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OPERATOR SUBADDITIVITY OF THE \mathcal{D} -LOGARITHMIC INTEGRAL TRANSFORM FOR POSITIVE OPERATORS IN HILBERT SPACES

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Abstract. For a continuous and positive function $w(\lambda)$, $\lambda > 0$ and μ a positive measure on $[0, \infty)$ we consider the following \mathcal{D} -logarithmic integral transform

$$\mathcal{D}\text{Log}(w, \mu)(T) := \int_0^\infty w(\lambda) \ln\left(\frac{\lambda + T}{\lambda}\right) d\mu(\lambda),$$

where the integral is assumed to exist for T a positive operator on a complex Hilbert space H .

We show among others that, if $A, B > 0$ with $BA + AB \geq 0$, then

$$\mathcal{D}\text{Log}(w, \mu)(A) + \mathcal{D}\text{Log}(w, \mu)(B) \geq \mathcal{D}\text{Log}(w, \mu)(A + B).$$

In particular we have

$$\frac{1}{6}\pi^2 + \text{dilog}(A + B) \geq \text{dilog}(A) + \text{dilog}(B),$$

where the *dilogarithmic function* $\text{dilog} : [0, \infty) \rightarrow \mathbb{R}$ is defined by

$$\text{dilog}(t) := \int_1^t \frac{\ln s}{1-s} ds, \quad t \geq 0.$$

Some examples for integral transform $\mathcal{D}\text{Log}(\cdot, \cdot)$ related to the operator monotone functions are also provided.

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1. Introduction

Consider a complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$. An operator T is said to be positive (denoted by $T \geq 0$) if $\langle Tx, x \rangle \geq 0$ for all $x \in H$ and also an operator T is said to be *strictly positive* (denoted by $T > 0$) if T is positive and invertible. A real valued continuous function f on $(0, \infty)$ is said to be operator monotone if $f(A) \geq f(B)$ holds for any $A \geq B > 0$.

We have the following representation of operator monotone functions ([7], [6]), see for instance [1, p. 144–145]:

THEOREM 1. *A function $f: [0, \infty) \rightarrow \mathbb{R}$ is operator monotone in $[0, \infty)$ if and only if it has the representation*

$$(1.1) \quad f(t) = f(0) + bt + \int_0^\infty \frac{t\lambda}{t + \lambda} d\mu(\lambda),$$

where $b \geq 0$ and a positive measure μ on $(0, \infty)$ such that

$$\int_0^\infty \frac{\lambda}{1 + \lambda} d\mu(\lambda) < \infty.$$

For some examples of operator monotone functions see [3]-[5], [8]-[9] and the references therein.

We have the following integral representation for the power function when $s > 0$, $r \in (0, 1]$, see for instance [1, p. 145]:

$$(1.2) \quad s^{r-1} = \frac{\sin(r\pi)}{\pi} \int_0^\infty \frac{\lambda^{r-1}}{\lambda + s} d\lambda.$$

Observe that for $s > 0$, $s \neq 1$, we have

$$\int_0^u \frac{d\lambda}{(\lambda + s)(\lambda + 1)} = \frac{\ln s}{s - 1} + \frac{1}{1 - s} \ln \left(\frac{u + s}{u + 1} \right)$$

for all $u > 0$. By taking the limit over $u \rightarrow \infty$ in this equality, we derive

$$(1.3) \quad \frac{\ln s}{s - 1} = \int_0^\infty \frac{d\lambda}{(\lambda + s)(\lambda + 1)},$$

which gives the representation for the logarithm

$$(1.4) \quad \ln s = (s - 1) \int_0^\infty \frac{d\lambda}{(\lambda + 1)(\lambda + s)}.$$

