Error bounds related to midpoint and trapezoid rules for the monotonic integral transform of positive operators in Hilbert spaces

Mathematica Pannonica ***New Series*** **27 /NS 1/ (2021) 2, 105-119**
DOI: 10.1556/314.2021.00011

AKADÉMIAI KIADÓ

ERROR BOUNDS RELATED TO MIDPOINT AND TRAPEZOID RULES FOR THE MONOTONIC INTEGRAL TRANSFORM OF POSITIVE OPERATORS IN HILBERT SPACES

Silvestru Sever Dragomir[1,2,*]

[1] Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia
[2] DST-NRF Centre of Excellence in the Mathematical, and Statistical Sciences, School of Computer Science, & Applied Mathematics, University of the Witwatersrand, Johannesburg, South Africa

Communicated by László Tóth

Original Research Paper
Received: Apr 27, 2021 • Accepted: May 3, 2021
First published online: October 6, 2021

ABSTRACT

For a continuous and positive function $w(\lambda)$, $\lambda > 0$ and μ a positive measure on $(0, \infty)$ we consider the following *monotonic integral transform*

$$\mathcal{M}(w,\mu)(T) := \int_0^\infty w(\lambda)T(\lambda + T)^{-1} d\mu(\lambda),$$

where the integral is assumed to exist for T a positive operator on a complex Hilbert space H. We show among others that, if $\beta \geq A, B \geq \alpha > 0$, and $0 < \delta \leq (B - A)^2 \leq \Delta$ for some constants α, β, δ, Δ, then

$$0 \leq -\frac{1}{24}\delta\mathcal{M}''(w,\mu)(\beta) \leq \mathcal{M}(w,\mu)\left(\frac{A+B}{2}\right) - \int_0^1 \mathcal{M}(w,\mu)\big((1-t)A + tB\big)dt \leq -\frac{1}{24}\Delta\mathcal{M}''(w,\mu)(\alpha)$$

and

$$0 \leq -\frac{1}{12}\delta\mathcal{M}''(w,\mu)(\beta) \leq \int_0^1 \mathcal{M}(w,\mu)\big((1-t)A + tB\big)dt - \frac{\mathcal{M}(w,\mu)(A) + \mathcal{M}(w,\mu)(B)}{2} \leq -\frac{1}{12}\Delta\mathcal{M}''(w,\mu)(\alpha),$$

where $\mathcal{M}''(w,\mu)$ is the second derivative of $\mathcal{M}(w,\mu)$ as a real function.

Applications for power function and logarithm are also provided.

KEYWORDS

operator monotone functions, Operator convex functions, Operator inequalities, Midpoint inequality, Trapezoid inequality

MATHEMATICS SUBJECT CLASSIFICATION (2020)

Primary 47A63, 47A60; Secondary 47B65

1. INTRODUCTION

Consider a complex Hilbert space $(H, \langle\cdot,\cdot\rangle)$. An operator T is said to be positive (denoted by $T \geq 0$) if $\langle Tx, x\rangle \geq 0$ for all $x \in H$ and also an operator T is said to be *strictly positive* (denoted by $T > 0$) if T is positive and invertible.

* Corresponding author. E-mail: sever.dragomir@vu.edu.au

We have the following integral representation for the power function when $t > 0$, $r \in (0, 1]$, see for instance [1, p. 145]

$$t^{r-1} = \frac{\sin(r\pi)}{\pi} \int_0^\infty \frac{\lambda^{r-1}}{\lambda + t} d\lambda. \tag{1.1}$$

Observe that for $t > 0$, $t \neq 1$, we have

$$\int_0^u \frac{d\lambda}{(\lambda + t)(\lambda + 1)} = \frac{\ln t}{t - 1} + \frac{1}{1 - t} \ln\left(\frac{u + t}{u + 1}\right)$$

for all $u > 0$.

By taking the limit over $u \longrightarrow \infty$ in this equality, we derive

$$\frac{\ln t}{t - 1} = \int_0^\infty \frac{d\lambda}{(\lambda + t)(\lambda + 1)},$$

which gives the representation for the logarithm

$$\ln t = (t - 1) \int_0^\infty \frac{d\lambda}{(\lambda + 1)(\lambda + t)} \tag{1.2}$$

for all $t > 0$.

