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Mathematical Modelling and Analysis
Volume 27, Issue 1, 1–18, 2022
https://doi.org/10.3846/mma.2022.13100

Evaluating Log-Tangent Integrals via Euler Sums

Anthony Sofo

College of Engineering and Science, Victoria University
Ballarat Rd., 8001 Melbourne, Australia
E-mail(*corresp.*): anthony.sofo@vu.edu.au

Received August 24, 2020; revised October 18, 2021; accepted October 18, 2021

Abstract. An investigation into the representation of integrals involving the product of the logarithm and the arctan functions, reducing to log-tangent integrals, will be undertaken in this paper. We will show that in many cases these integrals take an explicit form involving the Riemann zeta function, the Dirichlet eta function, Dirichlet lambda function and many other special functions. Some examples illustrating the theorems will be detailed.

Keywords: Dirichlet beta functions, log-tangent integral, Euler sums, Dirichlet lambda function, zeta functions.

AMS Subject Classification: 11M06; 11M35; 26B15; 33B15; 42A70; 65B10.

1 Introduction, preliminaries and notation

In this paper we investigate the representations of integrals of the type

$$I(\delta, a, b, p) = \int_0^1 \frac{x^a \ln^p(x)}{1+\delta x^2} \arctan\left(x^b\right) dx, \tag{1.1}$$

$$J(\delta, a, b, p) = \int_0^\infty \frac{x^a \ln^p(x)}{1+\delta x^2} \arctan\left(x^b\right) dx \tag{1.2}$$

in terms of special functions such as zeta functions, Dirichlet eta functions, beta functions and others. We note that $\delta = \pm 1, a$ and b are fixed real numbers and $p \in \mathbb{N}$ for the set of complex numbers $\mathbb{C}$, natural numbers $\mathbb{N}$, the set of real numbers $\mathbb{R}$ and the set of positive real numbers, $\mathbb{R}^+$. In fact, for $\delta = \pm 1, p \in \mathbb{N}$ and $b \in \mathbb{R}^+$, the integral $I(\delta, a, b, p)$ converges if and only if $a + b > -1$; also the integral $J(\delta, a, b, p)$ converges if and only if $a + b > -1$ and $a < 1$. In the two papers [9] and [8] the authors investigate log-tangent integrals and the Riemann zeta function. In the paper [9], they showed that for any square integrable function $f \in (0, \pi/2)$ the integral

$$L(f) := \int_0^{\frac{\pi}{2}} f(x) \ln(\tan x)\, dx$$

can be approximated by a finite sum involving the Riemann zeta function at odd positive integers. The authors prove that for any polynomial $P(x)$

$$\int_0^{\frac{\pi}{2}} P\left(\frac{2x}{\pi}\right) \ln(\tan x)\, dx = \sum_{k=1}^{\left[\frac{1}{2}(\deg P+1)\right]} \frac{(-1)^{k+1}}{\pi^{2k-1}} c_k(P)\, \zeta(2k+1),$$

where

$$c_k(P) = \left(1 - 2^{-1-2k}\right)\left(P^{(2k-1)}(1) + P^{(2k-1)}(0)\right)$$

and $P^{(k)}(x)$ denotes the k^{th} derivative of P at the point x. In the second paper [8], the authors show that the integrals involving the log-tangent function can be evaluated by some series involving the harmonic numbers, H_n. For certain square integrable functions $f \in (0, \pi/2)$, then

$$L(f) = -\sum_{n\geq 1} \frac{h_n}{n} b_n(f),$$

where

$$b_n(f) := \langle f, \sin(4n\cdot)\rangle = \frac{4}{\pi}\int_0^{\frac{\pi}{2}} f(x)\sin(4nx)\, dx,\;\; h_n = H_{2n} - \frac{1}{2}H_n = \sum_{j=1}^{n} \frac{1}{2j-1}.$$

The authors also point out that integrals involving log-tangent functions have important applications in many fields of mathematics including evaluation of classical, semi classical and quantum entropies of position and momentum. We can see that (1.1) and (1.2) reduce to

$$\int_0^1 \frac{x^a \ln^p(x)}{1+x^2} \arctan(x)\, dx = \int_0^{\frac{\pi}{4}} \theta\, (\tan\theta)^a \ln^p(\tan\theta)\, d\theta,$$
$$\int_0^\infty \frac{x^a \ln^p(x)}{1+x^2} \arctan(x)\, dx = \int_0^{\frac{\pi}{2}} \theta\, (\tan\theta)^a \ln^p(\tan\theta)\, d\theta.$$

The following notation and results will be useful in the subsequent sections of this paper. The generalized p-order harmonic numbers, $H_n^{(p)}(\alpha, \beta)$ are defined as the partial sums of the modified Hurwitz zeta function

$$\zeta(p, \alpha, \beta) = \sum_{n\geq 0} \frac{1}{(\alpha n + \beta)^p}.$$

The classical Hurwitz zeta function

$$\zeta(p, a) = \sum_{n\geq 0} \frac{1}{(n+a)^p}$$

for $Re\,(p)>1$ and by analytic continuation to other values of $p\neq 1$, where any term of the form $(n+a)=0$ is excluded. Therefore,

$$H_n^{(p)}\left(\alpha,\beta\right)=\sum_{j=1}^{n}\frac{1}{\left(\alpha j+\beta\right)^{p}}$$

and the "ordinary" p-order harmonic numbers $H_n^{(p)}=H_n^{(p)}\left(1,0\right).$ The Catalan constant

$$G=\beta\left(2\right)=\sum_{n=1}^{\infty}\frac{\left(-1\right)^{n+1}}{\left(2n-1\right)^{2}}\approx 0.91597$$

is a special case of the Dirichlet beta function, see [1]

$$\begin{aligned}\beta\left(z\right)=&\sum_{n=1}^{\infty}\frac{\left(-1\right)^{n+1}}{\left(2n-1\right)^{z}},\ \text{for } Re\left(z\right)>0 \\ =&\frac{1}{\left(-2\right)^{2z}\Gamma\left(z\right)}\left(\psi^{(z-1)}\left(\frac{1}{4}\right)-\psi^{(z-1)}\left(\frac{3}{4}\right)\right),\end{aligned} \tag{1.3}$$