If we integrate (1.2) over s from 0 to $t > 0$, we get by Fubini's theorem

$$\begin{aligned} \frac{t^r}{r} &= \frac{\sin(r\pi)}{\pi} \int_0^\infty \left(\int_0^t \left(\frac{1}{\lambda+s} \right) ds \right) \lambda^{r-1} d\lambda \\ &= \frac{\sin(r\pi)}{\pi} \int_0^\infty \lambda^{r-1} \ln \left(\frac{t+\lambda}{\lambda} \right) d\lambda \end{aligned}$$

giving the identity of interest

$$t^r = \frac{r \sin(r\pi)}{\pi} \int_0^\infty \lambda^{r-1} \ln \left(\frac{t+\lambda}{\lambda} \right) d\lambda, \quad t > 0 \text{ and } r \in (0, 1].$$

Recall the *dilogarithmic function* $\text{dilog}: [0, \infty) \rightarrow \mathbb{R}$ defined by

$$\text{dilog}(t) := \int_1^t \frac{\ln s}{1-s} ds, \quad t \geq 0.$$

Some particular values of interest are

$$\text{dilog}(1) = 0, \quad \text{dilog}(0) = \int_1^0 \frac{\ln s}{1-s} ds = \int_0^1 \frac{\ln s}{s-1} ds = \frac{1}{6}\pi^2,$$

and

$$\text{dilog}\left(\frac{1}{2}\right) = \frac{1}{12}\pi^2 - \frac{1}{2}(\ln 2)^2.$$

If we integrate the identity (1.3) over s from 0 to $t > 0$, we get by Fubini's theorem

$$\int_0^t \frac{\ln s}{s-1} ds = \int_0^\infty \left(\int_0^t \frac{1}{\lambda+s} ds \right) \frac{1}{(\lambda+1)} d\lambda = \int_0^\infty \frac{1}{(\lambda+1)} \ln \left(\frac{t+\lambda}{\lambda} \right) d\lambda$$

and since

$$\begin{aligned} \int_0^t \frac{\ln s}{s-1} ds &= \int_0^1 \frac{\ln s}{s-1} ds + \int_1^t \frac{\ln s}{s-1} ds = \frac{1}{6}\pi^2 - \int_1^t \frac{\ln s}{1-s} ds \\ &= \frac{1}{6}\pi^2 - \text{dilog}(t) \end{aligned}$$

then we get the identity of interest

$$\frac{1}{6}\pi^2 - \text{dilog}(t) = \int_0^\infty \frac{1}{(\lambda+1)} \ln \left(\frac{t+\lambda}{\lambda} \right) d\lambda, \quad t > 0.$$

Motivated by the above representations, we define the \mathcal{D} -logarithmic transform for a continuous and positive function $w(\lambda)$, $\lambda > 0$ by

$$(1.5) \quad \mathcal{D}\mathcal{L}og(w, \mu)(t) := \int_0^\infty w(\lambda) \ln\left(\frac{\lambda+t}{\lambda}\right) d\mu(\lambda),$$

where μ is a positive measure on $(0, \infty)$ and the integral (1.5) exists for all $t > 0$. Also, when μ is the usual Lebesgue measure, then

$$\mathcal{D}\mathcal{L}og(w)(t) := \int_0^\infty w(\lambda) \ln\left(\frac{\lambda+t}{\lambda}\right) d\lambda.$$

Obviously,

$$\begin{aligned} \mathcal{D}\mathcal{L}og(w, \mu)(t) &= \int_0^\infty w(\lambda) \ln\left(1 + \frac{t}{\lambda}\right) d\mu(\lambda) \\ &= \int_0^\infty w(\lambda) [\ln(\lambda+t) - \ln(\lambda)] d\mu(\lambda) \end{aligned}$$

and one can use either of these representations when is needed.

By utilising the continuous functional calculus for selfadjoint operators, we can define the operator \mathcal{D} -logarithmic transform by

$$\mathcal{D}\mathcal{L}og(w, \mu)(T) = \int_0^\infty w(\lambda) \ln\left(1 + \frac{1}{\lambda}T\right) d\mu(\lambda)$$

for $T > 0$.

