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OSTROWSKI AND TRAPEZOID TYPE INEQUALITIES FOR RIEMANN–LIOUVILLE FRACTIONAL INTEGRALS OF ABSOLUTELY CONTINUOUS FUNCTIONS WITH BOUNDED DERIVATIVES

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Abstract. In this paper we establish some Ostrowski and trapezoid type inequalities for the Riemann-Liouville fractional integrals of absolutely continuous functions with bounded derivatives. Applications for mid-point and trapezoid inequalities are provided as well. They generalize the known results holding for the classical Riemann integral. Some examples for convex functions are also given.

1. Introduction

In 2002 [12], we proved the following *Ostrowski type inequality* for convex functions $f:[a,b]\to\mathbb{R}$ and $x\in(a,b)$,

$$\begin{aligned}&\frac{1}{2}\left[(b-x)^2 f'_+(x)-(x-a)^2 f'_-(x)\right]\\ &\leqslant \int_a^b f(t)\,dt-(b-a)f(x)\\ &\leqslant \frac{1}{2}\left[(b-x)^2 f'_-(b)-(x-a)^2 f'_+(a)\right].\end{aligned}\tag{1.1}$$

In particular, we have the *mid-point inequalities*

$$\begin{aligned}&\frac{1}{8}\left[f'_+\left(\frac{a+b}{2}\right)-f'_-\left(\frac{a+b}{2}\right)\right](b-a)^2\\ &\leqslant \int_a^b f(t)\,dt-(b-a)f\left(\frac{a+b}{2}\right)\\ &\leqslant \frac{1}{8}\left[f'_-(b)-f'_+(a)\right](b-a)^2,\end{aligned}\tag{1.2}$$

with the constant $\frac{1}{8}$ as best possible in both inequalities.

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In the same year [13], we also obtained the following *generalized trapezoid type inequality* for convex functions $f:[a,b]\to\mathbb{R}$ and $x\in(a,b)$,

$$\begin{aligned}
&\frac{1}{2}\left[(b-x)^2 f'_+(x)-(x-a)^2 f'_-(x)\right]\\
&\leqslant (x-a)f(a)+(b-x)f(b)-\int_a^b f(t)\,dt\\
&\leqslant \frac{1}{2}\left[(b-x)^2 f'_-(b)-(x-a)^2 f'_+(a)\right].
\end{aligned} \tag{1.3}$$

In particular, we have the *trapezoid inequality*

$$\begin{aligned}
&\frac{1}{8}\left[f'_+\left(\frac{a+b}{2}\right)-f'_-\left(\frac{a+b}{2}\right)\right](b-a)^2\\
&\leqslant \frac{f(a)+f(b)}{2}(b-a)-\int_a^b f(t)\,dt\\
&\leqslant \frac{1}{8}\left[f'_-(b)-f'_+(a)\right](b-a)^2,
\end{aligned} \tag{1.4}$$

with the constant $\frac{1}{8}$ as best possible in both inequalities.

These results were generalized in the following manner:

THEOREM 1. (Dragomir, 2003 [14]) *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous function on $[a,b]$ and $x\in[a,b]$. Suppose that there exist the functions m_i, $M_i:[a,b]\to\mathbb{R}$ $\left(i=\overline{1,2}\right)$ with the properties:*

$$m_1(x)\leqslant f'(t)\leqslant M_1(x)\ \ \textit{for a.e. } t\in[a,x] \tag{1.5}$$

and

$$m_2(x)\leqslant f'(t)\leqslant M_2(x)\ \ \textit{for a.e. } t\in(x,b]. \tag{1.6}$$

Then we have the inequalities:

$$\begin{aligned}
&\frac{1}{2(b-a)}\left[m_1(x)(x-a)^2-M_2(x)(b-x)^2\right]\\
&\leqslant f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\\
&\leqslant \frac{1}{2(b-a)}\left[M_1(x)(x-a)^2-m_2(x)(b-x)^2\right].
\end{aligned} \tag{1.7}$$

The constant $\frac{1}{2}$ is sharp on both sides.