where the Gamma function

$$\Gamma\left(z\right)=\int_{0}^{+\infty}t^{z-1}e^{-t}dt$$

and with functional equation

$$\beta\left(1-z\right)=\left(\frac{2}{\pi}\right)^{z}\sin\left(\frac{\pi z}{2}\right)\Gamma\left(z\right)\beta\left(z\right)$$

extending the Dirichlet Beta function to the left hand side of the complex plane $\mathrm{Re}(z)\leq 0$. For the odd case $z=2q+1, q\in\mathbb{N}_0$

$$\beta\left(2q+1\right)=\sum_{n=1}^{\infty}\frac{\left(-1\right)^{n+1}}{\left(2n-1\right)^{2q+1}}=\frac{\left(-1\right)^{q}E_{2q}}{(2q)!2^{2q+2}}\pi^{2q+1},$$

where E_q are the Euler numbers, that is the integer numbers obtained as the coefficients of $z^q/q!$ in the Taylor series expansion of $\mathrm{sech}(z)\,, |z|<\pi/2$. For the even case, utilizing the following from Kölbig [13]

$$\begin{aligned}\psi^{(2q-1)}(\frac{1}{4})+\psi^{(2q-1)}(\frac{3}{4})=&2^{4q-1}\left(2^{2q}-1\right)\pi^{2q}\frac{|B_{2q}|}{2q},\\ \psi^{(2q-1)}(\frac{1}{4})-\psi^{(2q-1)}(\frac{3}{4})=&2^{4q}\left(2q-1\right)!\beta\left(2q\right),\end{aligned}$$

we have upon adding,

$$\beta\left(2q\right)=\sum_{n\geq 1}\frac{\left(-1\right)^{n+1}}{\left(2n-1\right)^{2q}}=\frac{\psi^{(2q-1)}\left(\frac{1}{4}\right)}{2^{4q-1}\left(2q-1\right)!}+\frac{\left(-1\right)^{q}\left(2^{2q}-1\right)B_{2q}}{2\left(2q\right)!}\pi^{2q},$$

where B_n are the Bernoulli numbers, the rational coefficients of $z^q/q!$ in the Taylor expansion of $z/(e^z-1)$, $|z|<2\pi$. The Lerch transcendent,

$$\Phi(z,t,a)=\sum_{m=0}^{\infty}\frac{z^m}{(m+a)^t}$$

is defined for $|z|<1$ and $Re\,(a)>0$ and satisfies the recurrence

$$\Phi(z,t,a)=z\,\Phi(z,t,a+1)+a^{-t}.$$

It is known that the Lerch transcendent extends by analytic continuation to a function $\Phi(z,t,a)$ which is defined for all complex t, $z\in\mathbb{C}-[1,\infty)$ and $a>0$. The Lerch transcendent generalizes the Hurwitz zeta function at $z=1$,

$$\Phi(1,t,a)=\zeta(t,a)=\sum_{m=0}^{\infty}\frac{1}{(m+a)^t}. \tag{1.4}$$

The Dirichlet lambda function, $\lambda(p)$ is

$$2\lambda(p)=\zeta(p)+\eta(p),$$

where

$$\eta(p)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^p}=\left(1-2^{1-p}\right)\zeta(p)$$

is the alternating zeta function and $\eta(1)=\ln 2$. We know that for $n\geq 1$, $\psi(n+1)-\psi(1)=H_n$ with $\psi(1)=-\gamma$, where γ is the Euler Mascheroni constant and $\psi(n)$ is the digamma function. For real values of x, $\psi(x)$ is the digamma (or psi) function defined by

$$\psi(x):=\frac{d}{dx}\{\log\Gamma(x)\}=\frac{\Gamma'(x)}{\Gamma(x)},$$

$$\psi(x)=-\gamma-\frac{1}{x}+\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{x+n}\right)=-\gamma+\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{x+n-1}\right),$$

leading to the telescoping sum:

$$\psi(1+x)-\psi(x)=\sum_{n=1}^{\infty}\left(\frac{1}{x+n-1}-\frac{1}{x+n}\right)=\frac{1}{x}.$$

The polygamma function

$$\psi^{(k)}(z)=\frac{d^k}{dz^k}\{\psi(z)\}=(-1)^{k+1}\,k!\sum_{r=0}^{\infty}\frac{1}{(r+z)^{k+1}}$$

and has the recurrence

$$\psi^{(k)}(z+1)=\psi^{(k)}(z)+\frac{(-1)^k\,k!}{z^{k+1}}.$$

The connection of the polygamma function with harmonic numbers is,

$$H_z^{(m+1)} = \zeta(m+1) + \frac{(-1)^m}{m!}\psi^{(m)}(z+1)$$
$$= \frac{(-1)^m}{m!}\int_0^1 \frac{(1-t^z)}{1-t}\ln^m(t)\ dt,\ z \neq \{-1,-2,-3,...\} \tag{1.5}$$

and the multiplication formula is, see [25]

$$\psi^{(k)}(pz) = \delta_{k,0}\ln p + \frac{1}{p^{k+1}}\sum_{j=0}^{p-1}\psi^{(k)}(z+\frac{j}{p}) \tag{1.6}$$

for p a positive integer and $\delta_{p,k}$ is the Kronecker delta. The following lemma will be useful.

Lemma 1. *We define*

$$W(3) := \sum_{n\geq 1}\frac{\sin\left(\frac{n\pi}{4}\right)}{2^{\frac{n}{2}}\ n^3} = \sum_{n\geq 1}\frac{(-1)^{n+1}}{2^{2n}}\left(\frac{2}{(4n-3)^3}+\frac{2}{(4n-2)^3}+\frac{1}{(4n-1)^3}\right)$$
$$= \sum_{n\geq 1}\frac{1}{2^{4n}}\begin{pmatrix}\frac{8}{(8n-7)^3}+\frac{8}{(8n-6)^3}+\frac{4}{(8n-5)^3}\\ -\frac{2}{(8n-3)^3}-\frac{2}{(8n-2)^3}-\frac{1}{(8n-1)^3}\end{pmatrix} \tag{1.7}$$
$$= \frac{1}{256}\left(2\Phi\left(-\frac{1}{4},3,\frac{1}{4}\right)+2\Phi\left(-\frac{1}{4},3,\frac{1}{2}\right)+\Phi\left(-\frac{1}{4},3,\frac{3}{4}\right)\right),$$

where $\Phi(\cdot,\cdot,\cdot)$ is the Lerch transcendent.

