On Hermite–Hadamard-Type Inequalities for Coordinated h-Convex Interval-Valued Functions

mathematics

Article

On Hermite–Hadamard-Type Inequalities for Coordinated h-Convex Interval-Valued Functions

Dafang Zhao [1,2,*], Guohui Zhao [2], Guoju Ye [3], Wei Liu [3] and Silvestru Sever Dragomir [4]

1 School of Mathematics and Statistics, Hubei Normal University, Huangshi 435002, China
2 Northwest Institute of Eco-Environment and Resources, Chinese Academy of Sciences, Lanzhou 730000, China; zhgh@lzb.ac.cn
3 College of Science, Hohai University, Nanjing 210098, China; ygjhhu@163.com (G.Y.); liuw626@hhu.edu.cn (W.L.)
4 Mathematics, College of Engineering and Science, Victoria University, Melbourne 8001, Australia; Sever.Dragomir@vu.edu.au
* Correspondence: dafangzhao@163.com; Tel.: +86-135-9768-0179

Abstract: This paper is devoted to establishing some Hermite–Hadamard-type inequalities for interval-valued functions using the coordinated h-convexity, which is more general than classical convex functions. We also discuss the relationship between coordinated h-convexity and h-convexity. Furthermore, we introduce the concepts of minimum expansion and maximum contraction of interval sequences. Based on these two new concepts, we establish some new Hermite–Hadamard-type inequalities, which generalize some known results in the literature. Additionally, some examples are given to illustrate our results.

Keywords: Hermite–Hadamard inequality; interval double integral; coordinated h-convex; interval-valued functions

check for updates

Citation: Zhao, D.; Zhao, G.; Ye, G.; Liu, W.; Dragomir, S.S. On Hermite–Hadamard-Type Inequalities for Coordinated h-Convex Interval-Valued Functions. *Mathematics* **2021**, *9*, 2352. https://doi.org/10.3390/math9192352

Academic Editor: Pedro J. Miana

Received: 23 August 2021
Accepted: 18 September 2021
Published: 22 September 2021

1. Introduction

The following double-inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)dt \leq \frac{f(a)+f(b)}{2},$$

is known as the Hermite–Hadamard (H–H) inequality, where $f : [a,b] \to \mathbb{R}$ is a convex function.

Due to its key role in convex analysis, the H–H inequality has been used as a powerful tool to acquire a large number of nice results in integral inequalities and optimization theory. Recently, it has been generalized by means of different types of convexity, such as s-convex functions [1–4], log-convex [5–7], harmonic convexity [8], and especially for h-convex functions [9]. Since 2007, various extensions and generalizations of H–H inequalities for h-convex functions have been established in [10–16].

On the other hand, the theory of interval analysis has a long history which can be traced back to Archimedes' computation of the circumference of a circle. However, it fell into oblivion for a long time because of lack of applications to other sciences. In 1924, Burkill [17] developed some elementary properties of functions of intervals. Shortly afterwards, Kolmogorov [18] generalized Burkill's results from single-valued functions to multi-valued functions. Of course, there are many other excellent results that have been achieved over the next two decades. Please note that Moore was the first to recognize how to use interval analysis to compute the error bounds of the numerical solutions of computer. Since the publication of the first monograph on interval analysis by Moore [19] in 1966, the theoretical and applied research on interval analysis has attracted much attention, and also has yielded fruitful results over the past 50 years. More recently, numerous famous inequalities have been extended to set-valued functions by Nikodem et al. [20], especially, to interval-valued functions by

Budak et al. [21], Chalco-Cano et al. [22,23], Costa et al. [24–26], Román-Flores et al. [27,28], Flores-Franulič et al. [29], Zhao et al. [30–33].

Motivated by Dragomir [34], Latif and Alomari [10], and Sarikaya et al. [15], we introduce the coordinated h-convex for interval-valued functions (IVFs). Additionally, we discuss the relationship between coordinated h-convexity and h-convexity. We introduce two new concepts of interval sequences, minimum expansion and maximum contraction. Using these two new concepts, we establish new interval version of Hermite–Hadamard-type inequalities, which are the main results of this paper. Finally, we give some examples to illustrate our main results. Furthermore, the present results can be considered to be tools for further research in generalized convexity, interval optimization, and inequalities for IVFs, among others.

The paper is organized as follows. Section 2 contains some necessary preliminaries. In Section 3, the coordinated h-convex concept for IVFs is given. Moreover, we establish some H–H-type inequalities for coordinated h-convex IVFs. We end with Section 4 of conclusions.

2. Preliminaries

In this section, we recall some basic definitions, notations, properties, and results on interval analysis, which are used throughout the paper. A real interval $[u]$ is the bounded, closed subset of $\mathbb{R}$ defined by

$$[u] = [\underline{u}, \overline{u}] = \{x \in \mathbb{R} |\ \underline{u} \le x \le \overline{u}\},$$

where $\underline{u}, \overline{u} \in \mathbb{R}$ and $\underline{u} \le \overline{u}$. The numbers $\underline{u}$ and $\overline{u}$ are called the left and the right endpoints of $[\underline{u}, \overline{u}]$, respectively. When $\underline{u}$ and $\overline{u}$ are equal, the interval $[u]$ is said to be degenerate. In this paper, the term interval will mean a nonempty interval. We call $[u]$ positive if $\underline{u} > 0$ or negative if $\overline{u} < 0$. The inclusion "$\subseteq$" is defined by

$$[\underline{u}, \overline{u}] \subseteq [\underline{v}, \overline{v}] \Longleftrightarrow \underline{v} \le \underline{u}, \overline{u} \le \overline{v}.$$

For an arbitrary real number λ and $[u]$, the interval $\lambda[u]$ is given by

$$\lambda[\underline{u}, \overline{u}] = \begin{cases} [\lambda\underline{u}, \lambda\overline{u}] & \text{if } \lambda > 0, \\ \{0\} & \text{if } \lambda = 0, \\ [\lambda\overline{u}, \lambda\underline{u}] & \text{if } \lambda < 0. \end{cases}$$

For $[u] = [\underline{u}, \overline{u}]$ and $[v] = [\underline{v}, \overline{v}]$, the four arithmetic operators (+, $-$, $\cdot$,/) are defined by

$$[u] + [v] = [\underline{u} + \underline{v}, \overline{u} + \overline{v}],$$

$$[u] - [v] = [\underline{u} - \overline{v}, \overline{u} - \underline{v}],$$

$$[u] \cdot [v] = \big[\min\{\underline{u}\underline{v}, \underline{u}\overline{v}, \overline{u}\underline{v}, \overline{u}\overline{v}\}, \max\{\underline{u}\underline{v}, \underline{u}\overline{v}, \overline{u}\underline{v}, \overline{u}\overline{v}\}\big],$$

$$\begin{aligned} [u]/[v] = \big[&\min\{\underline{u}/\underline{v}, \underline{u}/\overline{v}, \overline{u}/\underline{v}, \overline{u}/\overline{v}\}, \\ &\max\{\underline{u}/\underline{v}, \underline{u}/\overline{v}, \overline{u}/\underline{v}, \overline{u}/\overline{v}\}\big], \text{where } 0 \notin [\underline{v}, \overline{v}]. \end{aligned}$$

We denote by $\mathbb{R}_{\mathcal{I}}$ the set of all intervals of $\mathbb{R}$, and by $\mathbb{R}_{\mathcal{I}}^{+}$ and $\mathbb{R}_{\mathcal{I}}^{-}$ the set of all positive intervals and negative intervals of $\mathbb{R}$, respectively.