Motivated by these representations, we introduce, for a continuous and positive function $w(\lambda)$, $\lambda > 0$, the following *integral transform*

$$\mathcal{D}(w, \mu)(t) := \int_0^\infty \frac{w(\lambda)}{\lambda + t} d\mu(\lambda), \quad t > 0, \tag{1.3}$$

where μ is a positive measure on $(0, \infty)$ and the integral (1.3) exists for all $t > 0$.

For μ the Lebesgue usual measure, we put

$$\mathcal{D}(w)(t) := \int_0^\infty \frac{w(\lambda)}{\lambda + t} d\lambda, \quad t > 0. \tag{1.4}$$

If we take μ to be the usual Lebesgue measure and the kernel $w_r(\lambda) = \lambda^{r-1}$, $r \in (0, 1]$, then

$$t^{r-1} = \frac{\sin(r\pi)}{\pi} \mathcal{D}(w_r)(t), \quad t > 0. \tag{1.5}$$

For the same measure, if we take the kernel $w_{\ln}(\lambda) = (\lambda + 1)^{-1}$, $t > 0$, we have the representation

$$\ln t = (t - 1)\mathcal{D}(w_{\ln})(t), \quad t > 0. \tag{1.6}$$

Assume that $T > 0$, then by the continuous functional calculus for selfadjoint operators, we can define the positive operator

$$\mathcal{D}(w, \mu)(T) := \int_0^\infty w(\lambda)(\lambda + T)^{-1} d\mu(\lambda), \tag{1.7}$$

where w and μ are as above. Also, when μ is the usual Lebesgue measure, then

$$\mathcal{D}(w)(T) := \int_0^\infty w(\lambda)(\lambda + T)^{-1} d\lambda, \tag{1.8}$$

for $T > 0$.

A real valued continuous function f on $(0, \infty)$ is said to be operator monotone if $f(A) \geq f(B)$ holds for any $A \geq B > 0$.

We have the following representation of operator monotone functions [7], see for instance [1, p. 144-145]:

THEOREM 1.1. A function $f : [0, \infty) \longrightarrow \mathbb{R}$ is operator monotone in $[0, \infty)$ if and only if it has the representation

$$f(t) = f(0) + bt + \int_0^\infty \frac{t\lambda}{t + \lambda} d\mu(\lambda), \tag{1.9}$$

where $b \geq 0$ and a positive measure μ on $[0, \infty)$ such that

$$\int_0^\infty \frac{\lambda}{1 + \lambda} d\mu(\lambda) < \infty. \tag{1.10}$$

A real valued continuous function f on an interval I is said to be *operator convex (operator concave)* on I if

$$f\big((1-\lambda)A+\lambda B\big) \leq (\geq)(1-\lambda)f(A)+\lambda f(B) \tag{1.11}$$

in the operator order, for all $\lambda \in [0,1]$ and for every selfadjoint operator A and B on a Hilbert space H whose spectra are contained in I. Notice that a function f is operator concave if $-f$ is operator convex.

We have the following representation of operator convex functions [1, p. 147]:

THEOREM 1.2. A function $f: [0,\infty) \longrightarrow \mathbb{R}$ is operator convex in $[0,\infty)$ with $f'_+(0) \in \mathbb{R}$ if and only if it has the representation

$$f(t) = f(0) + f'_+(0)t + ct^2 + \int_0^\infty \frac{t^2\lambda}{t+\lambda} d\mu(\lambda), \tag{1.12}$$

where $c \geq 0$ and a positive measure μ on $[0,\infty)$ such that (1.2) holds.

For a continuous and positive function $w(\lambda)$, $\lambda > 0$ and a positive measure μ on $(0,\infty)$, we can define the following mapping, which we call *monotonic integral transform*, by

$$\mathcal{M}(w,\mu)(t) := t\mathcal{D}(w,\mu)(t), \quad t > 0. \tag{1.13}$$

For $t > 0$ we have

$$\begin{aligned} \mathcal{M}(w,\mu)(t) := t\mathcal{D}(w,\mu)(t) &= \int_0^\infty w(\lambda)t(t+\lambda)^{-1}d\mu(\lambda) \\ &= \int_0^\infty w(\lambda)\,(t+\lambda-\lambda)\,(t+\lambda)^{-1}d\mu(\lambda) \\ &= \int_0^\infty w(\lambda)\big[1-\lambda(t+\lambda)^{-1}\big]d\mu(\lambda). \end{aligned} \tag{1.14}$$

If $\int_0^\infty w(\lambda)d\mu(\lambda) < \infty$, then

$$\mathcal{M}(w,\mu)(t) = \int_0^\infty w(\lambda)d\mu(\lambda) - \mathcal{D}(\ell w,\mu)(t), \tag{1.15}$$

where $\ell(t) = t$, $t > 0$.