If we use the \mathcal{D} -logarithmic transform for the kernel $w_{\ell^{r-1}}(\lambda) := \frac{r \sin(r\pi)}{\pi} \lambda^{r-1}$, $r \in (0, 1]$ we have

$$\mathcal{D}\mathcal{L}og(w_{\ell^{r-1}})(T) = T^r, \quad T \geq 0$$

while for the kernel $w_{(\ell+1)^{-1}}(\lambda) := \frac{1}{\lambda+1}$ we have

$$\mathcal{D}\mathcal{L}og\left(w_{(\ell+1)^{-1}}\right)(T) = \frac{1}{6}\pi^2 - \text{dilog}(T), \quad T \geq 0.$$

In the recent paper [2] we introduced the following *integral transform*

$$(1.6) \quad \mathcal{D}(w, \mu)(s) := \int_0^\infty \frac{w(\lambda)}{\lambda+s} d\mu(\lambda), \quad s > 0,$$

for a continuous and positive function $w(\lambda)$, $\lambda > 0$, where μ is a positive measure on $(0, \infty)$ and the integral (1.6) exists for all $s > 0$.

For μ the Lebesgue usual measure, we put

$$\mathcal{D}(w)(s) := \int_0^\infty \frac{w(\lambda)}{\lambda + s} d\lambda, \quad s > 0.$$

Several examples of integral transforms $\mathcal{D}(w, \mu)$ have also been given in [2].

If we integrate the identity (1.3) over s from 0 to $t > 0$, we get by Fubini's theorem

$$\begin{aligned} \int_0^t \mathcal{D}(w, \mu)(s) ds &:= \int_0^\infty \left(\int_0^t \frac{1}{\lambda + s} ds \right) w(\lambda) d\mu(\lambda) \\ &= \int_0^\infty w(\lambda) \ln \left(\frac{\lambda + t}{\lambda} \right) d\mu(\lambda) \end{aligned}$$

for $t > 0$, which provides the equality of interest

$$\mathcal{D}\mathcal{L}og(w, \mu)(t) = \int_0^t \mathcal{D}(w, \mu)(s) ds, \quad t > 0,$$

provided that the integral on the right side exists for all $t > 0$.

2. Main results

We have the following identity of interest:

LEMMA 1. For all $A, B > 0$ and $\lambda > 0$ we have

$$\begin{aligned} (2.1) \quad &\ln \left(\frac{A + \lambda}{\lambda} \right) + \ln \left(\frac{B + \lambda}{\lambda} \right) - \ln \left(\frac{A + B + \lambda}{\lambda} \right) \\ &= \ln(A + \lambda) + \ln(B + \lambda) - \ln(A + B + \lambda) - \ln \lambda \\ &= \int_0^\infty \frac{1}{s + \lambda} (A + B + s + \lambda)^{-1} \\ &\quad \times [A(B + s + \lambda)^{-1}BA + B(A + s + \lambda)^{-1}AB] (A + B + s + \lambda)^{-1} ds \\ &\quad + \int_0^\infty \frac{1}{s + \lambda} (A + B + s + \lambda)^{-1} (BA + AB) (A + B + s + \lambda)^{-1} ds. \end{aligned}$$

PROOF. We have, by (1.4), that

$$\begin{aligned} \ln(T + \lambda) &= \int_0^\infty \frac{1}{(s+1)} (T + \lambda - 1)(s + T + \lambda)^{-1} ds \\ &= \int_0^\infty \frac{1}{(s+1)} (T + \lambda + s - 1 - s)(s + T + \lambda)^{-1} ds \\ &= \int_0^\infty \frac{1}{(s+1)} \left[1 - (1+s)(s + T + \lambda)^{-1} \right] ds \\ &= \int_0^\infty \left[\frac{1}{s+1} - (s + T + \lambda)^{-1} \right] ds. \end{aligned}$$

For $A, B > 0$ and $u \geq 0$, define

$$K_u := (A + u)^{-1} + (B + u)^{-1} - (A + B + u)^{-1}$$

and $W_u := 1 - uK_u$.