The Hausdorff distance between $[\underline{u}, \overline{u}]$ and $[\underline{v}, \overline{v}]$ is defined by

$$d\big([\underline{u}, \overline{u}], [\underline{v}, \overline{v}]\big) = \max\Big\{|\underline{u} - \underline{v}|, |\overline{u} - \overline{v}|\Big\}.$$

Then, $(\mathbb{R}_{\mathcal{I}}, d)$ is a complete metric space. For more basic notations on IVFs, see [30,35].

Definition 1 ([9]). *Let $[0,1] \subseteq I \subseteq \mathbb{R}$, and $h : I \to \mathbb{R}^+$ with $h \not\equiv 0$. Then $f : I \to \mathbb{R}^+$ is called h-convex (i.e., $f \in SX(h, I, \mathbb{R}^+)$), if for all $s, t \in I$ and $\alpha \in (0,1)$ we have*

$$f(\alpha s + (1-\alpha)t) \le h(\alpha)f(s) + h(1-\alpha)f(t). \tag{1}$$

h is called supermultiplicative if

$$h(st) \ge h(s)h(t) \tag{2}$$

for all $s, t \in I$. If "$\ge$" in (2) is replaced with "$\le$", then h is called submultiplicative. Additionally, we say that h is multiplicative if the equality holds in (2).

Definition 2 ([30]). *$f : I \to \mathbb{R}^+_{\mathcal{I}}$ is called h-convex if for all $s, t \in I$ and $\alpha \in (0,1)$ we have*

$$h(\alpha)f(s) + h(1-\alpha)f(t) \subseteq f(\alpha s + (1-\alpha)t). \tag{3}$$

If "$\subseteq$" in (3) is reversed, then f is called concave. Let $SX(h, I, \mathbb{R}^+_{\mathcal{I}})$ denote the family of all h-convex IVFs.

The Riemann integral for IVF is introduced in [35]. The set of all Riemann integrable IVFs and real-valued functions on $[a,b]$ are denoted by $\mathcal{IR}_{([a,b])}$ and $\mathcal{R}_{([a,b])}$, respectively. Moreover, we have

Theorem 1 ([35]). *Let $f : [a,b] \to \mathcal{K}_{\mathcal{I}}$ be an IVF with $f = \left[\underline{f}, \overline{f}\right]$. Then $f \in \mathcal{IR}_{([a,b])}$ iff $\underline{f}, \overline{f} \in \mathcal{R}_{([a,b])}$ and*

$$(IR)\int_a^b f(t)dt = \left[(R)\int_a^b \underline{f}(t)dt, (R)\int_a^b \overline{f}(t)dt\right].$$

A set of numbers $\{t_{i-1}, \xi_i, t_i\}_{i=1}^m$ is called a tagged partition P_1 of $[a,b]$ if

$$a = t_0 < t_1 < \cdots < t_m = b$$

and if $t_{i-1} \le \xi_i \le t_i$ for all $i = 1, 2, \ldots, m$. Moreover, if we let $\Delta t_i = t_i - t_{i-1}$, then P_1 is said to be δ-fine if $\Delta t_i < \delta$ for all i. Let $\mathcal{P}(\delta, [a,b])$ denote the set of all δ-fine partitions of $[a,b]$. If $\{t_{i-1}, \xi_i, t_i\}_{i=1}^m$ is a δ-fine P_1 of $[a,b]$, and if $\{s_{j-1}, \eta_j, s_j\}_{j=1}^n$ is a δ-fine P_2 of $[c,d]$, then the rectangles

$$\Delta_{i,j} = [t_{i-1}, t_i] \times [s_{j-1}, s_j]$$

partition the rectangle $\Delta = [a,b] \times [c,d]$ and the points (ξ_i, η_j) are inside the rectangles $[t_{i-1}, t_i] \times [s_{j-1}, s_j]$. Furthermore, by $\mathcal{P}(\delta, \Delta)$ we denote the set of all δ-fine partitions P of Δ with $P = P_1 \times P_2$, where $P_1 \in \mathcal{P}(\delta, [a,b])$ and $P_2 \in \mathcal{P}(\delta, [c,d])$. Let $\Delta A_{i,j}$ be the area of rectangle $\Delta_{i,j}$. In each rectangle $\Delta_{i,j}$, where $1 \le i \le m$, $1 \le j \le n$, choose arbitrary (ξ_i, η_j) and obtain

$$S(f, P, \delta, \Delta) = \sum_{i=1}^m \sum_{j=1}^n f(\xi_i, \eta_j)\Delta A_{i,j}.$$

We call $S(f, P, \delta, \Delta)$ an integral sum of f associated with $P \in \mathcal{P}(\delta, \Delta)$.

Now, we recall the concept of interval double integral.

Definition 3 ([32]). Let $f : \Delta \to \mathbb{R}_{\mathcal{I}}$. Then f is called ID-integrable on Δ with the ID-integral $U = (ID)\iint_\Delta f(t,s)dA$, if for any $\epsilon > 0$ there exists $\delta > 0$ such that

$$d\big(S(f, P, \delta, \Delta), U\big) < \epsilon$$

for any $P \in \mathcal{P}(\delta, \Delta)$. The collection of all ID-integrable functions on Δ will be denoted by $\mathcal{ID}_{(\Delta)}$.

Theorem 2 ([32]). *Let $\Delta = [a,b] \times [c,d]$. If $f : \Delta \to \mathbb{R}_{\mathcal{I}}$ is ID-integrable on Δ, then*

$$(ID) \iint_{\Delta} f(t,s)dA = (IR) \int_a^b (IR) \int_c^d f(t,s)dsdt.$$

Example 1. *Suppose that $\Delta = [0,1] \times [1,2]$. Let $f : \Delta \to \mathbb{R}_{\mathcal{I}}$ be defined by*

$$f(s,t) = [st, s+t].$$

Then $f(t,s)$ is ID-integrable on Δ and

$$\begin{aligned}
(ID) \iint_{\Delta} f(s,t)dA &= (ID) \int_0^1 \int_1^2 f(s,t)dsdt \\
&= (IR) \int_0^1 \left[(IR) \int_1^2 [st, s+t]ds \right] dt \\
&= \left[\frac{3}{4}, 2 \right].
\end{aligned}$$

3. Main Results

In this section, all considered Δ will mean $[a,b] \times [c,d]$, i.e., $\Delta = [a,b] \times [c,d]$. We begin by introducing some new concepts of IVFs.