If we assume global bounds for the derivative, then we have:

COROLLARY 1. (Dragomir, 2003 [14]) *If $f:[a,b]\to\mathbb{R}$ is absolutely continuous on $[a,b]$ and the derivative $f':[a,b]\to\mathbb{R}$ is bounded above and below, that is,*

$$-\infty<m\leqslant f'(t)\leqslant M<\infty\ \textit{for a.e. } t\in[a,b], \tag{1.8}$$

then we have the inequality

$$\frac{1}{2(b-a)}\left[m(x-a)^2 - M(b-x)^2\right] \leqslant f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \leqslant \frac{1}{2(b-a)}\left[M(x-a)^2 - m(b-x)^2\right] \tag{1.9}$$

for all $x\in[a,b]$. *The constant* $\frac{1}{2}$ *is the best in both inequalities.*

In particular, we have

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt - f\left(\frac{a+b}{2}\right)\right| \leqslant \frac{1}{8}(M-m)(b-a), \tag{1.10}$$

with $\frac{1}{8}$ *as the best possible constant.*

In order to extend these results for fractional integrals we need the following definitions.

Let $f:[a,b]\to\mathbb{C}$ be a complex valued Lebesgue integrable function on the real interval $[a,b]$. The *Riemann-Liouville fractional integrals* are defined for $\alpha>0$ by

$$J_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)\,dt$$

for $a<x\leqslant b$ and

$$J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)\,dt$$

for $a\leqslant x<b$, where Γ is the *Gamma function*. For $\alpha=0$, they are defined as

$$J_{a+}^{0} f(x) = J_{b-}^{0} f(x) = f(x) \text{ for } x\in(a,b).$$

For several Ostrowski type inequalities for Riemann-Liouville fractional integrals see [1]–[6], [19]–[29] and the references therein.

Motivated by the above results, we obtain in this paper some inequalities for the Riemann-Liouville fractional integrals of absolutely continuous functions with bounded derivatives and of convex functions. Applications for mid-point and trapezoid inequalities are provided as well. Some examples for convex functions are also given.

2. Some identities

We have the following representation:

LEMMA 1. *Let* $f:[a,b]\to\mathbb{C}$ *be an absolutely continuous function on* $[a,b]$.

(i) For any $x\in(a,b)$ *we have*

$$\begin{aligned} &J_{a+}^{\alpha} f(x) + J_{b-}^{\alpha} f(x) \\ &= \frac{1}{\Gamma(\alpha+1)}\left[(x-a)^{\alpha} f(a) + (b-x)^{\alpha} f(b)\right] \\ &\quad + \frac{1}{\Gamma(\alpha+1)}\left[\int_a^x (x-t)^{\alpha} f'(t)\,dt - \int_x^b (t-x)^{\alpha} f'(t)\,dt\right]. \end{aligned} \tag{2.1}$$

(ii) For any $x\in(a,b)$ we have

$$J_{x-}^{\alpha}f(a)+J_{x+}^{\alpha}f(b) \tag{2.2}$$
$$=\frac{1}{\Gamma(\alpha+1)}\left[(x-a)^{\alpha}+(b-x)^{\alpha}\right]f(x)$$
$$+\frac{1}{\Gamma(\alpha+1)}\left[\int_x^b(b-t)^{\alpha}f'(t)\,dt-\int_a^x(t-a)^{\alpha}f'(t)\,dt\right].$$

Proof. (i) Since $f:[a,b]\to\mathbb{C}$ is an absolutely continuous function on $[a,b]$, then the Lebesgue integrals

$$\int_a^x(x-t)^{\alpha}f'(t)\,dt \text{ and } \int_x^b(t-x)^{\alpha}f'(t)\,dt$$

exist and integrating by parts, we have

$$\frac{1}{\Gamma(\alpha+1)}\int_a^x(x-t)^{\alpha}f'(t)\,dt \tag{2.3}$$
$$=\frac{1}{\Gamma(\alpha)}\int_a^x(x-t)^{\alpha-1}f(t)\,dt-\frac{1}{\Gamma(\alpha+1)}(x-a)^{\alpha}f(a)$$
$$=J_{a+}^{\alpha}f(x)-\frac{1}{\Gamma(\alpha+1)}(x-a)^{\alpha}f(a)$$

for $a<x\leqslant b$ and

$$\frac{1}{\Gamma(\alpha+1)}\int_x^b(t-x)^{\alpha}f'(t)\,dt \tag{2.4}$$
$$=\frac{1}{\Gamma(\alpha+1)}(b-x)^{\alpha}f(b)-\frac{1}{\Gamma(\alpha)}\int_x^b(t-x)^{\alpha-1}f(t)\,dt$$
$$=\frac{1}{\Gamma(\alpha+1)}(b-x)^{\alpha}f(b)-J_{b-}^{\alpha}f(x)$$

for $a\leqslant x<b$.