Consider the kernel $e_{-a}(\lambda) := \exp(-a\lambda)$, $\lambda \geq 0$ and $a > 0$. Then after some calculations, we get

$$\mathcal{D}(e_{-a})(t) = \int_0^\infty \frac{\exp(-a\lambda)}{t+\lambda} d\lambda = E_1(at)\exp(at), \quad t \geq 0$$

and

$$\int_0^\infty w(\lambda)d\lambda = \int_0^\infty \exp(-a\lambda)d\lambda = \frac{1}{a},$$

where

$$E_1(t) := \int_t^\infty \frac{e^{-u}}{u} du.$$

This gives that

$$\mathcal{M}(e_{-a})(t) = t\mathcal{D}(w,\mu)(t) = tE_1(at)\exp(at), \quad t \geq 0.$$

By integration we also have

$$\mathcal{D}(\ell e_{-a},\mu)(t) = \int_0^\infty \frac{\lambda\exp(-a\lambda)}{t+\lambda} d\lambda = \frac{1}{a} - tE_1(at)\exp(at)$$

for $t > 0$.

One observes that

$$\mathcal{M}(e_{-a})(t) = \int_0^\infty w(\lambda)d\lambda - \mathcal{D}(\ell e_{-a},\mu)(t), \quad t > 0$$

and the equality (1.15) is verified in this case.

If we take $w_r(\lambda) = \lambda^{r-1}$, $r \in (0,1]$, then $\int_0^\infty w_r(\lambda)d\lambda = \infty$ and the equality (1.15) does not hold in this case.

For all $T > 0$ we have, by the continuous functional calculus for selfadjoint operators, that

$$\mathcal{M}(w,\mu)(T) = T\mathcal{D}(w,\mu)(T) = \int_0^\infty w(\lambda)\left[1 - \lambda(T+\lambda)^{-1}\right] d\mu(\lambda). \tag{1.16}$$

This gives the representation

$$T^r = \frac{\sin(r\pi)}{\pi}\mathcal{M}(w_r,\mu)(T),$$

for $T > 0$.

In this paper, we show among others that, if $\beta \geq A, B \geq \alpha > 0$ and $0 < \delta \leq (B-A)^2 \leq \Delta$ for some constants α, β, δ, Δ, then

$$\begin{aligned} 0 &\leq -\frac{1}{24}\delta\mathcal{M}''(w,\mu)(\beta) \\ &\leq \mathcal{M}(w,\mu)\left(\frac{A+B}{2}\right) - \int_0^1 \mathcal{M}(w,\mu)\left((1-t)A + tB\right) dt \\ &\leq -\frac{1}{24}\Delta\mathcal{M}''(w,\mu)(\alpha) \end{aligned}$$

and

$$\begin{aligned} 0 &\leq -\frac{1}{12}\delta\mathcal{M}''(w,\mu)(\beta) \\ &\leq \int_0^1 \mathcal{M}(w,\mu)\big((1-t) + tB\big) dt - \frac{\mathcal{M}(w,\mu)(A) + \mathcal{M}(w,\mu)(B)}{2} \\ &\leq -\frac{1}{12}\Delta\mathcal{M}''(w,\mu)(\alpha). \end{aligned}$$

Applications for power function and logarithm are also provided.

2. SOME REPRESENTATIONS

We have the following representation of the Fréchet derivative $D\big(\mathcal{M}(w,\mu)\big)$:

LEMMA 2.1. For all $A > 0$,

$$D\big(\mathcal{M}(w,\mu)\big)(A)(V) = \int_0^\infty \lambda w(\lambda)(\lambda + A)^{-1} V (\lambda + A)^{-1} d\mu(\lambda) \tag{2.1}$$

for all $V \in S(H)$, the class of all selfadjoint operators on H.