Definition 4. *Let $h : [0,1] \subseteq I \to \mathbb{R}^+$ with $h \not\equiv 0$. Then $f : \Delta \to \mathbb{R}^+_{\mathcal{I}}$ is called a coordinated h-convex IVF on Δ if the partial mappings*

$$f_t : [a,b] \to \mathbb{R}^+_{\mathcal{I}},\ f_t(s) = f(s,t),$$

and

$$f_s : [c,d] \to \mathbb{R}^+_{\mathcal{I}},\ f_s(t) = f(s,t),$$

are h-convex for all $t \in [c,d]$ and $s \in [a,b]$. Then the set of all coordinated h-convex IVFs on Δ is denoted by $SX(ch, \Delta, \mathbb{R}^+_{\mathcal{I}})$.

Definition 5. *Let $f : \Delta \to \mathbb{R}^+_{\mathcal{I}}$. Then f is called an h-convex IVF in Δ if for any $(s_1,t_1), (s_2,t_2) \in \Delta$ and $\alpha \in [0,1]$ we have*

$$h(\alpha)f(s_1,t_1) + h(1-\alpha)f(s_2,t_2) \subseteq f(\alpha s_1 + (1-\alpha)s_2, \alpha t_1 + (1-\alpha)t_2). \quad (4)$$

The set of all h-convex IVFs in Δ is denoted by $SX(h, \Delta, \mathbb{R}^+_{\mathcal{I}})$.

Theorem 3. *If $f \in SX(h, \Delta, \mathbb{R}^+_{\mathcal{I}})$, then*

$$f \in SX(ch, \Delta, \mathbb{R}^+_{\mathcal{I}}).$$

Proof. Assume that $f \in SX(h, \Delta, \mathbb{R}^+_{\mathcal{I}})$. Let

$$f_s : [c,d] \to \mathbb{R}^+_{\mathcal{I}},\ \ f_s(t) = f(s,t).$$

Then for all $\alpha \in [0,1]$ and $t_1, t_2 \in [c,d]$, we have

$$\begin{aligned}
f_s(\alpha t_1 + (1-\alpha)t_2) &= f(s, \alpha t_1 + (1-\alpha)t_2) \\
&= f(\alpha s + (1-\alpha)s, \alpha t_1 + (1-\alpha)t_2) \\
&\supseteq\ h(\alpha)f(s,t_1) + h(1-\alpha)f(s,t_2) \\
&= h(\alpha)f_s(t_1) + h(1-\alpha)f_s(t_2).
\end{aligned}$$

Hence,

$$f_s(t) = f(s,t)$$

is h-convex on $[c,d]$ for any $s \in [a,b]$. The fact that

$$f_t(s) = f(s,t)$$

is also h-convex on $[a,b]$ for all $t \in [c,d]$ goes likewise. $\square$

Remark 1. *The converse of Theorem 3 is not generally true.*

Example 2. *Let*

$$h(\alpha) = \alpha,\ \Delta_1 = [0,1] \times [0,1],$$

and $f : \Delta_1 \to \mathbb{R}^+_{\mathcal{I}}$ *be defined:*

$$f(s,t) = [s, 6 - e^s] \cdot [t, 6 - e^t] = [st, (6 - e^s)(6 - e^t)].$$

Obviously, we have that

$$f \in SX(ch, \Delta_1, \mathbb{R}^+_{\mathcal{I}}),$$

and

$$f \notin SX(h, \Delta_1, \mathbb{R}^+_{\mathcal{I}}).$$

Indeed, if

$$(s,0), (0,t) \in \Delta_1$$

and $\alpha \in [0,1]$*, we have:*

$$\begin{aligned} f(\alpha s + (1-\alpha)0, \alpha \cdot 0 + (1-\alpha)t) &= f(\alpha s, (1-\alpha)t) \\ &= [\alpha(1-\alpha)st, (6 - e^{\alpha s})(6 - e^{(1-\alpha)t})] \end{aligned}$$

and

$$\alpha f(s,0) + (1-\alpha) f(0,t) = [0, 30 - 5\alpha e^s - 5e^t + 5\alpha e^t]$$

If $s,t \neq 0$ *and* $\alpha \neq 0$*, then*

$$f(\alpha x, (1-\alpha)w) \not\supseteq \alpha f(x,0) + (1-\alpha) f(0,w).$$

Thus, $f \notin SX(h, \Delta_1, \mathbb{R}^+_{\mathcal{I}})$*.*

In what follows, without causing confusion, we will delete notations of (R), (IR), (ID).

Theorem 4. *Let* $f : \Delta \to \mathbb{R}^+_{\mathcal{I}}$ *and* $h : [0,1] \to \mathbb{R}^+$ *be continuous functions and* $h\left(\frac{1}{2}\right) \neq 0$*. If* $f \in SX(ch, \Delta, \mathbb{R}^+_{\mathcal{I}})$*, then*

$$\begin{aligned} &\frac{1}{4\left(h\left(\frac{1}{2}\right)\right)^2} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\ &\supseteq \frac{1}{4h\left(\frac{1}{2}\right)} \left[\frac{1}{b-a} \int_a^b f\left(s, \frac{c+d}{2}\right) ds + \frac{1}{d-c} \int_c^d f\left(\frac{a+b}{2}, t\right) dt\right] \\ &\supseteq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(s,t)\, ds dt \\ &\supseteq \frac{1}{2} \int_0^1 h(t) dt \left[\frac{1}{b-a} \int_a^b f(s,c) ds + \frac{1}{b-a} \int_a^b f(s,d) ds \right. \\ &\quad \left. + \frac{1}{d-c} \int_c^d f(a,t) dt + \frac{1}{d-c} \int_c^d f(b,t) dt\right] \\ &\supseteq \big(f(a,c) + f(a,d) + f(b,c) + f(b,d)\big) \left(\int_0^1 h(t) dt\right)^2. \end{aligned}$$

Proof. Assume that $f \in SX(ch, \Delta, \mathbb{R}_{\mathcal{I}}^{+})$. Then, the mapping

$$g_s : [c,d] \to \mathbb{R}_{\mathcal{I}}^{+}, g_s(t) = f(s,t)$$

is h-convex on $[c,d]$ for each $s \in [a,b]$. Consequently, we have

$$\begin{aligned}
&h\Big(\frac{1}{2}\Big)\Big[g_s(\alpha c + (1-\alpha)d) + g_s((1-\alpha)c + \alpha d)\Big] \\
&\subseteq g_s\Big(\frac{\alpha c + (1-\alpha)d}{2} + \frac{(1-\alpha)c + \alpha d}{2}\Big) \\
&= g_s\Big(\frac{c+d}{2}\Big).
\end{aligned}$$

Integrating both sides of above inequality over $[0,1]$, we have

$$\begin{aligned}
\int_0^1 g_s\Big(\frac{c+d}{2}\Big)dt \supseteq{}& h\Big(\frac{1}{2}\Big)\int_0^1 \big[\underline{g_s}(\alpha c + (1-\alpha)d) + \underline{g_s}((1-\alpha)c + \alpha d), \\
&\overline{g_s}(\alpha c + (1-\alpha)d) + \overline{g_s}((1-\alpha)c + \alpha d)\big]dt \\
={}& h\Big(\frac{1}{2}\Big)\Big[\int_0^1 \big[\underline{g_s}(\alpha c + (1-\alpha)d) + \underline{g_s}((1-\alpha)c + \alpha d)\big]dt, \\
&\int_0^1 \big[\overline{g_s}(\alpha c + (1-\alpha)d) + \overline{g_s}((1-\alpha)c + \alpha d)\big]dt\Big] \\
={}& h\Big(\frac{1}{2}\Big)\Big[\frac{2}{d-c}\int_c^d \underline{g_s}(t)dt, \frac{2}{d-c}\int_c^d \overline{g_s}(t)dt\Big] \\
={}& h\Big(\frac{1}{2}\Big)\frac{2}{d-c}\int_c^d g_s(t)dt.
\end{aligned}$$