From (2.3) we have

$$J_{a+}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha+1)}(x-a)^{\alpha}f(a)+\frac{1}{\Gamma(\alpha+1)}\int_a^x(x-t)^{\alpha}f'(t)\,dt$$

for $a<x\leqslant b$ and from (2.4) we have

$$J_{b-}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha+1)}(b-x)^{\alpha}f(b)-\frac{1}{\Gamma(\alpha+1)}\int_x^b(t-x)^{\alpha}f'(t)\,dt,$$

for $a\leqslant x<b$, which by addition give (2.1).

(ii) We have

$$J_{x+}^{\alpha}f(b)=\frac{1}{\Gamma(\alpha)}\int_x^b(b-t)^{\alpha-1}f(t)\,dt$$

for $a \leqslant x < b$ and

$$J_{x-}^{\alpha} f(a) = \frac{1}{\Gamma(\alpha)} \int_a^x (t-a)^{\alpha-1} f(t)\, dt$$

for $a < x \leqslant b$.

Since $f : [a,b] \to \mathbb{C}$ is an absolutely continuous function $[a,b]$, then the Lebesgue integrals

$$\int_a^x (t-a)^{\alpha} f'(t)\, dt \text{ and } \int_x^b (b-t)^{\alpha} f'(t)\, dt$$

exist and integrating by parts, we have

$$\begin{aligned} &\frac{1}{\Gamma(\alpha+1)} \int_a^x (t-a)^{\alpha} f'(t)\, dt \\ &= \frac{1}{\Gamma(\alpha+1)} (x-a)^{\alpha} f(x) - \frac{1}{\Gamma(\alpha)} \int_a^x (t-a)^{\alpha-1} f(t)\, dt \\ &= \frac{1}{\Gamma(\alpha+1)} (x-a)^{\alpha} f(x) - J_{x-}^{\alpha} f(a) \end{aligned} \tag{2.5}$$

for $a < x \leqslant b$ and

$$\begin{aligned} &\frac{1}{\Gamma(\alpha+1)} \int_x^b (b-t)^{\alpha} f'(t)\, dt \\ &= \frac{1}{\Gamma(\alpha)} \int_x^b (b-t)^{\alpha-1} f(t)\, dt - \frac{1}{\Gamma(\alpha+1)} (b-x)^{\alpha} f(x) \\ &= J_{x+}^{\alpha} f(b) - \frac{1}{\Gamma(\alpha+1)} (b-x)^{\alpha} f(x) \end{aligned} \tag{2.6}$$

for $a \leqslant x < b$.

From (2.5) we have

$$J_{x-}^{\alpha} f(a) = \frac{1}{\Gamma(\alpha+1)} (x-a)^{\alpha} f(x) - \frac{1}{\Gamma(\alpha+1)} \int_a^x (t-a)^{\alpha} f'(t)\, dt \tag{2.7}$$

for $a < x \leqslant b$ and from (2.6)

$$J_{x+}^{\alpha} f(b) = \frac{1}{\Gamma(\alpha+1)} (b-x)^{\alpha} f(x) + \frac{1}{\Gamma(\alpha+1)} \int_x^b (b-t)^{\alpha} f'(t)\, dt, \tag{2.8}$$

for $a \leqslant x < b$, which by addition produce (2.2). □

COROLLARY 2. *Let $f : [a,b] \to \mathbb{C}$ be an absolutely continuous function on $[a,b]$. We have the midpoint equalities*

$$\begin{aligned} &J_{a+}^{\alpha} f\left(\frac{a+b}{2}\right) + J_{b-}^{\alpha} f\left(\frac{a+b}{2}\right) \\ &= \frac{1}{2^{\alpha-1}\Gamma(\alpha+1)} \frac{f(a)+f(b)}{2} \\ &\quad + \frac{1}{\Gamma(\alpha+1)} \left[\int_a^{\frac{a+b}{2}} \left(\frac{a+b}{2} - t\right)^{\alpha} f'(t)\, d - \int_{\frac{a+b}{2}}^b \left(t - \frac{a+b}{2}\right)^{\alpha} f'(t)\, dt \right] \end{aligned} \tag{2.9}$$

and

$$J^{\alpha}_{\frac{a+b}{2}-}f(a)+J^{\alpha}_{\frac{a+b}{2}+}f(b) \tag{2.10}$$
$$=\frac{1}{2^{\alpha-1}\Gamma(\alpha+1)}f\left(\frac{a+b}{2}\right)(b-a)^{\alpha}$$
$$+\frac{1}{\Gamma(\alpha+1)}\left[\int_{\frac{a+b}{2}}^{b}(b-t)^{\alpha}f'(t)\,dt-\int_{a}^{\frac{a+b}{2}}(t-a)^{\alpha}f'(t)\,dt\right]$$