Proof. Consider the polylogarithms

$$\mathrm{Li}_3\left(\frac{1\pm i}{2}\right) = \sum_{n\geq 1}\frac{\exp\left(\pm\frac{in\pi}{4}\right)}{2^{\frac{n}{2}}\ n^3} = \sum_{n\geq 1}\frac{\cos\left(\frac{n\pi}{4}\right)\pm i\sin\left(\frac{n\pi}{4}\right)}{2^{\frac{n}{2}}\ n^3},$$

adding produces

$$\mathrm{Li}_3\left(\frac{1+i}{2}\right)+\mathrm{Li}_3\left(\frac{1-i}{2}\right) = 2\sum_{n\geq 1}\frac{\cos\left(\frac{n\pi}{4}\right)}{2^{\frac{n}{2}}\ n^3}$$
$$= \frac{35}{32}\zeta(3)+\frac{1}{24}\ln^3 2-\frac{5}{16}\zeta(2)\ln 2,$$

by a result from [11]. By the circular nature of the trigonometric function, the BBP type sum for

$$\sum_{n\geq 1}\frac{\cos\left(\frac{n\pi}{4}\right)}{2^{\frac{n}{2}}\ n^3} = \sum_{n\geq 1}\frac{(-1)^{n+1}}{2^{2n}}\left(\frac{2}{(4n-3)^3}-\frac{1}{(4n-1)^3}-\frac{1}{(4n)^3}\right)$$
$$= \frac{1}{256}\left(2\Phi\left(-\frac{1}{4},3,\frac{1}{4}\right)-\Phi\left(-\frac{1}{4},3,\frac{3}{4}\right)-\Phi\left(-\frac{1}{4},3,1\right)\right)$$

$$=\sum_{n\geq 1}\frac{1}{2^{4n}}\left(\begin{array}{c}\frac{8}{(8n-7)^3}+\frac{1}{(8n-1)^3}+\frac{1}{(8n)^3}\\ \\ -\frac{4}{(8n-5)^3}-\frac{4}{(8n-4)^3}-\frac{2}{(8n-3)^3}\end{array}\right).$$

By a similar argument we have

$$\mathrm{Li}_3(\frac{1+i}{2})-\mathrm{Li}_3(\frac{1-i}{2})=2i\sum_{n\geq 1}\frac{\sin\left(\frac{n\pi}{4}\right)}{2^{\frac{n}{2}}\ n^3}$$

and the

$$\begin{aligned}\Im\left(\mathrm{Li}_3(\frac{1+i}{2})\right)&=\sum_{n\geq 1}\frac{\sin\left(\frac{n\pi}{4}\right)}{2^{\frac{n}{2}}\ n^3}\\ &=\sum_{n\geq 1}\frac{(-1)^{n+1}}{2^{2n}}\left(\frac{2}{(4n-3)^3}+\frac{2}{(4n-2)^3}+\frac{1}{(4n-1)^3}\right)\\ &=\sum_{n\geq 1}\frac{1}{2^{4n}}\left(\begin{array}{c}\frac{8}{(8n-7)^3}+\frac{8}{(8n-6)^3}+\frac{4}{(8n-5)^3}\\ \\ -\frac{2}{(8n-3)^3}-\frac{2}{(8n-2)^3}-\frac{1}{(8n-1)^3}\end{array}\right)\\ &=\frac{1}{256}\left(2\Phi\left(-\frac{1}{4},3,\frac{1}{4}\right)+2\Phi\left(-\frac{1}{4},3,\frac{1}{2}\right)+\Phi\left(-\frac{1}{4},3,\frac{3}{4}\right)\right).\end{aligned}$$

□

The following Lemma deals with the representation of an alternating Euler sum and utilizes the result from Lemma 1.

Lemma 2.

$$\sum_{n\geq 1}\frac{(-1)^{n+1}H_n^{(2)}}{2n-1}=\pi-\frac{\pi^2}{12}-2\ln 2+2G\ln 2-\frac{11\pi^3}{96}-\frac{\pi}{8}\ln^2 2+4W(3).$$

Proof. From Knopp [12], using the definition of the Polylogarithm

$$\mathrm{Li}_t(z)=\sum_{n\geq 1}\frac{z^n}{n^t},$$

which is valid for $t\in\mathbb{C}$ and $|z|<1$ and can be extended to $|z|\geq 1$ by analytic continuation. We have, by series expansion

$$\frac{\mathrm{Li}_2(z)}{1-z}=\sum_{n\geq 0}H_n^{(2)}z^n$$

and replacing $z:=-x^2$, we get

$$\frac{\mathrm{Li}_2\left(-x^2\right)}{1+x^2}=\sum_{n\geq 0}(-1)^n H_n^{(2)}x^{2n}.$$

Integrating the LHS by parts, for $x \in (0,1)$, or using a mathematical package, reduces to

$$\sum_{n\geq 0} \frac{(-1)^n H_n^{(2)}}{2n+1} = 2G\ln 2 - \frac{11\pi^3}{96} - \frac{\pi}{8}\ln^2 2 + i\left(\frac{35}{16}\zeta(3) + \frac{1}{12}\ln^3 2 - \frac{5}{8}\zeta(2)\ln 2 - 4\mathrm{Li}_3\left(\frac{1+i}{2}\right)\right).$$