Proof. The proof follows directly from the fact that the Fréchet derivative of the map $\mathrm{Inv}(A) = A^{-1}$ is

$$D(\mathrm{Inv})(A)(V) = -A^{-1}VA^{-1}$$

for all $A > 0$ and $V \in S(H)$. □

For the case of second Fréchet derivative $D^2\big(\mathcal{M}(w,\mu)\big)$, we have the representation:

LEMMA 2.2. For all $A > 0$,

$$D^2\big(\mathcal{M}(w,\mu)\big)(A)(V,V) = -2\int_0^\infty \lambda w(\lambda)(\lambda + A)^{-1} V (\lambda + A)^{-1} V (\lambda + A)^{-1} d\mu(\lambda) \tag{2.2}$$

for all $V \in S(H)$.

Proof. The proof follows directly from the fact that the Fréchet second derivative of the map $\mathrm{Inv}(A) = A^{-1}$ is

$$D^2(\mathrm{Inv})(A)(V,V) = 2A^{-1}VA^{-1}V^{-1}$$

for all $A > 0$ and $V \in S(H)$. The details are omitted. □

We have the following representation for the transform $\mathcal{M}(w,\mu)$:

THEOREM 2.3. For all A, $B > 0$ we have

$$\begin{aligned}\mathcal{M}(w,\mu)(B) = \mathcal{M}(w,\mu)(A) + \int_0^\infty \lambda w(\lambda)(\lambda + A)^{-1}(B - A)(\lambda + A)^{-1} d\mu(\lambda) \\ - 2\int_0^1 (1-t)\left[\int_0^\infty \lambda w(\lambda)\big(\lambda + (1-t)A + tB\big)^{-1}(B-A)\right. \\ \left. \times \big(\lambda + (1-t)A + tB\big)^{-1}(B-A)\big(\lambda + (1-t)A + tB\big)^{-1} d\mu(\lambda)\right] dt.\end{aligned} \tag{2.3}$$

Proof. We use the Taylor's type formula with integral remainder, see for instance [2, p. 112],

$$f(E) = f(C) + D(f)(C)(E-C) + \int_0^1 (1-t)D^2(f)\big((1-t)C + tE\big)(E-C, E-C)dt \tag{2.4}$$

that holds for functions f which are of class C^2 on an open and convex subset $\mathcal{O}$ in the Banach algebra $B(H)$ and $C, E \in \mathcal{O}$.

If we write (2.4) for $\mathcal{M}(w,\mu)$ and A, $B > 0$, we get

$$\begin{aligned}\mathcal{M}(w,\mu)(B) = \mathcal{M}(w,\mu)(A) + D\big(\mathcal{M}(w,\mu)\big)(A)(B-A) \\ + \int_0^1 (1-t)D^2\big(\mathcal{M}(w,\mu)\big)\big((1-t)A + tB\big)(B-A, B-A)dt\end{aligned}$$

and by the representations (2.1) and (2.2) we obtain the desired result (2.3). □

We have the following representation of operator Jensen's gap for the n-tuple of positive operators $\mathbf{A} = (A_1, \dots, A_n)$ and probability density n-tuple $\mathbf{p} = (p_1, \dots, p_n)$,

$$J\big(\mathbf{A}, \mathbf{p}, \mathcal{M}(w,\mu)\big) := \mathcal{M}(w,\mu)\left(\sum_{k=1}^n p_k A_k\right) - \sum_{k=1}^n p_k \mathcal{M}(w,\mu)(A_k).$$

THEOREM 2.4. We have the representation

$$\begin{aligned}J\big(\mathbf{A}, \mathbf{p}, \mathcal{M}(w,\mu)\big) = 2\sum_{k=1}^n p_k \int_0^\infty \lambda w(\lambda)\left(\int_0^1 (1-t)\left(\lambda + (1-t)\sum_{j=1}^n p_j A_j + tA_k\right)^{-1}\right. \\ \times \left(A_k - \sum_{j=1}^n p_j A_j\right)\left(\lambda + (1-t)\sum_{j=1}^n p_j A_j + tA_k\right)^{-1} \\ \left.\times \left(A_k - \sum_{j=1}^n p_j A_j\right)\left(\lambda + (1-t)\sum_{j=1}^n p_j A_j + tA_k\right)^{-1} dt\right) d\mu(\lambda) \\ \geq 0\end{aligned} \tag{2.5}$$

for the n-tuple of positive operators $\mathbf{A} = (A_1, \dots, A_n)$ and probability density n-tuple $\mathbf{p} = (p_1, \dots, p_n)$. This also shows that $\mathcal{M}(w,\mu)$ is operator concave on $(0, \infty)$.

