{a+b}{2}]} |f'(R \exp(it))| + \sup_{t \in [\frac{a+b}{2}, b]} |f'(R \exp(it))| \right] \\
 & \leq \frac{1}{4} R (b-a)^2 \sup_{t \in [a,b]} |f'(R \exp(it))|.
 \end{aligned}$$

We give now examples for some fundamental complex functions. Consider the function  $f(z) = z^n$ ,  $z \in \mathbb{C}$  with  $n \geq 1$ . Then  $f'(z) = nz^{n-1}$ ,

$$f(R \exp(ix)) = R^n \exp(nxi),$$

$$|f'(R \exp(it))| = nR^{n-1} |\exp((n-1)it)| = nR^{n-1}, \quad t \in [a, b]$$

and

$$\int_a^b f(R \exp(it)) dt = R^n \int_a^b \exp(nit) dt = R^n \frac{\exp(nbi) - \exp(nai)}{ni}.$$

Making use of the inequality (3.23), we get

$$\left| (b-a) R^n \exp(nxi) - R^n \frac{\exp(nbi) - \exp(nai)}{ni} \right| \leq R \left[ \frac{1}{4} (b-a)^2 + \left( x - \frac{a+b}{2} \right)^2 \right] nR^{n-1},$$

which is equivalent to

$$(3.25) \quad \left| (b-a) \exp(nxi) - \frac{\exp(nbi) - \exp(nai)}{ni} \right| \leq n \left[ \frac{1}{2} (b-a)^2 + \left( x - \frac{a+b}{2} \right)^2 \right]$$

for  $x \in [a, b]$ . If we take in (3.25)  $x = \frac{a+b}{2}$ , then we get

$$(3.26) \quad \left| (b-a) \exp\left(n \left(\frac{a+b}{2}\right) i\right) - \frac{\exp(nbi) - \exp(nai)}{ni} \right| \leq \frac{1}{4} n (b-a)^2$$

for  $n \geq 1$ . Consider the exponential function  $f(z) = \exp(z)$ ,  $z \in \mathbb{C}$ . Then  $f'(z) = \exp(z)$ ,

$$|f'(R \exp(it))| = |\exp(R(\cos t + i \sin t))| = \exp(R \cos t), \quad t \in [a, b]$$

and by the inequality (3.17), we get

$$(3.27) \quad \left| (b-a) \exp(R \exp(ix)) - \int_a^b \exp(R \exp(it)) dt \right| \\ \leq R \left[ (x-a) \int_a^x \exp(R \cos t) dt + (b-x) \int_x^b \exp(R \cos t) dt \right],$$

while from the inequality (3.23), we get

$$(3.28) \quad \left| (b-a) \exp(R \exp(ix)) - \int_a^b \exp(R \exp(it)) dt \right| \\ \leq \frac{1}{2} R \left[ (x-a)^2 \sup_{t \in [a,x]} \exp(R \cos t) + (b-x)^2 \sup_{t \in [x,b]} \exp(R \cos t) \right] \\ \leq R \left[ \frac{1}{4} (b-a)^2 + \left( x - \frac{a+b}{2} \right)^2 \right] \sup_{t \in [a,b]} \exp(R \cos t)$$

for  $x \in [a, b]$ . From the inequality (3.20), we get

$$(3.29) \quad \left| (b-a) \exp(R \exp(ix)) - \int_a^b \exp(R \exp(it)) dt \right| \\ \leq \frac{1}{q+1} R \left[ (x-a)^{1+1/q} \left( \int_a^x \exp(pR \cos t) dt \right)^{1/p} \right. \\ \left. + (b-x)^{1+1/q} \left( \int_x^b \exp(pR \cos t) dt \right)^{1/p} \right]$$

for  $x \in [a, b]$ , where  $p, q > 1$  with  $\frac{1}{p} + \frac{1}{q} = 1$ . If in the inequality (3.27) we take  $x = \frac{a+b}{2}$ , then we get

$$(3.30) \quad \left| (b-a) \exp \left( R \exp \left( \frac{a+b}{2} i \right) \right) - \int_a^b \exp(R \exp(it)) dt \right| \\ \leq \frac{1}{2} R (b-a) \int_a^b \exp(R \cos t) dt,$$

while from the inequality (2.15), we get

$$(3.31) \quad \left| (b-a) \exp \left( R \exp \left( \frac{a+b}{2} i \right) \right) - \int_a^b \exp(R \exp(it)) dt \right| \\ \leq \frac{1}{8} (b-a)^2 R \left[ \sup_{t \in [a, \frac{a+b}{2}]} \exp(R \cos t) + \sup_{t \in [\frac{a+b}{2}, b]} \exp(R \cos t) \right] \\ \leq \frac{1}{4} R (b-a)^2 \sup_{t \in [a,b]} \exp(R \cos t).$$

From (3.29), we have

$$(3.32) \quad \left| (b-a) \exp \left( R \exp \left( \frac{a+b}{2} i \right) \right) - \int_a^b \exp (R \exp (it)) dt \right| \\ \leq \frac{1}{(q+1) 2^{1+1/q}} R (b-a)^{1+1/q} \\ \times \left[ \left( \int_a^{\frac{a+b}{2}} \exp (pR \cos t) dt \right)^{1/p} + \left( \int_{\frac{a+b}{2}}^b \exp (pR \cos t) dt \right)^{1/p} \right].$$

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