A shift in the counter n results in

$$\sum_{n\geq 1} \frac{(-1)^{n+1} H_n^{(2)}}{2n-1} = \pi - \frac{\pi^2}{12} - 2\ln 2 + 2G\ln 2 - \frac{11\pi^3}{96} - \frac{\pi}{8}\ln^2 2 + i\left(\frac{35}{16}\zeta(3) + \frac{1}{12}\ln^3 2 - \frac{5}{8}\zeta(2)\ln 2 - 4\mathrm{Li}_3\left(\frac{1+i}{2}\right)\right)$$

and making use of Lemma 1, we have that

$$\sum_{n\geq 1} \frac{(-1)^{n+1} H_n^{(2)}}{2n-1} = \pi - \frac{\pi^2}{12} - 2\ln 2 + 2G\ln 2 - \frac{11\pi^3}{96} - \frac{\pi}{8}\ln^2 2 + 4W(3).$$

□

We expect that integrals of the type (1.1) and (1.2) may be represented by Euler sums and therefore in terms of special functions such as the Riemann zeta function. A search of the current literature has found some examples for the representation of the log tangent integral in terms of Euler sums, see [2,8,9] and [15]. The following papers, [22,23] also examined some integrals in terms of Euler sums. Some other important sources of information on log-tangent integrals and Euler sums are the works in the excellent books [14,26] and [27]. Other useful references related to the representation of Euler sums in terms of special functions include [3,4,6,17,18,19,22,24]. Some examples will be given highlighting specific cases of the integrals, some of which are not amenable to a computer mathematical package.

2 Analysis of integrals

Theorem 1. *Let $\delta = 1, b \in \mathbb{R}^+, p \in \mathbb{N}$ and for $a+b > -1$, the following integral,*

$$I(1,a,b,p) = \int_0^1 \frac{x^a}{1+x^2} \ln^p(x) \arctan\left(x^b\right) dx = \frac{(-1)^p p!}{2^{2p+2}} \sum_{n\geq 1} \frac{(-1)^{n+1}}{(2n-1)} \left(H^{(p+1)}_{\frac{2bn+a-b-1}{4}} - H^{(p+1)}_{\frac{2bn+a-b-3}{4}}\right), \quad (2.1)$$

where $H^{(p+1)}_{\frac{2bn+a-b-1}{4}}$ are harmonic numbers of order $p+1$.

Proof. Applying the Taylor series expansion we can write

$$\int_0^1 \frac{x^a}{1+x^2}\arctan\left(x^b\right)dx = \int_0^1 \sum_{n\geq 1}\frac{(-1)^{n+1}}{(2n-1)}\sum_{j\geq 0}(-1)^j\, x^{2nb+2j+a-b}\; dx,$$

for $x \in (0,1)$, by Fubini's theorem, see [5], we are assured convergence and upon reversing the order of integration and summation we have

$$\int_0^1 \frac{x^a}{1+x^2}\arctan\left(x^b\right)dx = \sum_{n\geq 1}\frac{(-1)^{n+1}}{(2n-1)}\sum_{j\geq 0}(-1)^j\int_0^1 x^{2nb+2j+a-b}\; dx$$

$$=\sum_{n\geq 1}\frac{(-1)^{n+1}}{(2n-1)}\sum_{j\geq 0}\frac{(-1)^j}{(2bn+2j+a-b+1)}$$

$$=\frac{1}{2}\sum_{n\geq 1}\frac{(-1)^{n+1}}{(2n-1)}\Phi\left(-1,1,\frac{1}{2}(2bn+a-b+1)\right)$$

$$=\frac{1}{4}\sum_{n\geq 1}\frac{(-1)^{n+1}}{(2n-1)}\left(\psi\left(\frac{2bn+a-b+3}{4}\right)-\psi\left(\frac{2bn+a-b+1}{4}\right)\right).$$

Applying the Leibnitz differentiation rule, we have

$$\frac{d^p}{da^p}\int_0^1 \frac{x^a}{1+x^2}\arctan\left(x^b\right)dx = \int_0^1 \frac{x^a\ln^p(x)}{1+x^2}\arctan\left(x^b\right)dx$$

$$=\frac{1}{2^{2p+2}}\sum_{n\geq 1}\frac{(-1)^{n+1}}{(2n-1)}\left(\psi^{(p)}\left(\frac{2bn+a-b+3}{4}\right)-\psi^{(p)}\left(\frac{2bn+a-b+1}{4}\right)\right).$$

From the relation (1.5) we have,

$$\int_0^1\frac{x^a\ln^p(x)}{1+x^2}\arctan\left(x^b\right)dx=\frac{(-1)^p\,p!}{2^{2p+2}}\sum_{n\geq 1}\frac{(-1)^{n+1}}{(2n-1)}\left(H^{(p+1)}_{\frac{2bn+a-b-1}{4}}-H^{(p+1)}_{\frac{2bn+a-b-3}{4}}\right)$$

and Theorem 1 follows. □

The case $b=1$ gives us the following interesting relation.

Corollary 1. Let the conditions of Theorem 1 hold then, for $b=1$

$$I(1,a,1,p)=\int_0^1\frac{x^a\ln^p(x)}{1+x^2}\arctan(x)\,dx \;=\int_0^{\frac{\pi}{4}}\theta\,(\tan\theta)^a\ln^p(\tan\theta)\,d\theta$$

$$=(-1)^p\,p!\sum_{n\geq 1}\frac{(-1)^{n+1}\left(H_{2n}-\frac{1}{2}H_n\right)}{(2n+a)^{p+1}}. \tag{2.2}$$

Also

$$2^{2p+2}\sum_{n\geq 1}\frac{(-1)^{n+1}\left(H_{2n}-\frac{1}{2}H_n\right)}{(2n+a)^{p+1}}=\sum_{n\geq 1}\frac{(-1)^{n+1}\left(H^{(p+1)}_{\frac{n}{2}+\frac{a-2}{4}}-H^{(p+1)}_{\frac{n}{2}+\frac{a-4}{4}}\right)}{(2n-1)}. \tag{2.3}$$

Proof. First, by the substitution $x = \tan\theta$,

$$\int_0^1 \frac{x^a \ln^p(x)}{1+x^2} \arctan(x)\, dx = \int_0^{\frac{\pi}{4}} \theta (\tan\theta)^a \ln^p(\tan\theta)\, d\theta.$$