Therefore

$$\begin{aligned} (2.2) \quad & \ln(A + \lambda) + \ln(B + \lambda) - \ln(A + B + \lambda) - \ln \lambda \\ &= \int_0^\infty \left[\frac{1}{(s+1)} - (s + A + \lambda)^{-1} \right] ds \\ & \quad + \int_0^\infty \left[\frac{1}{(s+1)} - (s + B + \lambda)^{-1} \right] ds \\ & \quad - \int_0^\infty \left[\frac{1}{(s+1)} - (s + A + B + \lambda)^{-1} \right] ds \\ & \quad - \int_0^\infty \left[\frac{1}{(s+1)} - (s + \lambda)^{-1} \right] ds \\ &= \int_0^\infty \left[(s + A + B + \lambda)^{-1} + (s + \lambda)^{-1} \right. \\ & \quad \left. - (s + A + \lambda)^{-1} - (s + B + \lambda)^{-1} \right] ds \\ &= \int_0^\infty \left[(s + \lambda)^{-1} - K_{s+\lambda} \right] ds = \int_0^\infty \left(\frac{1}{s + \lambda} - K_{s+\lambda} \right) ds \\ &= \int_0^\infty \frac{1}{s + \lambda} [1 - (s + \lambda) K_{s+\lambda}] ds = \int_0^\infty \frac{1}{s + \lambda} W_{s+\lambda} ds. \end{aligned}$$

We have successively

$$\begin{aligned}
 & (A + B + \lambda) W_\lambda (A + B + \lambda) \\
 &= (A + B + \lambda) (1 - \lambda K_\lambda) (A + B + \lambda) \\
 &= (A + B + \lambda)^2 - \lambda (A + B + \lambda) K_\lambda (A + B + \lambda) \\
 &= (A + B + \lambda) (A + B + \lambda) \\
 &\quad - \lambda \left[B (A + \lambda)^{-1} B + A (B + \lambda)^{-1} A + 2(A + B) + \lambda \right] \\
 &= A^2 + BA + \lambda A + AB + B^2 + \lambda B + \lambda A + \lambda B + \lambda^2 \\
 &\quad - \lambda B (A + \lambda)^{-1} B - \lambda A (B + \lambda)^{-1} A - 2\lambda (A + B) - \lambda^2 \\
 &= A^2 + B^2 + BA + AB - \lambda B (A + \lambda)^{-1} B - \lambda A (B + \lambda)^{-1} A \\
 &= A (B + \lambda)^{-1} (B + \lambda) A - \lambda A (B + \lambda)^{-1} A \\
 &\quad + B (A + \lambda)^{-1} (A + \lambda) B - \lambda B (A + \lambda)^{-1} B + BA + AB \\
 &= A (B + \lambda)^{-1} BA + B (A + \lambda)^{-1} AB + BA + AB,
 \end{aligned}$$

therefore

$$\begin{aligned}
 (2.3) \quad W_\lambda &= (A + B + \lambda)^{-1} \left[A (B + \lambda)^{-1} BA \right. \\
 &\quad \left. + B (A + \lambda)^{-1} AB + BA + AB \right] (A + B + \lambda)^{-1}.
 \end{aligned}$$

From (2.3) we obtain

$$\begin{aligned}
 (2.4) \quad W_{s+\lambda} &= (A + B + s + \lambda)^{-1} \left[A (B + s + \lambda)^{-1} BA \right. \\
 &\quad \left. + B (A + s + \lambda)^{-1} AB + BA + AB \right] (A + B + s + \lambda)^{-1} \\
 &= (A + B + s + \lambda)^{-1} \left[A (B + s + \lambda)^{-1} BA \right. \\
 &\quad \left. + B (A + s + \lambda)^{-1} AB \right] (A + B + s + \lambda)^{-1} \\
 &\quad + (A + B + s + \lambda)^{-1} (BA + AB) (A + B + s + \lambda)^{-1}.
 \end{aligned}$$