Proof. From the identity (2.3) we get

$$\begin{aligned}D\big(\mathcal{M}(w,\mu)\big)\left(\sum_{j=1}^n p_j A_j\right)\left(A_k - \sum_{j=1}^n p_j A_j\right) + \mathcal{M}(w,\mu)\left(\sum_{j=1}^n p_j A_j\right) - \mathcal{M}(w,\mu)(A_k) \\ = 2\int_0^\infty w(\lambda)\left(\int_0^1 (1-t)\left(\lambda + (1-t)\sum_{j=1}^n p_j A_j + tA_k\right)^{-1}\right. \\ \times \left(A_k - \sum_{j=1}^n p_j A_j\right)\left(\lambda + (1-t)\sum_{j=1}^n p_j A_j + tA_k\right)^{-1}\end{aligned}$$

$$\times\left(A_k-\sum_{j=1}^{n}p_jA_j\right)\left(\lambda+(1-t)\sum_{j=1}^{n}p_jA_j+tA_k\right)^{-1}dt\Bigg)d\mu(\lambda)\geq 0$$

for all $k\in\{1,\dots,n\}$.

If we multiply this inequality with $p_k\geq 0$, take into account that $\sum_{k=1}^{n}p_k=1$ and

$$\begin{aligned}\sum_{k=1}^{n}p_kD\big(\mathcal{M}(w,\mu)\big)\left(\sum_{j=1}^{n}p_jA_j\right)\left(A_k-\sum_{j=1}^{n}p_jA_j\right)\\ =D\big(\mathcal{M}(w,\mu)\big)\left(\sum_{j=1}^{n}p_jA_j\right)\left(\sum_{k=1}^{n}p_kA_k-\sum_{j=1}^{n}p_jA_j\right)\\ =D\big(\mathcal{M}(w,\mu)\big)\left(\sum_{j=1}^{n}p_jA_j\right)(0)=0,\end{aligned}$$

then we obtain the desired result (2.5). □

For a continuous function f on $(0,\infty)$ and A, $B>0$ we consider the auxiliary function

$$f_{A,B}:\ [0,1]\longrightarrow\mathbb{R}$$

defined by

$$f_{A,B}(t)\ :=f\big((1-t)A+tB\big),\quad t\in[0,1].$$

We have the following representations of the derivatives:

LEMMA 2.5. Assume that the operator function generated by f is twice Fréchet differentiable in each $A>0$, then for $B>0$ we have that $f_{A,B}$ is twice differentiable on $[0,1]$,

$$\frac{df_{A,B}(t)}{dt}=D(f)\big((1-t)A+tB\big)(B-A)\tag{2.6}$$

and

$$\frac{d^2f_{A,B}(t)}{dt^2}=D^2(f)\big((1-t)A+tB\big)(B-A,B-A)\tag{2.7}$$

for $t\in[0,1]$, where in 0 and 1 the derivatives are the right and left derivatives.

Proof. We prove only for the interior points $t\in(0,1)$. Let h be in a small interval around 0 such that $t+h\in(0,1)$. Then for $h\neq 0$,

$$\begin{aligned}\frac{f_{A,B}(t+h)-f(t)}{h}&=\frac{f\Big(\big(1-(t+h)\big)A+(t+h)B\Big)-f\big((1-t)A+tB\big)}{h}\\ &=\frac{f\big((1-t)A+tB+h(B-A)\big)-f\big((1-t)A+tB\big)}{h}\end{aligned}$$

and by taking the limit over $h\longrightarrow 0$, we get

$$\begin{aligned}\frac{df_{A,B}(t)}{dt}&=\lim_{h\to 0}\frac{f_{A,B}(t+h)-f(t)}{h}\\ &=\lim_{h\to 0}\left[\frac{f\big((1-t)A+tB+h(B-A)\big)-f\big((1-t)A+tB\big)}{h}\right]\\ &=D(f)\big((1-t)A+tB\big)(B-A),\end{aligned}$$

which proves (2.6).