and the trapezoid equality

$$\frac{J^{\alpha}_{b-}f(a)+J^{\alpha}_{a+}f(b)}{2}=\frac{1}{\Gamma(\alpha+1)}\frac{f(b)+f(a)}{2}(b-a)^{\alpha}$$
$$+\frac{1}{\Gamma(\alpha+1)}\int_{a}^{b}\frac{(b-t)^{\alpha}-(t-a)^{\alpha}}{2}f'(t)\,dt. \tag{2.11}$$

Proof. Equality (2.9) follows by (2.1) for $x=\frac{a+b}{2}$ while the equality (2.10) follows by (2.2).

For $x=b$ in (2.7) we have

$$J^{\alpha}_{b-}f(a)=\frac{1}{\Gamma(\alpha+1)}(b-a)^{\alpha}f(b)-\frac{1}{\Gamma(\alpha+1)}\int_{a}^{b}(t-a)^{\alpha}f'(t)\,dt$$

while from (2.8) we have for $x=a$ that

$$J^{\alpha}_{a+}f(b)=\frac{1}{\Gamma(\alpha+1)}(b-a)^{\alpha}f(a)+\frac{1}{\Gamma(\alpha+1)}\int_{a}^{b}(b-t)^{\alpha}f'(t)\,dt.$$

If we add these two equalities and divide by 2, we get (2.11). □

3. Inequalities for functions with bounded derivatives

We have the following result that provides upper and lower bounds for the Ostrowski and trapezoid differences:

THEOREM 2. *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous function on $[a,b]$. If $x\in(a,b)$ and there exists the real numbers $m_1(x)$, $M_1(x)$, $m_2(x)$, $M_2(x)$ such that*

$$m_1(x)\leqslant f'(t)\leqslant M_1(x) \text{ for a.e. } t\in(a,x) \tag{3.1}$$

and

$$m_2(x)\leqslant f'(t)\leqslant M_2(x) \text{ for a.e. } t\in(x,b) \tag{3.2}$$

then

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha+2)}\left[m_2(x)(b-x)^{\alpha+1}-M_1(x)(x-a)^{\alpha+1}\right] \qquad (3.3)\\
&\leqslant \frac{1}{\Gamma(\alpha+1)}\left[(x-a)^{\alpha}f(a)+(b-x)^{\alpha}f(b)\right]-J_{a+}^{\alpha}f(x)-J_{b-}^{\alpha}f(x)\\
&\leqslant \frac{1}{\Gamma(\alpha+2)}\left[M_2(x)(b-x)^{\alpha+1}-m_1(x)(x-a)^{\alpha+1}\right]
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha+2)}\left[m_2(x)(b-x)^{\alpha+1}-M_1(x)(x-a)^{\alpha+1}\right] \qquad (3.4)\\
&\leqslant J_{x-}^{\alpha}f(a)+J_{x+}^{\alpha}f(b)-\frac{1}{\Gamma(\alpha+1)}\left[(x-a)^{\alpha}+(b-x)^{\alpha}\right]f(x)\\
&\leqslant \frac{1}{\Gamma(\alpha+2)}\left[M_2(x)(b-x)^{\alpha+1}-m_1(x)(x-a)^{\alpha+1}\right].
\end{aligned}$$

Proof. We have

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha+1)}\left[(x-a)^{\alpha}f(a)+(b-x)^{\alpha}f(b)\right]-J_{a+}^{\alpha}f(x)-J_{b-}^{\alpha}f(x) \qquad (3.5)\\
&=\frac{1}{\Gamma(\alpha+1)}\left[\int_x^b (t-x)^{\alpha}f'(t)\,dt-\int_a^x (x-t)^{\alpha}f'(t)\,dt\right]
\end{aligned}$$

for any $x\in(a,b)$.