It is known that the Cauchy product of two convergent series, see [7]

$$\left(\sum_{n\geq 0} a_n x^n\right)\left(\sum_{n\geq 0} b_n x^n\right) = \sum_{n\geq 0} c_n x^n,$$

where $c_n = \sum_{j=0}^{n} a_j b_{n-j}$. For $x \in (-1,1)$ then by a Taylor series expansion,

$$\arctan(x) = \sum_{n\geq 0} \frac{(-1)^n x^{2n+1}}{2n+1}, \quad \frac{1}{1+x^2} = \sum_{n\geq 0} (-1)^n x^{2n},$$

so that

$$\frac{\arctan(x)}{1+x^2} = \sum_{n\geq 1} (-1)^{n+1} \sum_{j=1}^{n} \frac{1}{2j-1} x^{2n-1} = \sum_{n\geq 1} (-1)^{n+1} h_n x^{2n-1},$$

where $h_n = H_{2n} - \frac{1}{2} H_n$. By expansion,

$$\begin{aligned}\int_0^1 \frac{x^a \ln^p(x)}{1+x^2} \arctan(x)\, dx &= \sum_{n\geq 1} (-1)^{n+1} \left(H_{2n} - \frac{1}{2} H_n\right) \\ \int_0^1 x^{2n-1+a} \ln^p(x)\, dx &= (-1)^p\, p! \sum_{n\geq 1} \frac{(-1)^{n+1}\left(H_{2n} - \frac{1}{2}H_n\right)}{(2n+a)^{p+1}}.\end{aligned}$$

If we know match (2.2) with (2.1) we obtain (2.3). □

There are some interesting special cases of Theorem 1 and we present mainly the case of $a = 0, b = 1$. Consider the following.

Corollary 2. Let the conditions of Theorem 1 hold, then, for $a = 0, b = 1$

$$\begin{aligned}I(1,0,1,p) = \int_0^1 \frac{\ln^p(x)}{1+x^2} \arctan(x)\, dx &= \int_0^{\frac{\pi}{4}} \theta \ln^p(\tan\theta)\, d\theta \\ &= \frac{(-1)^p\, p!}{2^{2p+1}} \sum_{n\geq 1} \frac{(-1)^{n+1}}{(2n-1)} \left(2^p H_n^{(p+1)} - H_{\frac{n}{2}}^{(p+1)}\right) - (-1)^p\, p! \sum_{k=2}^{p+1} \frac{\eta(k)}{2^k} \\ &\quad + \frac{(-1)^p\, p!}{2} \left(\frac{1}{2}\pi - \ln 2\right) - \frac{(-1)^p\, p!}{2^{p+3}} \pi \eta(p+1), \end{aligned} \tag{2.4}$$

where $\eta(\cdot)$ is the Dirichlet eta (or alternating zeta) function.

Proof. If we make the substitution $x = \tan\theta$, we obtain

$$\int_0^1 \frac{\ln^p(x)}{1+x^2} \arctan(x)\, dx = \int_0^{\frac{\pi}{4}} \theta \ln^p(\tan\theta)\, d\theta.$$

From

$$I(1,0,1,p)=\int_0^1 \frac{\ln^p(x)}{1+x^2}\arctan(x)\,dx$$
$$=\frac{(-1)^p\,p!}{2^{2p+2}}\sum_{n\geq 1}(-1)^{n+1}\left(H^{(p+1)}_{\frac{n}{2}-\frac{1}{2}}-H^{(p+1)}_{\frac{n}{2}-1}\right)/(2n-1),$$

from the multiplication formula (1.6), see [20], we know

$$H^{(p+1)}_{\frac{n}{2}-\frac{1}{2}}=2^{p+1}H^{(p+1)}_n-2^{p+1}\eta(p+1)-H^{(p+1)}_{\frac{n}{2}},$$

therefore the simplified integral is

$$=\frac{(-1)^p\,p!}{2^{2p+2}}\sum_{n\geq 1}\frac{(-1)^{n+1}\left(2^{p+1}H^{(p+1)}_n-2^{p+1}\eta(p+1)-2H^{(p+1)}_{\frac{n}{2}}+\left(\frac{2}{n}\right)^{p+1}\right)}{(2n-1)}$$
$$=\frac{(-1)^p\,p!}{2^{2p+1}}\sum_{n\geq 1}\frac{(-1)^{n+1}\left(2^pH^{(p+1)}_n-H^{(p+1)}_{\frac{n}{2}}\right)}{(2n-1)}+\frac{(-1)^p\,p!}{2^{p+1}}\sum_{n\geq 1}\frac{(-1)^{n+1}}{(2n-1)\,n^{p+1}}$$
$$-\frac{(-1)^p\,p!}{2^{p+1}}\sum_{n\geq 1}\frac{(-1)^{n+1}\,\eta(p+1)}{(2n-1)},$$

apply partial fraction decomposition to the second sum and the corollary follows. □

Remark 1. From Theorem 1, using the functional equation

$$\arctan\left(x^b\right)+\arctan\left(x^{-b}\right)=\frac{\pi}{2},$$

we can evaluate, for $b>0$, $I(1,a,-b,p)$. Consider the case $a=1, b>0, p\in\mathbb{N}$, then

$$I(1,1,-b,p)=\int_0^1\frac{x}{1+x^2}\ln^p(x)\arctan\left(x^{-b}\right)dx=\frac{\pi}{2}\int_0^1\frac{x}{1+x^2}\ln^p(x)\,dx$$
$$-\int_0^1\frac{x}{1+x^2}\ln^p(x)\arctan\left(x^b\right)dx=\frac{(-1)^p\,p!\pi}{2^{p+2}}\eta(p+1)-I(1,1,b,p),$$

where $I(1,1,b,p)$ is obtained from (2.1).

Some illustrative examples of Theorem 1 and its corollaries follow. To evaluate the resultant Euler sums for these examples we need a mixture of identities, including (1.6), Lemma 1 and some from [21].