The identity (2.7) follows in a similar way. □

For the transform $\mathcal{M}(w,\mu)(t)$ defined in the introduction, we consider the auxiliary function

$$\mathcal{M}(w,\mu)_{A,B}(t)\ :=\mathcal{M}(w,\mu)\big((1-t)A+tB\big)$$

where A, $B>0$ and $t\in[0,1]$.

COROLLARY 2.6. For all $A, B > 0$ and $t \in [0,1]$,

$$\begin{aligned}\frac{d\mathcal{M}(w,\mu)_{A,B}(t)}{dt} &= D\big(\mathcal{M}(w,\mu)\big)\big((1-t)A+tB\big)(B-A)\\ &= \int_0^\infty \lambda w(\lambda)\big(\lambda+(1-t)A+tB\big)^{-1}(B-A)\times\big(\lambda+(1-t)A+tB\big)^{-1}d\mu(\lambda)\end{aligned} \tag{2.8}$$

and

$$\begin{aligned}\frac{d^2\mathcal{M}(w,\mu)_{A,B}(t)}{dt^2} &= D^2\big(\mathcal{M}(w,\mu)\big)\big((1-t)A+tB\big)(B-A,B-A)\\ &= -2\int_0^\infty \lambda w(\lambda)\big(\lambda+(1-t)A+tB\big)^{-1}(B-A)\\ &\quad\times\big(\lambda+(1-t)A+tB\big)^{-1}(B-A)\big(\lambda+(1-t)A+tB\big)^{-1}d\mu(\lambda).\end{aligned} \tag{2.9}$$

We observe that if $f(t) = \mathcal{M}(w,\mu)(t)$, $t > 0$, in Lemma 2.5, then by the representations from Lemma 2.1 and Lemma 2.2 we obtain the desired equalities (2.8) and (2.9).

3. MIDPOINT AND TRAPEZOID INEQUALITIES

We have the following identity for the midpoint rule:

THEOREM 3.1. For all $A, B > 0$ we have the identity

$$\begin{aligned}&\mathcal{M}(w,\mu)\left(\frac{A+B}{2}\right)-\int_0^1\mathcal{M}(w,\mu)\big((1-t)A+tB\big)dt\\ &= 2\int_0^1\left(t-\frac{1}{2}\right)^2\left\{\int_0^1(1-s)\right.\\ &\quad\times\left[\int_0^\infty\lambda w(\lambda)\left(\lambda+(1-s)\frac{A+B}{2}+s\big((1-t)A+tB\big)\right)^{-1}(B-A)\right.\\ &\quad\times\left(\lambda+(1-s)\frac{A+B}{2}+s\big((1-t)A+tB\big)\right)^{-1}(B-A)\\ &\quad\times\left.\left.\left(\lambda+(1-s)\frac{A+B}{2}+s\big((1-t)A+tB\big)\right)^{-1}d\mu(\lambda)\right]ds\right\}dt.\end{aligned} \tag{3.1}$$

Proof. From (2.3) we have for $B = E > 0$ and $A = C > 0$ that

$$\begin{aligned}\mathcal{M}(w,\mu)(E) &= \mathcal{M}(w,\mu)(C)+\int_0^\infty\lambda w(\lambda)(\lambda+C)^{-1}(E-C)(\lambda+C)^{-1}d\mu(\lambda)\\ &\quad-2\int_0^1(1-s)\left[\int_0^\infty\lambda w(\lambda)\left(\lambda+(1-s)C+sE\right)^{-1}(E-C)\right.\\ &\quad\left.\times\big(\lambda+(1-s)C+sE\big)^{-1}(E-C)\big(\lambda+(1-s)C+sE\big)^{-1}d\mu(\lambda)\right]ds,\end{aligned}$$