Using the conditions (3.1) and (3.2) we have

$$m_2(x)\int_x^b (t-x)^{\alpha}dt\leqslant\int_x^b (t-x)^{\alpha}f'(t)\,dt\leqslant M_2(x)\int_x^b (t-x)^{\alpha}dt$$

and

$$m_1(x)\int_a^x (x-t)^{\alpha}dt\leqslant\int_a^x (x-t)^{\alpha}f'(t)\,dt\leqslant M_1(x)\int_a^x (x-t)^{\alpha}dt$$

namely

$$\frac{1}{\alpha+1}m_2(x)(b-x)^{\alpha+1}\leqslant\int_x^b (t-x)^{\alpha}f'(t)\,dt\leqslant\frac{1}{\alpha+1}M_2(x)(b-x)^{\alpha+1}$$

and

$$\frac{1}{\alpha+1}m_1(x)(x-a)^{\alpha+1}\leqslant\int_a^x (x-t)^{\alpha}f'(t)\,dt\leqslant\frac{1}{\alpha+1}M_1(x)(x-a)^{\alpha+1}.$$

These imply that

$$\begin{aligned}
&\frac{1}{\alpha+1}\left[m_2(x)(b-x)^{\alpha+1}-M_1(x)(x-a)^{\alpha+1}\right]\\
&\leqslant\int_x^b (t-x)^{\alpha}f'(t)\,dt-\int_a^x (x-t)^{\alpha}f'(t)\,dt\\
&\leqslant\frac{1}{\alpha+1}\left[M_2(x)(b-x)^{\alpha+1}-m_1(x)(x-a)^{\alpha+1}\right]
\end{aligned}$$

that is equivalent to

$$\begin{aligned}&\frac{1}{\Gamma(\alpha+2)}\left[m_2(x)(b-x)^{\alpha+1}-M_1(x)(x-a)^{\alpha+1}\right]\\&\leqslant\frac{1}{\Gamma(\alpha+1)}\left[\int_x^b(t-x)^{\alpha}f'(t)\,dt-\int_a^x(x-t)^{\alpha}f'(t)\,dt\right]\\&\leqslant\frac{1}{\Gamma(\alpha+2)}\left[M_2(x)(b-x)^{\alpha+1}-m_1(x)(x-a)^{\alpha+1}\right].\end{aligned}$$

By using the equality (3.5) we get (3.3).

From (2.2) we have

$$\begin{aligned}&J_{x-}^{\alpha}f(a)+J_{x+}^{\alpha}f(b)-\frac{1}{\Gamma(\alpha+1)}\left[(x-a)^{\alpha}+(b-x)^{\alpha}\right]f(x)\\&=\frac{1}{\Gamma(\alpha+1)}\left[\int_x^b(b-t)^{\alpha}f'(t)\,dt-\int_a^x(t-a)^{\alpha}f'(t)\,dt\right].\end{aligned}\tag{3.6}$$

In a similar way, we have

$$\frac{1}{\alpha+1}m_2(x)(b-x)^{\alpha+1}\leqslant\int_x^b(b-t)^{\alpha}f'(t)\,dt\leqslant\frac{1}{\alpha+1}M_2(x)(b-x)^{\alpha+1}$$

and

$$\frac{1}{\alpha+1}m_1(x)(x-a)^{\alpha+1}\leqslant\int_a^x(t-a)^{\alpha}f'(t)\,dt\leqslant\frac{1}{\alpha+1}M_1(x)(x-a)^{\alpha+1},$$

which implies that

$$\begin{aligned}&\frac{1}{\Gamma(\alpha+2)}\left[m_2(x)(b-x)^{\alpha+1}-M_1(x)(x-a)^{\alpha+1}\right]\\&\leqslant\frac{1}{\Gamma(\alpha+1)}\left[\int_x^b(b-t)^{\alpha}f'(t)\,dt-\int_a^x(t-a)^{\alpha}f'(t)\,dt\right]\\&\leqslant\frac{1}{\Gamma(\alpha+2)}\left[M_2(x)(b-x)^{\alpha+1}-m_1(x)(x-a)^{\alpha+1}\right]\end{aligned}$$

and by (3.6) we get (3.4). □

REMARK 1. If we take $\alpha=1$ in (3.3), then we get

$$\begin{aligned}&\frac{1}{2}\left[m_2(x)(b-x)^2-M_1(x)(x-a)^2\right]\\&\leqslant(x-a)f(a)+(b-x)f(b)-\int_a^b f(t)\,dt\\&\leqslant\frac{1}{2}\left[M_2(x)(b-x)^2-m_1(x)(x-a)^2\right]\end{aligned}\tag{3.7}$$

for any $x\in(a,b)$. If we take $\alpha=1$ in (3.4), then we get (1.7).