Similarly, we obtain

$$\frac{1}{d-c}\int_c^d g_s(t)dt \supseteq \big(g_s(c) + g_s(d)\big)\int_0^1 h(t)dt.$$

Then,

$$\frac{1}{2h\big(\frac{1}{2}\big)}g_s\Big(\frac{c+d}{2}\Big) \supseteq \frac{1}{d-c}\int_c^d g_s(t)dt \supseteq \big(g_s(c) + g_s(d)\big)\int_0^1 h(t)dt.$$

That is

$$\frac{1}{2h\big(\frac{1}{2}\big)}f\Big(s, \frac{c+d}{2}\Big) \supseteq \frac{1}{d-c}\int_c^d f(s,t)dt \supseteq \big(f(s,c) + f(s,d)\big)\int_0^1 h(t)dt.$$

Integrating over $[a,b]$, we have

$$\begin{aligned}
\frac{1}{2h\big(\frac{1}{2}\big)}\frac{1}{b-a}\int_a^b f\Big(s, \frac{c+d}{2}\Big)ds \supseteq{}& \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(s,t)dsdt \\
\supseteq{}& \int_0^1 h(t)dt\Big[\frac{1}{b-a}\int_a^b f(s,c)ds + \frac{1}{b-a}\int_a^b f(s,d)ds\Big].
\end{aligned}$$

Similarly, for

$$g_t : [a,b] \to \mathbb{R}_{\mathcal{I}}^{+},\ g_t(s) = f(s,t),$$

we have

$$\begin{aligned}
\frac{1}{2h\big(\frac{1}{2}\big)}\frac{1}{d-c}\int_c^d f\Big(\frac{a+b}{2}, t\Big)dt \supseteq{}& \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(s,t)dsdt \\
\supseteq{}& \int_0^1 h(t)dt\Big[\frac{1}{d-c}\int_c^d f(a,t)dt + \frac{1}{d-c}\int_c^d f(b,t)dt\Big].
\end{aligned}$$

Finally, we obtain

$$\begin{aligned}
&\frac{1}{4\left(h\left(\frac{1}{2}\right)\right)^2} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\
&= \frac{1}{4h\left(\frac{1}{2}\right)}\left[\frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) + \frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\right] \\
&\supseteq \frac{1}{4h\left(\frac{1}{2}\right)}\left[\frac{1}{b-a}\int_a^b f\left(s, \frac{c+d}{2}\right)ds + \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2}, t\right)dt\right] \\
&\supseteq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(s,t)dsdt \\
&\supseteq \frac{1}{2}\int_0^1 h(t)dt\left[\frac{1}{b-a}\int_a^b f(s,c)ds + \frac{1}{b-a}\int_a^b f(s,d)ds\right. \\
&\qquad \left. + \frac{1}{d-c}\int_c^d f(a,t)dt + \frac{1}{d-c}\int_c^d f(b,t)dt\right] \\
&\supseteq \frac{1}{2}\left(\int_0^1 h(t)dt\right)^2 (f(a,c) + f(b,c) + f(a,d) + f(b,d) \\
&\qquad + f(a,c) + f(b,c) + f(a,d) + f(b,d)) \\
&\supseteq (f(a,c) + f(a,d) + f(b,c) + f(b,d))\left(\int_0^1 h(t)dt\right)^2.
\end{aligned}$$

This concludes the proof. □

Remark 2. *Theorem 4 gives an interval generalization of ([34] [Theorem 1]) and ([10] [Theorem 7]).*

Lemma 1 ([30])**.** *Let $g \in \mathcal{R}_{([c,d])}$ such that $g : [c,d] \to [m,M]$, $h : I \to [0,\infty)$ be a multiplicative function and $f : [m,M] \to \mathbb{R}_{\mathcal{I}}^+$ be h-convex and continuous. If the following limit exists, is finite and*

$$\lim_{t \to 0^+} \frac{h(t)}{t} = k > 0,$$

then

$$f\left(\frac{\int_c^d g(t)dt}{d-c}\right) \supseteq \frac{k}{d-c}\int_c^d f(g(t))dt.$$

Let $\left\{\left[\underline{u_i}, \overline{u^i}\right]\right\}$ be a sequence of intervals. If there exists an interval $[\underline{u}, \overline{u}]$ such that

$$\left[\underline{u_i}, \overline{u^i}\right] \subseteq [\underline{u}, \overline{u}]$$

for all i, then $[\underline{u}, \overline{u}]$ is an expansion of interval sequence $\left\{\left[\underline{u_i}, \overline{u^i}\right]\right\}$. If $[\underline{u}, \overline{u}]$ such that

$$[\underline{u}, \overline{u}] \subseteq [\underline{u'}, \overline{u'}]$$

for any expansion $[\underline{u'}, \overline{u'}]$ of $\left\{\left[\underline{u_i}, \overline{u^i}\right]\right\}$, then $[\underline{u}, \overline{u}]$ is called the *minimum expansion* of $\left\{\left[\underline{u_i}, \overline{u^i}\right]\right\}$ and is denoted by

$$[\underline{u}, \overline{u}] = E_i^{\min}\left\{\left[\underline{u_i}, \overline{u^i}\right]\right\}.$$

If there exists an interval $[\underline{v}, \overline{v}]$ such that

$$\left[\underline{u_i}, \overline{u^i}\right] \supseteq [\underline{v}, \overline{v}]$$

for all i, then $[\underline{v}, \overline{v}]$ is called a contraction of $\left\{ [\underline{u_i}, \overline{u^i}] \right\}$. If $[\underline{v}, \overline{v}]$ such that

$$[\underline{v}, \overline{v}] \supseteq [\underline{v'}, \overline{v'}]$$

for every contraction $[\underline{v'}, \overline{v'}]$ of $\left\{ [\underline{u_i}, \overline{u^i}] \right\}$, then $[\underline{v}, \overline{v}]$ is the *maximum contraction* of $\left\{ [\underline{u_i}, \overline{u^i}] \right\}$ and is denoted by

$$[\underline{v}, \overline{v}] = C_i^{\max}\left\{ [\underline{u_i}, \overline{u^i}] \right\}.$$

Now, we denote the mapping

$$H : [0,1] \times [0,1] \to \mathbb{R}^+_{\mathcal{I}}$$

by

$$[\underline{u}, \overline{u}] = E_i^{\min}\left\{ [\underline{u_i}, \overline{u^i}] \right\}.$$

If there exists an interval $[\underline{v}, \overline{v}]$ such that

$$[\underline{v}, \overline{v}] = C_i^{\max}\left\{ [\underline{u_i}, \overline{u^i}] \right\}.$$

$$H(\mu,\nu) = \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f\Big(\mu s + (1-\mu)\frac{a+b}{2},\ \nu t + (1-\nu)\frac{c+d}{2}\Big) dsdt.$$

The next theorem generalizes the result of Dragomir ([34] [Theorem 2]).