On making use of (2.2) and (2.4) we obtain the desired result (2.1). \square

THEOREM 2. For all $A, B > 0$ we have

$$\begin{aligned}
 (2.5) \quad & \mathcal{D}\mathcal{L}og(w, \mu)(A) + \mathcal{D}\mathcal{L}og(w, \mu)(B) - \mathcal{D}\mathcal{L}og(w, \mu)(A + B) \\
 &= \int_0^\infty w(\lambda) \left(\int_0^\infty \frac{1}{s + \lambda} (A + B + s + \lambda)^{-1} \right. \\
 &\quad \times \left. \left[A(B + s + \lambda)^{-1} BA + B(A + s + \lambda)^{-1} AB \right] \right. \\
 &\quad \times (A + B + s + \lambda)^{-1} ds \Big) d\mu(\lambda) \\
 &+ \int_0^\infty w(\lambda) \left(\int_0^\infty \frac{1}{s + \lambda} (A + B + s + \lambda)^{-1} \right. \\
 &\quad \times (BA + AB) (A + B + s + \lambda)^{-1} ds \Big) d\mu(\lambda) \\
 &\geq \int_0^\infty w(\lambda) \left(\int_0^\infty \frac{1}{s + \lambda} (A + B + s + \lambda)^{-1} \right. \\
 &\quad \times (BA + AB) (A + B + s + \lambda)^{-1} ds \Big) d\mu(\lambda).
 \end{aligned}$$

If $BA + AB \geq 0$, then

$$(2.6) \quad \mathcal{D}\mathcal{L}og(w, \mu)(A) + \mathcal{D}\mathcal{L}og(w, \mu)(B) \geq \mathcal{D}\mathcal{L}og(w, \mu)(A + B).$$

PROOF. The identity (2.5) follows by multiplying the equality (2.1) with $w(\lambda)$ and integrating on $[0, \infty)$ over the measure $d\mu(\lambda)$.

Let $s, \lambda \geq 0$. Since $(B + s + \lambda)^{-1} B > 0$ and $(A + s + \lambda)^{-1} A > 0$ hence $A(B + s + \lambda)^{-1} BA > 0$ and $B(A + s + \lambda)^{-1} AB > 0$. Therefore

$$A(B + s + \lambda)^{-1} BA + B(A + s + \lambda)^{-1} AB > 0$$

and by multiplying both sides by $(A + B + s + \lambda)^{-1}$ we get

$$\begin{aligned}
 & (A + B + s + \lambda)^{-1} \left[A(B + s + \lambda)^{-1} BA + B(A + s + \lambda)^{-1} AB \right] \\
 & \times (A + B + s + \lambda)^{-1} > 0.
 \end{aligned}$$

By multiplying with $\frac{1}{s+\lambda}$ and $w(\lambda)$ and integrating twice, we obtain

$$\begin{aligned} & \int_0^\infty w(\lambda) \left(\int_0^\infty \frac{1}{s+\lambda} (A+B+s+\lambda)^{-1} \right. \\ & \quad \times \left[A(B+s+\lambda)^{-1} BA + B(A+s+\lambda)^{-1} AB \right] \\ & \quad \left. \times (A+B+s+\lambda)^{-1} ds \right) d\mu(\lambda) \geq 0, \end{aligned}$$

which proves the inequality in (2.5).

If $BA + AB \geq 0$, then by multiplying both sides by $(A+B+s+\lambda)^{-1}$ we get

$$(A+B+s+\lambda)^{-1} (BA + AB) (A+B+s+\lambda)^{-1} \geq 0$$

for $s, \lambda \geq 0$ and by integration twice, we derive

$$\begin{aligned} & \int_0^\infty w(\lambda) \left(\int_0^\infty \frac{1}{s+\lambda} (A+B+s+\lambda)^{-1} \right. \\ & \quad \left. \times (BA + AB) (A+B+s+\lambda)^{-1} ds \right) d\mu(\lambda) \geq 0 \end{aligned}$$

and the subadditivity property (2.6) is proved. \square

REMARK 1. If we write the inequality (2.6) for the transform $\mathcal{DLog}(w_{\ell^{r-1}})$ we get

$$(2.7) \quad A^r + B^r \geq (A+B)^r, \quad r \in (0, 1]$$

provided $A, B > 0$ with $BA + AB \geq 0$.