Example 1. For $p=1$

$$I(1,0,1,1)=\int_0^1\frac{\ln(x)}{1+x^2}\arctan(x)\,dx=\int_0^{\frac{\pi}{4}}\theta\ln(\tan\theta)\,d\theta$$

$$= -\frac{1}{8}\sum_{n\geq 1}\frac{(-1)^{n+1}}{(2n-1)}\left(2H_n^{(2)} - H_{\frac{n}{2}}^{(2)}\right) + \frac{\eta(2)}{4} + \frac{1}{16}\pi\eta(2) - \frac{1}{2}\left(\frac{1}{2}\pi - \ln 2\right),$$

here we require, from Lemma 2

$$\sum_{n\geq 1}\frac{(-1)^{n+1}H_n^{(2)}}{(2n-1)} = \pi - \frac{1}{2}\zeta(2) - \frac{11}{96}\pi^3 - 2\ln 2 + 2G\ln 2 - \frac{1}{8}\pi\ln^2 2 + 4W(3),$$

where $W(3)$ is given by (1.7), and in a similar fashion

$$\begin{aligned}\sum_{n\geq 1}\frac{(-1)^{n+1}H_{\frac{n}{2}}^{(2)}}{(2n-1)} =& 4\pi - 2\pi G - 2\zeta(2) - \frac{13}{48}\pi^3 - 8\ln 2\\ &+ 4G\ln 2 - \frac{1}{4}\pi\ln^2 2 + 8W(3) + \frac{7}{2}\zeta(3),\end{aligned}$$

so that

$$\int_0^1 \frac{\ln(x)}{1+x^2}\arctan(x)\,dx = \frac{7}{16}\zeta(3) - \frac{\pi}{4}G.$$

The degenerate case

$$\int_0^1 \frac{\arctan(x)}{1+x^2}dx = \frac{3}{16}\zeta(2).$$

For $a = 0, b = 1, p = 2$,

$$\int_0^1 \frac{\ln^2(x)}{1+x^2}\arctan(x)\,dx = \int_0^{\frac{\pi}{4}}\theta\ln^2(\tan\theta)\,d\theta = \frac{1}{2}L(3) - \frac{25}{128}\zeta(4),$$

where, from the work of Flajolet and Salvy [10]

$$\begin{aligned}L(3) = \sum_{n\geq 1}\frac{(-1)^{n+1}H_n}{n^3} &= \frac{11}{4}\zeta(4) - \frac{7}{4}\zeta(3)\ln 2\\ &+ \frac{1}{2}\zeta(2)\ln^2 2 - \frac{1}{12}\ln^4 2 - 2Li_4\left(\frac{1}{2}\right).\end{aligned}$$

For $a = 1, b = 1, p = 2$,

$$\int_0^1 \frac{x\ln^2(x)}{1+x^2}\arctan(x)\,dx = \int_0^{\frac{\pi}{4}}\theta(\tan\theta)\ln^2(\tan\theta)\,d\theta = \frac{7\pi}{64}\zeta(3) + \beta(4) - \frac{\pi^3}{16}\ln 2.$$

For $a = 3, b = 2, p = 1$,

$$\begin{aligned}I(1,3,2,1) &= \int_0^1 \frac{x^3\ln(x)}{1+x^2}\arctan\left(x^2\right)dx\\ &= -\frac{1}{8}\sum_{n\geq 1}\frac{(-1)^{n+1}}{(2n-1)}\left(H_n^{(2)} - 2H_{2n}^{(2)} + 2\eta(2)\right).\end{aligned}$$

Here we need the following result which can be evaluated by the method of Lemma 2:

$$\sum_{n\geq 1}\frac{(-1)^{n+1}H_{2n}^{(2)}}{(2n-1)}=\frac{\pi}{4}-\frac{1}{8}\zeta(2)-\frac{1}{48}\pi^3-\frac{1}{2}\ln 2+\frac{1}{2}G\ln 2-\frac{1}{16}\pi\ln^2 2+2W(3),$$

$$\int_0^1 \frac{x^3\ln(x)}{1+x^2}\arctan\left(x^2\right)dx=\frac{1}{256}\pi^3-\frac{\pi}{16}+\frac{1}{32}\zeta(2)+\frac{1}{8}\ln 2-\frac{1}{8}G\ln 2.$$

Now consider the integral (1.2).

Theorem 2. *Let* $a=0,\ b=1,\ p\in\mathbb{N}$, *the following integral,*

$$\begin{aligned}J(1,0,1,p)=\int_0^\infty f(p,x)\,dx &= \int_0^{\frac{\pi}{2}}\theta\ln^p(\tan\theta)\,d\theta\\ &=(1-(-1)^p)\,I(1,0,1,p)+p!\frac{\pi}{2}\beta(p+1),\end{aligned}$$

where

$$f(p,x)=\frac{\ln^p(x)}{1+x^2}\arctan(x).$$

For $p=2q, q\in\mathbb{N}$

$$J(1,0,1,2q)=\frac{\pi^{2q+2}}{2^{2q+3}}A(2q),$$

where $A(2q):=(-1)^q E_{2q}$ *are the Euler (or secant or Zig-Zag) numbers.*
For $p=2q-1, q\in\mathbb{N}$

$$J(1,0,1,2q-1)=2I(1,0,1,2q-1)+(2q-1)!\frac{\pi}{2}\beta(2q)$$

where $\beta(\cdot)$ *is the Dirichlet beta function (1.3) and* $I(1,0,1,2q-1)$ *is obtained from (2.4).*

Proof. Consider

$$I(1,0,1,p)=\int_0^1 f(p,x)\,dx=\int_0^\infty f(p,x)\,dx-\int_1^\infty f(p,x)\,dx,$$

we notice that $f(p,x)$ is continuous, bounded and differentiable on the interval $x\in(0,1]$, with $\lim_{x\to 0^+} f(p,x)=\lim_{x\to 1} f(p,x)=0$, then

$$\int_0^1 f(p,x)\,dx=\int_0^\infty f(p,x)\,dx-\int_0^1\frac{1}{y^2}f\Big(p,\frac{1}{y}\Big)dy,$$

where in the third integral we have made the transformation $xy=1$, now use the relation $\arctan(x)+\arctan\left(\frac{1}{x}\right)=\frac{\pi}{2}$, for $x>0$, so that

$$\begin{aligned}\int_0^1 f(p,x)\,dx&=\int_0^\infty f(p,x)\,dx+(-1)^p\int_0^1 f(p,x)\,dx-(-1)^p\,\frac{\pi}{2}\int_0^1\frac{\ln^p(x)}{1+x^2}dx\\ &=\int_0^\infty f(p,x)\,dx+(-1)^p\int_0^1 f(p,x)\,dx-p!\frac{\pi}{2}\beta(p+1),\end{aligned}$$

therefore

$$\int_0^\infty f(p,x)\,dx = \left(1-(-1)^p\right) I(1,0,1,p) + p!\frac{\pi}{2}\beta(p+1).$$