Theorem 5. *Let $\Delta = [a,b] \times [c,d]$ and $\Delta_1 = [0,1] \times [0,1]$. Suppose that $f \in SX(ch, \Delta, \mathbb{R}^+_{\mathcal{I}})$. Then*

(1) $H \in SX(ch, \Delta_1, \mathbb{R}^+_{\mathcal{I}})$.
(2) We have :

$$E^{\min}_{(\mu,\nu)\in\Delta_1}\{H(\mu,\nu)\} = \frac{1}{k^2} f\Big(\frac{a+b}{2}, \frac{c+d}{2}\Big) = \frac{1}{k^2} H(0,0).$$

$$C^{\max}_{(\mu,\nu)\in\Delta_1}\{H(\mu,\nu)\} = h_{\Delta_1} \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(s,t) dsdt = h_{\Delta_1} H(1,1),$$

where

$$h_{\Delta_1} = \sup_{(\mu,\nu)\in\Delta_1} \left\{ h(\mu)h(\nu) + 2h(\tfrac{1}{2})\Big(h(\mu)h(1-\nu) + h(1-\mu)h(\nu) + 2h(1-\mu)h(1-\nu)h(\tfrac{1}{2})\Big) \right\}.$$

Proof. (1) Fix $\nu \in [0,1]$. Then for all $\alpha, \beta \geq 0$ with

$$\alpha + \beta = 1$$

and $\mu_1, \mu_2 \in [0,1]$, we have

$$\begin{aligned}
&H(\alpha\mu_1 + \beta\mu_2, \nu) \\
&= \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f\Big((\alpha\mu_1 + \beta\mu_2)s + (1 - (\alpha\mu_1 + \beta\mu_2))\frac{a+b}{2}, \nu t + (1-\nu)\frac{c+d}{2}\Big) dsdt \\
&= \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f\Big(\alpha\Big(\mu_1 s + (1-\mu_1)\frac{a+b}{2}\Big) + \beta\Big(\mu_2 s + (1-\mu_2)\frac{a+b}{2}\Big), \\
&\quad \nu t + (1-\mu)\frac{c+d}{2}\Big) dsdt \\
&\supseteq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d \Big(h(\alpha) f\Big(\mu_1 s + (1-\mu_1)\frac{a+b}{2}, \nu t + (1-\nu)\frac{c+d}{2}\Big) +
\end{aligned}$$

$$
\begin{aligned}
& h(\beta)f\Big(\mu_2 s+(1-\mu_2)\frac{a+b}{2},\nu t+(1-\nu)\frac{c+d}{2}\Big)\Big)dsdt \\
&= h(\alpha)\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f\Big(\mu_1 s+(1-\mu_1)\frac{a+b}{2},\nu t+(1-\nu)\frac{c+d}{2}\Big)dsdt+ \\
& h(\beta)\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f\Big(\mu_2 s+(1-\mu_2)\frac{a+b}{2},\nu t+(1-\nu)\frac{c+d}{2}\Big)dsdt \\
&= h(\alpha)H(\mu_1,\nu)+h(\beta)H(\mu_2,\nu).
\end{aligned}
$$

Similarly, we can have $H_\mu(\nu)$ is h-convex. This shows that

$$H\in SX(ch,\Delta_1,\mathbb{R}^+_{\mathcal{I}}).$$

(2) Since $f\in SX(ch,\Delta,\mathbb{R}^+_{\mathcal{I}})$, from Lemma 1 we have

$$
\begin{aligned}
H(\mu,\nu) &= \frac{1}{b-a}\int_a^b\left[\frac{1}{d-c}\int_c^d f\Big(\mu s+(1-\mu)\frac{a+b}{2},\nu t+(1-\nu)\frac{c+d}{2}\Big)dt\right]ds \\
&\subseteq \frac{1}{k(b-a)}\int_a^b f\Big(\mu s+(1-\mu)\frac{a+b}{2},\frac{1}{d-c}\int_c^d\Big[\nu t+(1-\nu)\frac{c+d}{2}\Big]dt\Big)ds \\
&= \frac{1}{k(b-a)}\int_a^b f\Big(\mu s+(1-\mu)\frac{a+b}{2},\frac{c+d}{2}\Big)ds \\
&\subseteq \frac{1}{k^2}f\Big(\frac{1}{b-a}\int_a^b\Big[\mu s+(1-\mu)\frac{a+b}{2}\Big]ds,\frac{c+d}{2}\Big) \\
&= \frac{1}{k^2}f\Big(\frac{a+b}{2},\frac{c+d}{2}\Big).
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
H(\mu,\nu) &\supseteq \frac{1}{(b-a)(d-c)}\int_a^b\Big[h(\nu)\int_c^d f\Big(\mu s+(1-\mu)\frac{a+b}{2},t\Big)dt \\
&\quad +h(1-\nu)\int_c^d f\Big(\mu s+(1-\mu)\frac{a+b}{2},\frac{c+d}{2}\Big)dt\Big]ds \\
&\supseteq \frac{1}{(b-a)(d-c)}\int_a^b\Big[h(\nu)\int_c^d\Big(h(\mu)f(s,t)+h(1-\mu)f\Big(\frac{a+b}{2},t\Big)\Big)dt \\
&\quad +h(1-\nu)\int_c^d\Big(h(\mu)f\Big(s,\frac{c+d}{2}\Big)+h(1-\mu)f\Big(\frac{a+b}{2},\frac{c+d}{2}\Big)dt\Big]ds \\
&= \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d\Big[h(\nu)h(\mu)f(s,t)dsdt+h(\nu)h(1-\mu)f\Big(\frac{a+b}{2},t\Big)dsdt \\
&\quad +h(1-\nu)h(\mu)f\Big(s,\frac{c+d}{2}\Big)dsdt+h(1-\nu)h(1-\mu)f\Big(\frac{a+b}{2},\frac{c+d}{2}\Big)dsdt\Big] \\
&= \frac{h(\nu)h(\mu)}{(b-a)(d-c)}\int_a^b\int_c^d f(s,t)dsdt+\frac{h(\nu)h(1-\mu)}{d-c}\int_c^d f\Big(\frac{a+b}{2},t\Big)dt \\
&\quad +\frac{h(1-\nu)h(\mu)}{b-a}\int_a^b f\Big(s,\frac{c+d}{2}\Big)ds+h(1-\nu)h(1-\mu)f\Big(\frac{a+b}{2},\frac{c+d}{2}\Big) \\
&\supseteq \frac{h(\mu)h(\nu)}{(b-a)(d-c)}\int_a^b\int_c^d f(s,t)dsdt+\frac{h(\nu)h(1-\mu)}{d-c}\int_c^d\Big(\frac{2h(\frac{1}{2})}{b-a}\int_a^b f(s,t)ds\Big)dt \\
&\quad +\frac{h(1-\nu)h(\mu)}{b-a}\int_a^b\Big(\frac{2h(\frac{1}{2})}{d-c}\int_c^d f(s,t)dt\Big)ds \\
&\quad +\frac{4h(1-\mu)h(1-\nu)\Big(h(\frac{1}{2})\Big)^2}{(b-a)(d-c)}\int_a^b\int_c^d f(s,t)dsdt \\
&= \Big[h(\mu)h(\nu)+2h(\frac{1}{2})\Big(h(\nu)h(1-\mu)+h(1-\nu)h(\mu)+2h(1-\nu)h(1-\mu)h(\frac{1}{2})\Big)\Big] \\
&\quad \cdot\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(s,t)dsdt.
\end{aligned}
$$