COROLLARY 3. *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous function on $[a,b]$. If there exists the real numbers $m_1,\ M_1,\ m_2,\ M_2$ such that*

$$m_1 \leqslant f'(t) \leqslant M_1 \text{ for a.e. } t\in\left(a,\frac{a+b}{2}\right) \tag{3.8}$$

and

$$m_2 \leqslant f'(t) \leqslant M_2 \text{ for a.e. } t\in\left(\frac{a+b}{2},b\right) \tag{3.9}$$

then

$$\begin{aligned}&\frac{1}{2^{\alpha+1}\Gamma(\alpha+2)}(b-a)^{\alpha+1}(m_2-M_1)\\&\leqslant \frac{1}{2^{\alpha-1}\Gamma(\alpha+1)}\frac{f(a)+f(b)}{2}(b-a)^{\alpha}-J^{\alpha}_{a+}f\left(\frac{a+b}{2}\right)-J^{\alpha}_{b-}f\left(\frac{a+b}{2}\right)\\&\leqslant \frac{1}{2^{\alpha+1}\Gamma(\alpha+2)}(b-a)^{\alpha+1}(M_2-m_1)\end{aligned} \tag{3.10}$$

and

$$\begin{aligned}&\frac{1}{2^{\alpha+1}\Gamma(\alpha+2)}(b-a)^{\alpha+1}(m_2-M_1)\\&\leqslant J^{\alpha}_{\frac{a+b}{2}-}f(a)+J^{\alpha}_{\frac{a+b}{2}+}f(b)-\frac{1}{2^{\alpha-1}\Gamma(\alpha+1)}f\left(\frac{a+b}{2}\right)(b-a)^{\alpha}\\&\leqslant \frac{1}{2^{\alpha+1}\Gamma(\alpha+2)}(b-a)^{\alpha+1}(M_2-m_1).\end{aligned} \tag{3.11}$$

In particular, we have the simpler inequalities:

COROLLARY 4. *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous function on $[a,b]$. If there exists the real numbers $m,\ M$, such that $m\leqslant f'(t)\leqslant M$ for a.e. $t\in(a,b)$, then*

$$\begin{aligned}&\left|\frac{1}{2^{\alpha-1}\Gamma(\alpha+1)}\frac{f(a)+f(b)}{2}(b-a)^{\alpha}-J^{\alpha}_{a+}f\left(\frac{a+b}{2}\right)-J^{\alpha}_{b-}f\left(\frac{a+b}{2}\right)\right|\\&\leqslant \frac{1}{2^{\alpha+1}\Gamma(\alpha+2)}(b-a)^{\alpha+1}(M-m)\end{aligned} \tag{3.12}$$

and

$$\begin{aligned}&\left|J^{\alpha}_{\frac{a+b}{2}-}f(a)+J^{\alpha}_{\frac{a+b}{2}+}f(b)-\frac{1}{2^{\alpha-1}\Gamma(\alpha+1)}f\left(\frac{a+b}{2}\right)(b-a)^{\alpha}\right|\\&\leqslant \frac{1}{2^{\alpha+1}\Gamma(\alpha+2)}(b-a)^{\alpha+1}(M-m).\end{aligned} \tag{3.13}$$

REMARK 2. If we take $\alpha=1$ in (3.12), then we get

$$\left|\frac{f(a)+f(b)}{2}(b-a)-\int_a^b f(t)\,dt\right|\leqslant\frac{1}{8}(b-a)^2(M-m) \tag{3.14}$$

while from (3.13) we get (1.10).

We also have the following trapezoid type result:

THEOREM 3. *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous function on $[a,b]$. If there exists the real numbers $m,\ M$, such that $m\leqslant f'(t)\leqslant M$ for a.e. $t\in(a,b)$, then*

$$\begin{aligned}&\left|\frac{1}{\Gamma(\alpha+1)}\frac{f(b)+f(a)}{2}(b-a)^{\alpha}-\frac{J_{b-}^{\alpha}f(a)+J_{a+}^{\alpha}f(b)}{2}\right| \\ &\leqslant \frac{2^{\alpha}-1}{2^{\alpha+1}\Gamma(\alpha+2)}(M-m)(b-a)^{\alpha+1}.\end{aligned} \tag{3.15}$$

Proof. We have by (2.11) that

$$\begin{aligned}&\frac{1}{\Gamma(\alpha+1)}\frac{f(b)+f(a)}{2}(b-a)^{\alpha}-\frac{J_{b-}^{\alpha}f(a)+J_{a+}^{\alpha}f(b)}{2} \\ &=\frac{1}{\Gamma(\alpha+1)}\int_a^b\frac{(t-a)^{\alpha}-(b-t)^{\alpha}}{2}f'(t)\,dt.\end{aligned}$$