which implies for $E = (1-t)A + tB$, $t \in [0,1]$ and $C = \frac{A+B}{2}$, that

$$\begin{aligned}
&\mathcal{M}(w,\mu)\big((1-t)A+tB\big)\\
&= \mathcal{M}(w,\mu)\left(\frac{A+B}{2}\right) + \left(t-\frac{1}{2}\right)\int_0^\infty \lambda w(\lambda)\left(\lambda+\frac{A+B}{2}\right)^{-1}(B-A)\left(\lambda+\frac{A+B}{2}\right)^{-1} d\mu(\lambda)\\
&- 2\left(t-\frac{1}{2}\right)^2\int_0^1 (1-s)\times\left[\int_0^\infty w(\lambda)\left(\lambda+(1-s)\frac{A+B}{2}+s\big((1-t)A+tB\big)\right)^{-1}(B-A)\right.\\
&\times\left(\lambda+(1-s)\frac{A+B}{2}+s\big((1-t)A+tB\big)\right)^{-1}(B-A)\\
&\left.\left(\lambda+(1-s)\frac{A+B}{2}+s\big((1-t)A+tB\big)\right)^{-1} d\mu(\lambda)\right] ds.
\end{aligned} \tag{3.2}$$

If we integrate (3.2) over $t \in [0,1]$, then we get

$$\begin{aligned}
\int_0^1 \mathcal{M}(w,\mu)\big((1-t)A+tB\big)\,dt &= \mathcal{M}(w,\mu)\left(\frac{A+B}{2}\right)\\
&+ \int_0^1 \left(t-\frac{1}{2}\right) dt\\
&\times \int_0^\infty \lambda w(\lambda)\left(\lambda+\frac{A+B}{2}\right)^{-1}(B-A)\left(\lambda+\frac{A+B}{2}\right)^{-1} d\mu(\lambda)\\
&- 2\int_0^1 \left(t-\frac{1}{2}\right)^2 \left\{\int_0^1 (1-s)\right.\\
&\times\left[\int_0^\infty w(\lambda)\left(\lambda+(1-s)\frac{A+B}{2}+s\big((1-t)A+tB\big)\right)^{-1}(B-A)\right.\\
&\times\left(\lambda+(1-s)\frac{A+B}{2}+s\big((1-t)A+tB\big)\right)^{-1}(B-A)\\
&\times\left.\left.\left(\lambda+(1-s)\frac{A+B}{2}+s\big((1-t)A+tB\big)\right)^{-1} d\mu(\lambda)\right] ds\right\} dt
\end{aligned}$$

and since $\int_0^1 \left(t-\frac{1}{2}\right) dt = 0$, hence the identity (3.1) is proved. □

COROLLARY 3.2. Assume that $\beta \geq A, B \geq \alpha > 0$ and $0 < \delta \leq (B-A)^2 \leq \Delta$ for some constants α, β, δ, Δ, then

$$\begin{aligned}
0 &\leq -\frac{1}{24}\delta\mathcal{M}''(w,\mu)(\beta)\\
&\leq \mathcal{M}(w,\mu)\left(\frac{A+B}{2}\right) - \int_0^1 \mathcal{M}(w,\mu)\big((1-t)A+tB\big)\,dt\\
&\leq -\frac{1}{24}\Delta\mathcal{M}''(w,\mu)(\alpha).
\end{aligned} \tag{3.3}$$

Proof. Since $\beta \geq A, B \geq \alpha > 0$, hence

$$\lambda+\alpha \leq \lambda+(1-s)\frac{A+B}{2}+s\big((1-t)A+tB\big) \leq \lambda+\beta,$$

for all $\lambda \geq 0$ and $t \in [0,1]$. This implies that

$$(\lambda+\beta)^{-1} \leq \left(\lambda+(1-s)\frac{A+B}{2}+s\big((1-t)A+tB\big)\right)^{-1} \leq (\lambda+\alpha)^{-1} \tag{3.4}$$

for all $\lambda \geq 0$ and $t \in [0,1]$. If we multiply this both sides with $B - A$, then we obtain

$$\begin{aligned}(\lambda+\beta)^{-1}(B-A)^2 &\leq (B-A)\left(\lambda+(1-s)\frac{A+B}{2}+s\big((1-t)A+tB\big)\right)^{-1}(B-A) \\ &\leq (\lambda+\alpha)^{-1}(B-A)^2\end{aligned} \tag{3.5}$$

for all $\lambda \geq 0$ and $t \in [0,1]$. Since $0 < \delta \leq (B-A)^2 \leq \Delta$, hence $(\lambda+\beta)^{-1}(B-A)^2 \geq \delta(\lambda+\beta)^{-1}$ and $(\lambda+\alpha)^{-1}(B-A)^2 \leq (\lambda+\alpha)^{-1}\Delta$, then by (3.5)