Let

$$h_{\Delta_1} = \sup_{(\mu,\nu)\in\Delta_1} \left\{ h(\mu)h(\nu) + 2h(\frac{1}{2})\Big(h(\mu)h(1-\nu) + h(1-\mu)h(\nu) + 2h(1-\mu)h(1-\nu)h(\frac{1}{2})\Big) \right\}.$$

The intended result follows. □

Remark 3. *If $h(t) = t$, then*

$$k = 1, h_{\Delta_1} = 1.$$

In addition, If $\underline{f} = \overline{f}$, then Theorem 5 reduces to Theorem 2 given by Dragomir [34].

Example 3. *Suppose that $\Delta = [0,2] \times [0,2]$, $h(t) = t$, and $\Delta_1 = [0,1] \times [0,1]$. Let $f : \Delta \to \mathbb{R}_{\mathcal{I}}$ be defined by*

$$f(s,t) = \Big[s^2t^2, 8\sqrt{st}\Big].$$

For $(\mu,\nu) \in (0,1] \times (0,1]$. Then $H \in SX(ch, \Delta_1, \mathbb{R}^+_{\mathcal{I}})$ and

$$\begin{aligned} H(\mu,\nu) &= \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f\Big(\mu s + (1-\mu)\frac{a+b}{2},\ \nu t + (1-\nu)\frac{c+d}{2}\Big) dsdt \\ &= \frac{1}{4}\int_0^2 \int_0^2 f\Big(\mu s + 1 - \mu, \nu t + 1 - \nu\Big) dsdt \\ &= \frac{1}{4}\int_0^2 \int_0^2 \Big[(\mu s + 1 - \mu)^2(\nu t + 1 - \nu)^2,\ 8\sqrt{(\mu s + 1 - \mu)(\nu t + 1 - \nu)}\Big] dsdt \\ &= \left[\frac{(6+2\mu^2)(6+2\nu^2)}{36}, \frac{8}{9\mu\nu}\Big((1+\mu)^{\frac{3}{2}} - (1-\mu)^{\frac{3}{2}}\Big) \cdot \Big((1+\nu)^{\frac{3}{2}} - (1-\nu)^{\frac{3}{2}}\Big)\right]. \end{aligned}$$

Furthermore, by the following Figures 1 and 2, we have

$$\inf_{(\mu,\nu)\in\Delta_1} \left\{ \frac{(6+2\mu^2)(6+2\nu^2)}{36} \right\} = 1,$$

$$\sup_{(\mu,\nu)\in\Delta_1} \left\{ \frac{(6+2\mu^2)(6+2\nu^2)}{36} \right\} = \frac{16}{9},$$

$$\inf_{(\mu,\nu)\in\Delta_1} \left\{ \frac{8}{9\mu\nu}\Big((1+\mu)^{\frac{3}{2}} - (1-\mu)^{\frac{3}{2}}\Big) \cdot \Big((1+\nu)^{\frac{3}{2}} - (1-\nu)^{\frac{3}{2}}\Big) \right\} = \frac{64}{9},$$

$$\sup_{(\mu,\nu)\in\Delta_1} \left\{ \frac{8}{9\mu\nu}\Big((1+\mu)^{\frac{3}{2}} - (1-\mu)^{\frac{3}{2}}\Big) \cdot \Big((1+\nu)^{\frac{3}{2}} - (1-\nu)^{\frac{3}{2}}\Big) \right\} = 8,$$

and

$$H(0,0) = [1,8], \quad H(1,1) = \left[\frac{16}{9}, \frac{64}{9}\right].$$

Then, we obtain

$$E^{\min}_{(\mu,\nu)\in\Delta_1}\{H(\mu,\nu)\} = f(1,1) = H(0,0).$$

$$\begin{aligned} C^{\max}_{(\mu,\nu)\in\Delta_1}\{H(\mu,\nu)\} &= h_{\Delta_1}\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(s,t) dsdt \\ &= \frac{1}{4}\int_0^2 \int_0^2 \Big[s^2t^2, 8\sqrt{st}\Big] dsdt \\ &= \left[\frac{16}{9}, \frac{64}{9}\right] \\ &= H(1,1). \end{aligned}$$

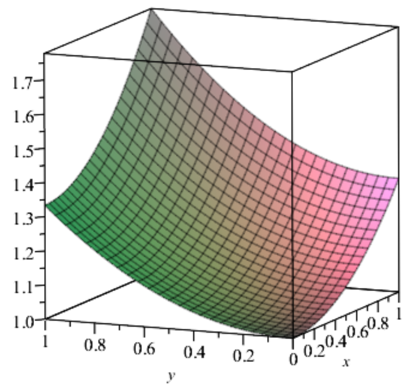

Figure 1. Illustration of Example 3: Let $f(s,t) = \left[s^2t^2, 8\sqrt{st}\right]$, $(\mu,\nu) \in \Delta_1 = [0,1]\times[0,1]$. then $\underline{H}(\mu,\nu) = \frac{(6+2\mu^2)(6+2\nu^2)}{36}$. From the above graph of the function $\underline{H}(\mu,\nu)$, we have $1 \leq \underline{H}(\mu,\nu) \leq \frac{16}{9}$.

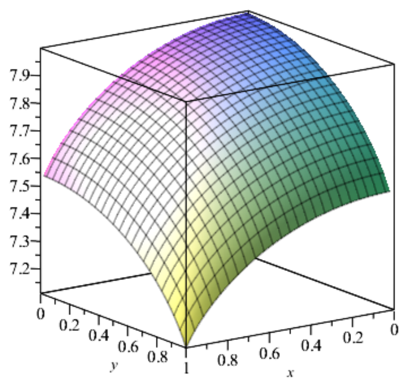

Figure 2. Illustration of Example 3: Let $f(s,t) = \left[s^2t^2, 8\sqrt{st}\right]$, $(\mu,\nu) \in \Delta_1 = [0,1]\times[0,1]$. then $\overline{H}(\mu,\nu) = \frac{8\left((1+\mu)^{\frac{3}{2}}-(1-\mu)^{\frac{3}{2}}\right)\cdot\left((1+\nu)^{\frac{3}{2}}-(1-\nu)^{\frac{3}{2}}\right)}{9\mu\nu}$. From the above graph of the function $\overline{H}(\mu,\nu)$, we have $\frac{64}{9} \leq \underline{H}(\mu,\nu) \leq 8$.

Remark 4. *Please note that if $h(t) = t$, then we have $k = 1$ and $h_{\Delta_1} = 1$, and we obtain*

$$E^{\min}_{(\mu,\nu)\in\Delta_1}\{H(\mu,\nu)\} = H(0,0),$$

and

$$C^{\max}_{(\mu,\nu)\in\Delta_1}\{H(\mu,\nu)\} = H(1,1).$$

The next theorem generalizes the result of Dragomir ([34] [Theorem 3]).