For the even case, let $p = 2q, q \in \mathbb{N}$

$$\begin{aligned}\int_0^\infty f(2q,x)\,dx &= \int_0^{\frac{\pi}{2}} \theta \ln^{2q}(\tan\theta)\,d\theta = (2q)!\frac{\pi}{2}\beta(2q+1)\\ &= \frac{(-1)^q E_{2q}}{2^{2q+3}}\pi^{2q+2} = \frac{\pi^{2q+2}}{2^{2q+3}} A(2q),\end{aligned}$$

where E_q are the Euler numbers and $A(2q)$ are the Euler (or secant) numbers A000364, given in the online Encyclopedia of integer sequences, [16].
For the odd case, let $p = 2q-1, q \in \mathbb{N}$

$$\begin{aligned}\int_0^\infty f(2q-1,x)\,dx &= \int_0^{\frac{\pi}{2}} \theta \ln^{2q-1}(\tan\theta)\,d\theta\\ &= 2I(1,0,1,2q-1) + \frac{\pi}{2}(2q-1)!\beta(2q)\\ &= 2\int_0^1 \frac{\ln^{2q-1}(x)}{1+x^2}\arctan(x)\,dx + \frac{\pi}{2}(2q-1)!\beta(2q)\end{aligned}$$

and using $x = \tan\left(\frac{\theta}{2}\right)$ on the second integral, we have

$$\int_0^{\frac{\pi}{2}} \theta\left(\ln^{2q-1}(\tan\theta) - \frac{1}{2}\ln^{2q-1}\left(\tan\frac{\theta}{2}\right)\right) d\theta = \frac{\pi}{2}(2q-1)!\beta(2q).$$

The proof of the theorem is finished. □

We know consider the second case of the integral (1.1).

Theorem 3. *Let $\delta = -1, b \in \mathbb{R}^+, p \in \mathbb{N}$ and for $a+b > -1$, the following integral,*

$$\begin{aligned}I(-1,a,b,p) &= \int_0^1 \frac{x^a \ln^p(x)}{1-x^2}\arctan\left(x^b\right) dx\\ &= \frac{(-1)^{p+1} p!}{2^{p+1}} \sum_{n\geq 1} \frac{(-1)^{n+1} H^{(p+1)}_{bn+\frac{a-b-1}{2}}}{2n-1} + \frac{(-1)^p p!}{2^{p+3}}\pi\zeta(p+1),\end{aligned}$$

where $H^{(p+1)}_{bn+\frac{a-b-1}{2}}$ are harmonic numbers of order $p+1$.

Proof.

$$I(-1,a,b,p) = \int_0^1 \frac{x^a \ln^p(x)}{1-x^2}\arctan\left(x^b\right) dx$$

$$
\begin{aligned}
&= \sum_{n\geq 1} \frac{(-1)^{n+1}}{(2n-1)} \times \sum_{j\geq 0} \int_0^1 x^{2nb+2j+a-b} \ln^p(x)\,dx \\
&= (-1)^p p! \sum_{n\geq 1} \frac{(-1)^{n+1}}{(2n-1)} \sum_{j\geq 0} \frac{1}{(2bn+2j+a-b+1)^{p+1}} \\
&= \frac{(-1)^p p!}{2^{p+1}} \sum_{n\geq 1} \frac{(-1)^{n+1}}{(2n-1)} \zeta\left(p+1, bn+\frac{a-b+1}{2}\right) \\
&= -\frac{1}{2^{p+1}} \sum_{n\geq 1} \frac{(-1)^{n+1}}{(2n-1)} \psi^{(p)}\left(bn+\frac{a-b+1}{2}\right) \\
&= \frac{(-1)^{p+1} p!}{2^{p+1}} \sum_{n\geq 1} \frac{(-1)^{n+1}}{(2n-1)} \left(H^{(p+1)}_{bn+\frac{a-b-1}{2}} - \zeta(p+1)\right)
\end{aligned}
$$

and Theorem 3 follows. □

Some examples follow.

Example 2. For $a=0, b=1, p=1$,

$$I(-1,0,1,1) = \int_0^1 \frac{\ln(x)}{1-x^2} \arctan(x)\,dx = \frac{1}{2} G \ln 2 + W(3) - \frac{\pi}{32} \ln^2 2 - \frac{5\pi^3}{128}.$$

For $a=0, b=1, p=2$,

$$I(-1,0,1,2) = \int_0^1 \frac{\ln^2(x)}{1-x^2} \arctan(x)\,dx = \frac{7\pi}{16} \zeta(3) - \frac{1}{4} \zeta(2) G - \beta(4).$$

For $a=3, b=2, p=1$,

$$
\begin{aligned}
I(-1,3,2,1) &= \int_0^1 \frac{x^3 \ln(x)}{1-x^2} \arctan\left(x^2\right) dx \\
&= \frac{\pi}{16} - \frac{1}{32} \zeta(2) - \frac{\pi^3}{64} - \frac{1}{8} \ln 2 + \frac{1}{8} G \ln 2 - \frac{1}{64} \pi \ln^2 2 + \frac{1}{2} W(3).
\end{aligned}
$$

For $a=0, b=1, p=4$,

$$
\begin{aligned}
I(-1,0,1,4) &= \int_0^1 \frac{\ln^4(x)}{1-x^2} \arctan(x)\,dx = \int_0^{\frac{\pi}{4}} \theta \sec(2\theta) \ln^4(\tan\theta)\,d\theta \\
&= \frac{93\pi}{16} \zeta(5) - \frac{21}{16} \zeta(4) G - 3\zeta(2)\beta(4) - 12\beta(6).
\end{aligned}
$$