If we write the inequality (2.6) for the transform $\mathcal{DLog}(w_{(\ell+1)^{-1}})$ we get

$$\frac{1}{6}\pi^2 + \text{dilog}(A+B) \geq \text{dilog}(A) + \text{dilog}(B)$$

provided $A, B > 0$ with $BA + AB \geq 0$.

We define the function

$$(2.8) \quad G_{w,\mu}(t) := \int_0^\infty w(\lambda) \left(\int_0^\infty \frac{ds}{(s+\lambda)(s+t+\lambda)^2} \right) d\mu(\lambda), \quad t > 0.$$

Observe that for $a, b > 0$ we have

$$\int_0^\infty \frac{ds}{(s+a)(s+b)^2} = \frac{\ln b - \ln a}{(b-a)^2} - \frac{1}{b(b-a)}.$$

This gives that

$$\int_0^\infty \frac{ds}{(s+\lambda)(s+t+\lambda)^2} = \frac{\ln(t+\lambda) - \ln \lambda}{t^2} - \frac{1}{t(t+\lambda)}, \quad t > 0.$$

Therefore

$$\begin{aligned} (2.9) \quad G_{w,\mu}(t) &= \int_0^\infty w(\lambda) \left(\int_0^\infty \frac{1}{(s+\lambda)(s+t+\lambda)^2} ds \right) d\mu(\lambda) \\ &= \int_0^\infty w(\lambda) \left(\frac{\ln(t+\lambda) - \ln \lambda}{t^2} - \frac{1}{t(t+\lambda)} \right) d\mu(\lambda) \\ &= \frac{1}{t^2} \int_0^\infty w(\lambda) \ln \left(\frac{t+\lambda}{\lambda} \right) d\mu(\lambda) - \frac{1}{t} \int_0^\infty \frac{w(\lambda)}{t+\lambda} d\mu(\lambda) \\ &= \frac{1}{t^2} \mathcal{D}\mathcal{L}\log(w, \mu)(t) - \frac{1}{t} \mathcal{D}(w)(t) \geq 0, \end{aligned}$$

for all $t > 0$.

COROLLARY 1. *If $A, B > 0$ with $BA + AB \geq k$, where k is a real number, then*

$$(2.10) \quad \begin{aligned} \mathcal{D}\mathcal{L}\log(w, \mu)(A) + \mathcal{D}\mathcal{L}\log(w, \mu)(B) - \mathcal{D}\mathcal{L}\log(w, \mu)(A+B) \\ \geq k(\mathcal{D}\mathcal{L}\log(w, \mu)(A+B) - (A+B)\mathcal{D}(w)(t))(A+B)^{-2}. \end{aligned}$$

PROOF. If $BA + AB \geq k$, then by multiplying both sides by $(A+B+s+\lambda)^{-1}$, we get

$$\begin{aligned} (A+B+s+\lambda)^{-1}(BA+AB)(A+B+s+\lambda)^{-1} \\ \geq k(A+B+s+\lambda)^{-2}, \end{aligned}$$

for $s, \lambda \geq 0$, which by integration gives that

$$\begin{aligned} & \int_0^\infty w(\lambda) \left(\int_0^\infty \frac{1}{s+\lambda} (A+B+s+\lambda)^{-1} \right. \\ & \quad \left. \times (BA+AB) (A+B+s+\lambda)^{-1} ds \right) d\mu(\lambda) \\ & \geq k \int_0^\infty w(\lambda) \left(\int_0^\infty \frac{1}{s+\lambda} (A+B+s+\lambda)^{-2} \right) d\mu(\lambda). \end{aligned}$$