Observe also that

$$\begin{aligned}&\int_a^b\frac{(t-a)^{\alpha}-(b-t)^{\alpha}}{2}\left(f'(t)-\frac{m+M}{2}\right)dt \\ &=\int_a^b\frac{(t-a)^{\alpha}-(b-t)^{\alpha}}{2}f'(t)\,dt-\frac{m+M}{2}\int_a^b\frac{(t-a)^{\alpha}-(b-t)^{\alpha}}{2}dt\end{aligned}$$

and since

$$\int_a^b\left[(t-a)^{\alpha}-(b-t)^{\alpha}\right]dt=\frac{(b-a)^{\alpha+1}}{\alpha+1}-\frac{(b-a)^{\alpha+1}}{\alpha+1}=0,$$

then we have the following identity of interest

$$\begin{aligned}&\frac{1}{\Gamma(\alpha+1)}\frac{f(b)+f(a)}{2}(b-a)^{\alpha}-\frac{J_{b-}^{\alpha}f(a)+J_{a+}^{\alpha}f(b)}{2} \\ &=\frac{1}{\Gamma(\alpha+1)}\int_a^b\frac{(t-a)^{\alpha}-(b-t)^{\alpha}}{2}\left(f'(t)-\frac{m+M}{2}\right)dt.\end{aligned} \tag{3.16}$$

By taking the modulus in (3.16), we get

$$\begin{aligned}&\left|\frac{1}{\Gamma(\alpha+1)}\frac{f(b)+f(a)}{2}(b-a)^{\alpha}-\frac{J_{b-}^{\alpha}f(a)+J_{a+}^{\alpha}f(b)}{2}\right| \\ &\leqslant\frac{1}{\Gamma(\alpha+1)}\int_a^b\left|\frac{(t-a)^{\alpha}-(b-t)^{\alpha}}{2}\right|\left|f'(t)-\frac{m+M}{2}\right|dt \\ &\leqslant\frac{1}{4}(M-m)\frac{1}{\Gamma(\alpha+1)}\int_a^b\left|(b-t)^{\alpha}-(t-a)^{\alpha}\right|dt.\end{aligned} \tag{3.17}$$

The function $h:[a,b]\to[0,\infty)$, $h(t)=\left|(b-t)^{\alpha}-(t-a)^{\alpha}\right|$ is symmetric on $[a,b]$, then

$$\begin{aligned}
&\int_a^b \left|(b-t)^{\alpha}-(t-a)^{\alpha}\right| dt \\
&=2\int_a^{\frac{a+b}{2}} \left[(b-t)^{\alpha}-(t-a)^{\alpha}\right] dt \\
&=2\left[-\left.\frac{(b-t)^{\alpha+1}}{\alpha+1}\right|_a^{\frac{a+b}{2}}-\left.\frac{(t-a)^{\alpha+1}}{\alpha+1}\right|_a^{\frac{a+b}{2}}\right] \\
&=2\left[-\frac{\left(b-\frac{a+b}{2}\right)^{\alpha+1}}{\alpha+1}+\frac{(b-a)^{\alpha+1}}{\alpha+1}-\frac{\left(\frac{a+b}{2}-a\right)^{\alpha+1}}{\alpha+1}\right] \\
&=2\left[\frac{(b-a)^{\alpha+1}}{\alpha+1}-\frac{(b-a)^{\alpha+1}}{2^{\alpha}(\alpha+1)}\right]=\frac{2^{\alpha}-1}{2^{\alpha-1}(\alpha+1)}(b-a)^{\alpha+1}.
\end{aligned}$$

By making use of (3.17) we then obtain (3.15). □

4. Inequalities for convex functions

We have the following result for convex functions:

THEOREM 4. *Let $f:[a,b]\to\mathbb{R}$ be a convex function and $x\in(a,b)$, then we have the inequalities*