For $a=\frac{3}{2}, b=\frac{1}{2}, p=1$,

$$I\left(-1, \frac{3}{2}, \frac{1}{2}, 1\right) = \int_0^1 \frac{x^{\frac{3}{2}} \ln(x)}{1-x^2} \arctan\left(x^{\frac{1}{2}}\right) dx$$

$$= \pi - \frac{\pi}{2} G - \frac{1}{2} \zeta(2) - \frac{5\pi^3}{64} - 2\ln 2 + G \ln 2 - \frac{1}{16} \pi \ln^2 2 + \frac{7}{8} \zeta(3) + 2W(3).$$

For $a = 4, b = 1, p = 2,$

$$I(-1,4,1,2) = \int_0^1 \frac{x^4 \ln^2(x)}{1-x^2} \arctan(x)\,dx$$
$$= \frac{19}{108} - \frac{14}{27}\pi - \frac{1}{4}\zeta(2)\,G + \frac{2}{9}\zeta(2) - \beta(4) + \frac{26}{27}\ln 2 + \frac{1}{8}\zeta(3) + \frac{7}{16}\pi\zeta(3).$$

Now we consider the corresponding integral (1.2).

Theorem 4. *Let* $\delta = -1, a = 0,\ b = 1,\ p \in \mathbb{N},$ *the following integral,*

$$J(-1,0,1,p) = \int_0^\infty g(p,x)\,dx = \int_0^{\frac{\pi}{2}} \theta \sec(2\theta) \ln^p(\tan\theta)\,d\theta \tag{2.5}$$
$$= (1+(-1)^p)\, I(-1,0,1,p) - p!\frac{\pi}{2}\lambda(p+1),$$

where

$$g(p,x) = \frac{\ln^p(x)}{1-x^2} \arctan(x)$$

and the lambda function $2\lambda(\cdot) = \zeta(\cdot) + \eta(\cdot)$.

For $p = 2q-1,\ q \in \mathbb{N}$

$$J(-1,0,1,2q-1) = \int_0^\infty \frac{\ln^{2q-1}(x)}{1-x^2} \arctan(x)\,dx = -(2q-1)!\frac{\pi}{2}\lambda(2q).$$

For $p = 2q,\ q \in \mathbb{N}$

$$J(-1,0,1,2q) = \int_0^\infty \frac{\ln^{2q}(x)}{1-x^2} \arctan(x)\,dx = 2I(-1,0,1,2q)$$
$$-(2q)!\frac{\pi}{2}\lambda(2q+1) = \frac{(2q)!}{2^{2q}} \sum_{n\geq 0} \frac{(-1)^{n+1} H_n^{(2q+1)}}{2n+1} + \frac{(2q)!}{2^{2q+2}}\pi\zeta(2q+1)$$
$$-(2q)!\frac{\pi}{2}\lambda(2q+1),$$

where $\lambda(2q+1)$ is the Dirichlet lambda function and $H_0^{(2q+1)} = 0$.

Proof. Consider

$$I(-1,0,1,p) = \int_0^1 g(p,x)\,dx = \int_0^\infty g(p,x)\,dx - \int_1^\infty g(p,x)\,dx,$$

in the third integral make the transformation $xy = 1$ and use the relation $\arctan(x) + \arctan\left(\frac{1}{x}\right) = \frac{\pi}{2}$, for $x > 0$, so that

$$\int_0^1 g(p,x)\,dx = \int_0^\infty g(p,x)\,dx \ - (-1)^p \int_0^1 g(p,x)\,dx + (-1)^p \frac{\pi}{2} \int_0^1 \frac{\ln^p(x)}{1-x^2} dx$$
$$= \int_0^\infty g(p,x)\,dx \ - (-1)^p \int_0^1 g(p,x)\,dx + \frac{\pi}{2} p!\lambda(p+1),$$

therefore

$$\int_0^\infty g(p,x)\,dx = (1+(-1)^p)\,I(-1,0,1,p) - \frac{\pi}{2}p!\lambda(p+1)$$

and (2.5) is finished.
For the odd case, let $p = 2q-1, q \in \mathbb{N}$, then

$$\begin{aligned}\int_0^\infty g(2q-1,x)\ dx &= \int_0^1 \frac{\ln^{2q-1}(x)}{1-x^2}\arctan(x)\,dx \\ &= -\frac{\pi}{2}(2q-1)!\lambda(2q).\end{aligned}$$

For $p = 2q, q \in \mathbb{N}$

$$J(-1,0,1,2q) = \int_0^\infty \frac{\ln^{2q}(x)}{1-x^2}\arctan(x)\,dx = 2I(-1,0,1,2q)$$
$$-(2q)!\frac{\pi}{2}\lambda(2q+1) = \frac{(2q)!}{2^{2q}}\sum_{n\geq 0}\frac{(-1)^{n+1}H_n^{(2q+1)}}{2n+1} + \frac{(2q)!}{2^{2q+2}}\pi\zeta(2q+1)$$
$$-(2q)!\frac{\pi}{2}\lambda(2q+1).$$

□

Some examples follow.

Example 3.

$$J(-1,0,1,2) = \int_0^\infty \frac{\ln^2(x)}{1-x^2}\arctan(x)\,dx = -\frac{1}{12}\pi^2 G - 2\beta(4),$$
$$\begin{aligned}J(-1,0,1,6) &= \int_0^\infty \frac{\ln^6(x)}{1-x^2}\arctan(x)\,dx \\ &= \frac{31}{1344}\pi^6 G - \frac{7}{8}\pi^4\beta(4) - 30\pi^2\beta(6) - 720\beta(8).\end{aligned}$$

3 Conclusions

We have carried out a systematic study of integrals containing a log tangent function in terms of Euler sums. We believe most of our results are new in the literature and have given many examples some of which are not amenable to a mathematical computer package.

Acknowledgements

The author expresses his sincere thanks to a referee whose keen eye and constructive suggestions has led to a better presented paper.

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