Observe that, by continuous functional calculus and by (2.8) and (2.9), we get

$$\begin{aligned} & \int_0^\infty w(\lambda) \left(\int_0^\infty \frac{1}{s+\lambda} (A+B+s+\lambda)^{-2} \right) d\mu(\lambda) \\ & = G_{w,\mu}(A+B) \\ & = (\mathcal{D}\mathcal{L}og(w, \mu)(A+B) - (A+B)\mathcal{D}(w)(t))(A+B)^{-2} \end{aligned}$$

and the inequality (2.10) is proved. \square

REMARK 2. If $A, B > 0$ with $BA+AB \geq k \geq 0$, then we have the following refinement of (2.6)

$$\begin{aligned} & \mathcal{D}\mathcal{L}og(w, \mu)(A) + \mathcal{D}\mathcal{L}og(w, \mu)(B) - \mathcal{D}\mathcal{L}og(w, \mu)(A+B) \\ & \geq k(\mathcal{D}\mathcal{L}og(w, \mu)(A+B) - (A+B)\mathcal{D}(w)(t))(A+B)^{-2} \geq 0. \end{aligned}$$

If we write the inequality (2.10) for the transform $\mathcal{D}\mathcal{L}og(w_{\ell^{r-1}})$ we get for $r \in (0, 1]$ that

$$A^r + B^r - (A+B)^r \geq (1-r)k(A+B)^{r-2},$$

provided $A, B > 0$ with $BA+AB \geq k$. If $k \geq 0$, then we obtain the following refinement of (2.7)

$$A^r + B^r - (A+B)^r \geq (1-r)k(A+B)^{r-2} \geq 0.$$

3. Some examples via operator monotone functions

We have the following class of examples that are of interest:

LEMMA 2. Assume that function $f: [0, \infty) \rightarrow \mathbb{R}$ is operator monotone in $[0, \infty)$ and has the representation (1.1), where $b \geq 0$ and μ is a positive measure on $[0, \infty)$. Then

$$\mathcal{D}\text{Log}(\ell, \mu)(t) = F_f(t) - bt$$

provided the function

$$F_f(t) := \int_0^t \frac{f(s) - f(0)}{s} ds$$

is defined for all $t \in (0, \infty)$.

PROOF. From (1.1) we have

$$\frac{f(s) - f(0)}{s} - b = \int_0^\infty \frac{\lambda}{s + \lambda} d\mu(\lambda) = \mathcal{D}(\ell, \mu)(s),$$

where $\ell(\lambda) = \lambda$, $\lambda \geq 0$.

By taking the integral over s on $(0, t)$, we have

$$F_f(t) = \int_0^t \frac{f(s) - f(0)}{s} ds - bt = \int_0^t \mathcal{D}(\ell, \mu)(s) ds = \mathcal{D}\text{Log}(\ell, \mu)(t)$$

for $t > 0$, and the proposition is proved. \square

REMARK 3. If we take $f(t) = \ln(t + a)$, for $a, t > 0$, then we have

$$F_{\ln(t+a)}(t) := \int_0^t \frac{\ln(s+a) - \ln(a)}{s} ds = \int_0^t \frac{1}{s} \ln\left(\frac{s}{a} + 1\right) ds.$$

If we change the variable $u = \frac{s}{a}$, then we get

$$\begin{aligned} \int_0^t \frac{1}{s} \ln\left(\frac{s}{a} + 1\right) ds &= \int_0^{t/a} \frac{1}{ua} \ln(u+1) a du = \int_0^{t/a} \frac{1}{u} \ln(u+1) du \\ &= -\text{dilog}\left(\frac{t}{a} + 1\right), \end{aligned}$$

which gives

$$F_{\ln(t+a)}(t) = -\operatorname{dilog}\left(\frac{t}{a} + 1\right), \quad t > 0.$$

If $f(t) = t^r$, $r \in (0, 1]$, then $F_f(t) := \frac{t^r}{r}$, $t > 0$.