Theorem 6. *Suppose that $f \in SX(h,\Delta,\mathbb{R}^+_{\mathcal{I}})$. Then*
(1) $H \in SX(h,\Delta_1,\mathbb{R}^+_{\mathcal{I}})$.
(2) Define the mapping

$$\widetilde{h} : [0,1] \to \mathbb{R}^+_{\mathcal{I}},\ \widetilde{h}(\mu) = H(\mu,\mu).$$

Then $\widetilde{h} \in SX(h,[0,1],\mathbb{R}^+_{\mathcal{I}})$ and

$$E^{\min}_{\mu\in[0,1]}\{\widetilde{h}(\mu)\} = \frac{1}{k^2}\widetilde{h}(1),\quad \inf_{\mu\in[0,1]}\widetilde{h}(\mu) = h_{\Delta_1}\widetilde{h}(0),$$

where

$$h_{\Delta_1} = \sup_{(\mu,\mu)\in\Delta_1} \left\{ 4h(\frac{1}{2})\Big(h(\mu)h(1-\mu) + h^2(1-\mu)h(\frac{1}{2})\Big) + h^2(\mu) \right\}.$$

Proof. (1) Suppose that

$$(\mu_1, \nu_1), (\mu_2, \nu_2) \in \Delta_1.$$

Then for all $\alpha \in [0,1]$, we have

$$\begin{aligned}
&H(\alpha(\mu_1,\nu_1) + (1-\alpha)(\mu_2,\nu_2)) \\
&= H(\alpha\mu_1 + (1-\alpha)\mu_2, \alpha\nu_1 + (1-\alpha)\nu_2) \\
&= \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f\Big((\alpha\mu_1 + (1-\alpha)\mu_2)s + [1-(\alpha\mu_1 + (1-\alpha)\mu_2)]\frac{a+b}{2}, \\
&\quad (\alpha\nu_1 + (1-\alpha)\nu_2)t + [1-(\alpha\nu_1 + (1-\alpha)\nu_2)]\frac{c+d}{2}\Big) dsdt \\
&= \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f\Big(\alpha\Big(\mu_1 s + (1-\mu_1)\frac{a+b}{2}, \nu_1 t + (1-\nu_1)\frac{c+d}{2}\Big) \\
&\quad + (1-\alpha)\Big(\mu_2 s + (1-\mu_2)\frac{a+b}{2}, \nu_2 t + (1-\nu_2)\frac{c+d}{2}\Big) dsdt \\
&\supseteq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d \Bigg(h(\alpha) f\Big(\mu_1 s + (1-\mu_1)\frac{a+b}{2}, \nu_1 t + (1-\nu_1)\frac{c+d}{2}\Big) \\
&\quad + h(1-\alpha) f\Big(\mu_2 s + (1-\mu_2)\frac{a+b}{2}, \nu_2 t + (1-\nu_2)\frac{c+d}{2}\Big)\Bigg) dsdt \\
&= h(\alpha)\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f\Big(\mu_1 s + (1-\mu_1)\frac{a+b}{2}, \nu_1 t + (1-\nu_1)\frac{c+d}{2}\Big) dsdt \\
&\quad + h(1-\alpha)\frac{1}{(b-a)(d-c)} \cdot \int_a^b \int_c^d f\Big(\mu_2 s + (1-\mu_2)\frac{a+b}{2}, \nu_2 t + (1-\nu_2)\frac{c+d}{2}\Big) dsdt \\
&= h(\alpha)H(\mu_1,\nu_1) + h(1-\alpha)H(\mu_2,\nu_2).
\end{aligned}$$

This shows that $H \in SX(h, \Delta_1, \mathbb{R}^+_{\mathcal{I}})$.

(2) Let $\mu_1, \mu_2 \in [0,1]$. Then for all $\alpha \in [0,1]$, we have

$$\begin{aligned}
\widetilde{h}(\alpha\mu_1 + (1-\alpha)\mu_2) &= H(\alpha\mu_1 + (1-\alpha)\mu_2, \alpha\mu_1 + (1-\alpha)\mu_2) \\
&= H(\alpha(\mu_1,\mu_1) + (1-\alpha)(\mu_2,\mu_2)) \\
&\supseteq h(\alpha)H(\mu_1,\mu_1) + h(1-\alpha)H(\mu_2,\mu_2) \\
&= h(\alpha)\widetilde{h}(\mu_1) + h(1-\alpha)\widetilde{h}(\mu_2).
\end{aligned}$$

Then

$$\widetilde{h} \in SX(h, [0,1], \mathbb{R}^+_{\mathcal{I}}).$$

From Theorem 5, we obtain

$$E^{\min}_{\mu\in[0,1]}\{\widetilde{h}(\mu)\} = \frac{1}{k^2}\widetilde{h}(1), \ \inf_{\mu\in[0,1]} \widetilde{h}(\mu) = h_{\Delta_1}\widetilde{h}(0),$$

where

$$h_{\Delta_1} = \sup_{(\mu,\mu)\in\Delta_1} \left\{ 4h(\frac{1}{2})\Big(h(\mu)h(1-\mu) + h^2(1-\mu)h(\frac{1}{2})\Big) + h^2(\mu) \right\}.$$

and the result follows. □

Remark 5. *If* $h(t) = t$, *then*

$$k = 1, h_{\Delta_1} = 1.$$

In addition, If $\underline{f} = \overline{f}$, *then Theorem 6 reduces to Theorem 3 given by Dragomir [34].*

4. Conclusions

The classical Hermite–Hadamard inequality is one of the most named inequalities and it is deeply connected with the research of convex analysis and inequality theory. This inequality and some generalizations have been exhaustively explored in the past few decades. On the other hand, interval-valued functions play an important role in interval optimization, interval differential equations, among others filed of mathematics.

In this work, we introduced coordinated h-convexity for interval-valued functions. With the help of minimum expansion and maximum contraction of interval sequences, we established some Hermite–Hadamard-type inequalities for interval-valued functions. Since these main inequalities are given using new assumptions than those used in the previous research articles [10,15,34], our results are original and more general. As a future research direction, we intend to use more general convexity to investigate new interval Hermite–Hadamard-type inequalities, and some potential applications in interval optimization.