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha+2)}\left[f'_+(x)(b-x)^{\alpha+1}-f'_-(x)(x-a)^{\alpha+1}\right] \\
&\leqslant \frac{1}{\Gamma(\alpha+1)}\left[(x-a)^{\alpha}f(a)+(b-x)^{\alpha}f(b)\right]-J^{\alpha}_{a+}f(x)-J^{\alpha}_{b-}f(x) \\
&\leqslant \frac{1}{\Gamma(\alpha+2)}\left[f'_-(b)(b-x)^{\alpha+1}-f'_+(a)(x-a)^{\alpha+1}\right]
\end{aligned} \tag{4.1}$$

and

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha+2)}\left[f'_+(x)(b-x)^{\alpha+1}-f'_-(x)(x-a)^{\alpha+1}\right] \\
&\leqslant J^{\alpha}_{x-}f(a)+J^{\alpha}_{x+}f(b)-\frac{1}{\Gamma(\alpha+1)}\left[(x-a)^{\alpha}+(b-x)^{\alpha}\right]f(x) \\
&\leqslant \frac{1}{\Gamma(\alpha+2)}\left[f'_-(b)(b-x)^{\alpha+1}-f'_+(a)(x-a)^{\alpha+1}\right],
\end{aligned} \tag{4.2}$$

where $f'_{\pm}(\cdot)$ are the lateral derivatives of f.

Proof. Since f is convex, then the derivative f' exists almost everywhere on $[a,b]$ and

$$f'_+(a)\leqslant f'(t)\leqslant f'_-(x) \text{ for a.e. } t\in(a,x)$$

and

$$f'_+(x) \leqslant f'(t) \leqslant f'_-(b) \text{ for a.e. } t \in (x,b).$$

Now, writing the inequalities (3.3) and (3.4) for $m_1(x) = f'_+(a)$, $M_1(x) = f'_-(x)$, $m_2(x) = f'_+(x)$ and $M_2(x) = f'_-(b)$ we get the desired results (4.1) and (4.2). □

REMARK 3. If we take $\alpha = 1$ in (4.1) and (4.2), then we recapture (1.3) and (1.1) that hold for convex functions.

COROLLARY 5. *Let $f : [a,b] \to \mathbb{R}$ be a convex function, then we have the inequalities*

$$\begin{aligned} 0 &\leqslant \frac{1}{2^{\alpha+1}\Gamma(\alpha+2)}\left[f'_+\left(\frac{a+b}{2}\right) - f'_-\left(\frac{a+b}{2}\right)\right](b-a)^{\alpha+1} \\ &\leqslant \frac{1}{2^{\alpha-1}\Gamma(\alpha+1)}\frac{f(a)+f(b)}{2}(b-a)^{\alpha} - J^{\alpha}_{a+}f\left(\frac{a+b}{2}\right) - J^{\alpha}_{b-}f\left(\frac{a+b}{2}\right) \\ &\leqslant \frac{1}{2^{\alpha+1}\Gamma(\alpha+2)}\left[f'_-(b) - f'_+(a)\right](b-a)^{\alpha+1}, \end{aligned} \tag{4.3}$$

$$\begin{aligned} 0 &\leqslant \frac{1}{2^{\alpha+1}\Gamma(\alpha+2)}\left[f'_+\left(\frac{a+b}{2}\right) - f'_-\left(\frac{a+b}{2}\right)\right](b-a)^{\alpha+1} \\ &\leqslant J^{\alpha}_{\frac{a+b}{2}-}f(a) + J^{\alpha}_{\frac{a+b}{2}+}f(b) - \frac{1}{2^{\alpha-1}\Gamma(\alpha+1)}f\left(\frac{a+b}{2}\right)(b-a)^{\alpha} \\ &\leqslant \frac{1}{2^{\alpha+1}\Gamma(\alpha+2)}\left[f'_-(b) - f'_+(a)\right](b-a)^{\alpha+1}, \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} 0 &\leqslant \frac{1}{\Gamma(\alpha+1)}\frac{f(b)+f(a)}{2}(b-a)^{\alpha} - \frac{J^{\alpha}_{b-}f(a) + J^{\alpha}_{a+}f(b)}{2} \\ &\leqslant \frac{2^{\alpha}-1}{2^{\alpha+1}\Gamma(\alpha+2)}\left(f'_-(b) - f'_+(a)\right)(b-a)^{\alpha+1}. \end{aligned} \tag{4.5}$$

If we take $\alpha = 1$ in (4.3) and (4.4), then we recapture the midpoint and trapezoid inequalities for convex functions mentioned in the introduction.

5. Conclusion

In this paper, by making use of some fundamental identities for the Riemann-Liouville fractional integrals, we established some Ostrowski and trapezoid type inequalities for these integrals of absolutely continuous functions with bounded derivatives. Applications for mid-point and trapezoid inequalities were provided as well. They generalize the known results holding for the classical Riemann integral. Some natural applications for the important case of convex functions were also given.

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