PROPOSITION 1. Assume that function $f: [0, \infty) \rightarrow \mathbb{R}$ is operator monotone in $[0, \infty)$. If $A, B > 0$ with $BA + AB \geq 0$, then

$$(3.1) \quad F_f(A) + F_f(B) \geq F_f(A + B).$$

PROOF. By Lemma 2 we have for all $A, B > 0$ that

$$\begin{aligned} & \mathcal{D}\mathcal{L}og(\ell, \mu)(A) + \mathcal{D}\mathcal{L}og(\ell, \mu)(B) - \mathcal{D}\mathcal{L}og(\ell, \mu)(A + B) \\ &= F_f(A) - bA + F_f(B) - bB - F_f(A + B) + b(A + B) \\ &= F_f(A) + F_f(B) - F_f(A + B). \end{aligned}$$

By making use of (2.6) we derive the desired result (3.1). \square

PROPOSITION 2. If $A, B > 0$ with $BA + AB \geq k$, where k is a real number, then

$$\begin{aligned} & F_f(A) + F_f(B) - F_f(A + B) \\ & \geq k [F_f(A + B) - f(A + B) + f(0)] (A + B)^{-2}. \end{aligned}$$

If $k \geq 0$, then we have the refinement of (3.1)

$$\begin{aligned} & F_f(A) + F_f(B) - F_f(A + B) \\ & \geq k [F_f(A + B) - f(A + B) + f(0)] (A + B)^{-2} \geq 0. \end{aligned}$$

REMARK 4. If we take $f(t) = \ln(t + a)$, for $a, t > 0$, then we have

$$\begin{aligned} & \operatorname{dilog}\left(\frac{1}{a}(A + B) + 1\right) - \operatorname{dilog}\left(\frac{1}{a}A + 1\right) - \operatorname{dilog}\left(\frac{1}{a}B + 1\right) \\ & \geq k \left[\ln a - \operatorname{dilog}\left(\frac{1}{a}(A + B) + 1\right) - \ln(A + B) \right] (A + B)^{-2} \geq 0 \end{aligned}$$

provided $A, B > 0$ with $BA + AB \geq k \geq 0$.

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References

- [1] R. Bhatia, *Matrix Analysis*, Graduate Texts in Mathematics, 169, Springer-Verlag, New York, 1997.
- [2] S.S. Dragomir, *Operator monotonicity of an integral transform of positive operators in Hilbert spaces with applications*, Preprint RGMIA Res. Rep. Coll. **23** (2020), Art. 65. Available at <https://rgmia.org/papers/v23/v23a65.pdf>.
- [3] J.I. Fujii and Y. Seo, *On parametrized operator means dominated by power ones*, Sci. Math. **1** (1998), no. 3, 301–306.
- [4] T. Furuta, *Concrete examples of operator monotone functions obtained by an elementary method without appealing to Löwner integral representation*, Linear Algebra Appl. **429** (2008), no. 5-6, 972–980.
- [5] T. Furuta, *Precise lower bound of $f(A) - f(B)$ for $A > B > 0$ and non-constant operator monotone function f on $[0, \infty)$* , J. Math. Inequal. **9** (2015), no. 1, 47–52.
- [6] E. Heinz, *Beiträge zur Störungstheorie der Spektralzerlegung*, Math. Ann. **123** (1951), 415–438.
- [7] K. Löwner, *Über monotone Matrixfunktionen*, Math. Z. **38** (1934), no. 1, 177–216.
- [8] M.S. Moslehian and H. Najafi, *An extension of the Löwner-Heinz inequality*, Linear Algebra Appl. **437** (2012), no. 9, 2359–2365.
- [9] H. Zuo and G. Duan, *Some inequalities of operator monotone functions*, J. Math. Inequal. **8** (2014), no. 4, 777–781.

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