Author Contributions: Investigation, G.Y. and D.Z.; methodology, G.Z. and S.S.D.; resources, W.L.; writing–original draft, D.Z.; writing—review and editing, D.Z. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by Key Projects of Educational Commission of Hubei Province of China grant number D20192501, Philosophy and Social Sciences of Educational Commission of Hubei Province of China grant number 20Y109 and Natural Science Foundation of Jiangsu Province grant number BK20180500.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The authors are grateful to four anonymous reviewers for several valuable comments, which helped them to improve the manuscript.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Kórus, P. An extension of the Hermite-Hadamard inequality for convex and s-convex functions. *Aequ. Math.* **2019**, *93*, 527–534. [CrossRef]
2. Latif, M.A. On some new inequalities of Hermite-Hadamard type for functions whose derivatives are s-convex in the second sense in the absolute value. *Ukr. Math. J.* **2016**, *67*, 1552–1571. [CrossRef]
3. Sarikaya, M.Z.; Kiris, M.E. Some new inequalities of Hermite-Hadamard type for s-convex functions. *Miskolc Math. Notes* **2015**, *16*, 491–501. [CrossRef]
4. Set, E.; İşcan, İ.; Kara, H.H. Hermite-Hadamard-Fejer type inequalities for s-convex function in the second sense via fractional integrals. *Filomat* **2016**, *30*, 3131–3138. [CrossRef]
5. Dragomir, S.S. New inequalities of Hermite-Hadamard type for log-convex functions. *Khayyam J. Math.* **2017**, *3*, 98–115.
6. Niculescu, C.P. The Hermite-Hadamard inequality for log-convex functions. *Nonlinear Anal.* **2012**, *75*, 662–669. [CrossRef]
7. Set, E.; Ardıç, M.A. Inequalities for log-convex functions and P-functions. *Miskolc Math. Notes* **2017**, *18*, 1033–1041. [CrossRef]
8. Mohsen, B.B.; Awan, M.U.; Noor, M.A.; Riahi, L.; Noor, K.I.; Almutairi, B. New quantum Hermite-Hadamard inequalities utilizing harmonic convexity of the functions. *IEEE Access* **2019**, *7*, 20479–20483. [CrossRef]
9. Varošanec, S. On h-convexity. *J. Math. Anal. Appl.* **2007**, *326*, 303–311. [CrossRef]
10. Latif, M.A.; Alomari, M. On Hadamard-type inequalities for h-convex functions on the co-ordinates. *Int. J. Math. Anal.* **2009**, *3*, 1645–1656.
11. Matłoka, M. On Hadamard's inequality for h-convex function on a disk. *Appl. Math. Comput.* **2014**, *235*, 118–123. [CrossRef]
12. Mihai, M.V.; Noor, M.A.; Awan, M.U. Trapezoidal like inequalities via harmonic h-convex functions on the co-ordinates in a rectangle from the plane. *RACSAM* **2017**, *111*, 257–262. [CrossRef]
13. Mihai, M.V.; Noor, M.A.; Noor, K.I.; Awan, M.U. Some integral inequalities for harmonic h-convex functions involving hypergeometric functions. *Appl. Math. Comput.* **2015**, *252*, 257–262. [CrossRef]
14. Noor, M.A.; Noor, K.I.; Awan, M.U. A new Hermite-Hadamard type inequality for h-convex functions. *Creat. Math. Inform.* **2015**, *24*, 191–197.

15. Sarikaya, M.Z.; Saglam, A.; Yildirim, H. On some Hadamard-type inequalities for h-convex functions. *J. Math. Inequal.* **2008**, *2*, 335–341. [CrossRef]
16. Sarikaya, M.Z.; Set, E.; Özdemir, M.E. On some new inequalities of Hadamard-type involving h-convex functions. *Acta Math. Univ. Comenian LXXIX* **2010**, *2*, 265–272.
17. Burkill, J.C. Functions of Intervals. *Proc. Lond. Math. Soc.* **1924**, *22*, 275–310. [CrossRef]
18. Kolmogorov, A.N. Untersuchungen über Integralbegriff. *Math. Ann.* **1930**, *103*, 654–696. [CrossRef]
19. Moore, R.E. *Interval Analysis*; Prentice-Hall, Inc.: Englewood Cliffs, NJ, USA, 1966.
20. Nikodem, K.; Snchez, J.L.; Snchez, L. Jensen and Hermite-Hadamard inequalities for strongly convex set-valued maps. *Math. Aeterna* **2014**, *4*, 979–987.
21. Budak, H.; Tunç, T.; Sarikaya, M.Z. Fractional Hermite-Hadamard-type inequalities for interval-valued functions. *Proc. Am. Math. Soc.* **2020**, *148*, 705–718. [CrossRef]
22. Chalco-Cano, Y.; Flores-Franulič, A.; Román-Flores, H. Ostrowski type inequalities for interval-valued functions using generalized Hukuhara derivative. *Comput. Appl. Math.* **2012**, *31*, 457–472.
23. Chalco-Cano, Y.; Lodwick, W.A.; Condori-Equice, W. Ostrowski type inequalities and applications in numerical integration for interval-valued functions. *Soft Comput.* **2015**, *19*, 3293–3300. [CrossRef]
24. Costa, T.M.; Silva, G.N.; Chalco-Cano, Y.; Román-Flores, H. Gauss-type integral inequalities for interval and fuzzy-interval-valued functions. *Comput. Appl. Math.* **2019**, *38*, 13. [CrossRef]
25. Costa, T.M. Jensen's inequality type integral for fuzzy-interval-valued functions. *Fuzzy Sets Syst.* **2017**, *327*, 31–47. [CrossRef]
26. Costa, T.M.; Román-Flores, H.; Chalco-Cano, Y. Opial-type inequalities for interval-valued functions. *Fuzzy Sets Syst.* **2019**, *358*, 48–63. [CrossRef]
27. Román-Flores, H.; Chalco-Cano, Y.; Silva, G.N. A note on Gronwall type inequality for interval-valued functions. In Proceedings of the IFSA World Congress and NAFIPS Annual Meeting IEEE, Edmonton, AB, Canada, 24–28 June 2013; pp. 1455–1458.
28. Román-Flores, H.; Chalco-Cano, Y.; Lodwick, W.A. Some integral inequalities for interval-valued functions. *Comput. Appl. Math.* **2018**, *37*, 1306–1318. [CrossRef]
29. Flores-Franulič, A.; Chalco-Cano, Y.; Román-Flores, H. An Ostrowski type inequality for interval-valued functions. In Proceedings of the IFSA World Congress and NAFIPS Annual Meeting IEEE, Edmonton, AB, Canada, 24–28 June 2013; pp. 1459–1462.
30. Zhao, D.F.; An, T.Q.; Ye, G.J.; Liu, W. New Jensen and Hermite-Hadamard type inequalities for h-convex interval-valued functions. *J. Inequal. Appl.* **2018**, *2018*, 302. [CrossRef]
31. Zhao, D.F.; Ye, G.J.; Liu, W.; Torres, D.F.M. Some inequalities for interval-valued functions on time scales. *Soft Comput.* **2019**, *23*, 6005–6015. [CrossRef]
32. Zhao, D.F.; An, T.Q.; Ye, G.J.; Liu, W. Chebyshev type inequalities for interval-valued functions. *Fuzzy Sets Syst.* **2020**, *396*, 82–101. [CrossRef]
33. Zhao, D.F.; An, T.Q.; Ye, G.J.; Torres, D.F.M. On Hermite-Hadamard type inequalities for harmonical h-convex interval-valued functions. *Math. Inequal. Appl.* **2020**, *23*, 95–105.
34. Dragomir, S.S. On the Hadamard's inequality for convex functions on the co-ordinates in a rectangle from the plane. *Taiwan. J. Math.* **2001**, *5*, 775–788. [CrossRef]
35. Moore, R.E.; Kearfott, R.B.; Cloud, M.J. *Introduction to Interval Analysis*; SIAM: Philadelphia, PA, USA, 2009.