$$\delta(\lambda+\beta)^{-1} \leq (B-A)\left(\lambda+(1-s)\frac{A+B}{2}+s\big((1-t)A+tB\big)\right)^{-1}(B-A) \leq \Delta(\lambda+\alpha)^{-1} \tag{3.6}$$

for all $\lambda \geq 0$ and $t \in [0,1]$. If we multiply both sides with $\left(\lambda+(1-s)\frac{A+B}{2}+s\big((1-t)A+tB\big)\right)^{-1}$ we derive

$$\begin{aligned}&\delta(\lambda+\beta)^{-1}\left(\lambda+(1-s)\frac{A+B}{2}+s\big((1-t)A+tB\big)\right)^{-2} \\ &\leq \left(\lambda+(1-s)\frac{A+B}{2}+s\big((1-t)A+tB\big)\right)^{-1}(B-A) \\ &\times\left(\lambda+(1-s)\frac{A+B}{2}+s\big((1-t)A+tB\big)\right)^{-1}(B-A) \\ &\times\left(\lambda+(1-s)\frac{A+B}{2}+s\big((1-t)A+tB\big)\right)^{-1} \\ &\leq \Delta(\lambda+\alpha)^{-1}\left(\lambda+(1-s)\frac{A+B}{2}+s\big((1-t)A+tB\big)\right)^{-2}\end{aligned}$$

for all $\lambda \geq 0$ and $t \in [0,1]$. By utilising (3.4) we further obtain the bounds

$$\begin{aligned}\delta(\lambda+\beta)^{-3} &\leq \left(\lambda+(1-s)\frac{A+B}{2}+s\big((1-t)A+tB\big)\right)^{-1}(B-A) \\ &\times\left(\lambda+(1-s)\frac{A+B}{2}+s\big((1-t)A+tB\big)\right)^{-1}(B-A) \\ &\times\left(\lambda+(1-s)\frac{A+B}{2}+s\big((1-t)A+tB\big)\right)^{-1} \\ &\leq \Delta(\lambda+\alpha)^{-3}\end{aligned}$$

for all $\lambda \geq 0$ and $t \in [0,1]$. If we multiply by $2\lambda w(\lambda)\left(t-\frac{1}{2}\right)^2(1-s) \geq 0$ and integrate, then we get

$$\begin{aligned}&2\delta\int_0^\infty \lambda w(\lambda)(\lambda+\beta)^{-3}d\mu(\lambda)\int_0^1\left(t-\frac{1}{2}\right)^2 dt\int_0^1(1-s)ds \\ &\leq 2\int_0^1\left(t-\frac{1}{2}\right)^2\left\{\int_0^1(1-s)\right. \\ &\times\left[\int_0^\infty \lambda w(\lambda)\left(\lambda+(1-s)\frac{A+B}{2}+s\big((1-t)A+tB\big)\right)^{-1}(B-A)\right. \\ &\times\left(\lambda+(1-s)\frac{A+B}{2}+s\big((1-t)A+tB\big)\right)^{-1}(B-A) \\ &\times\left.\left.\left(\lambda+(1-s)\frac{A+B}{2}+s\big((1-t)A+tB\big)\right)^{-1}d\mu(\lambda)\right]ds\right\}dt \\ &\leq 2\Delta\int_0^\infty \lambda w(\lambda)(\lambda+\alpha)^{-3}d\mu(\lambda)\int_0^1\left(t-\frac{1}{2}\right)^2 dt\int_0^1(1-s)ds\end{aligned} \tag{3.7}$$

and by the identity (3.1) and the fact that

$$\int_0^1 \left(t-\frac{1}{2}\right)^2 dt = \frac{1}{12} \text{ and } \int_0^1 (1-s)ds = \frac{1}{2}$$

we obtain

$$\begin{aligned} \frac{1}{12}\delta \int_0^\infty \lambda w(\lambda)(\lambda+\beta)^{-3} d\mu(\lambda) &\le \mathcal{M}(w,\mu)\left(\frac{A+B}{2}\right) - \int_0^1 \mathcal{M}(w,\mu)\big((1-t)A+tB\big)dt \\ &\le \frac{1}{12}\Delta \int_0^\infty \lambda w(\lambda)(\lambda+\alpha)^{-3} d\mu(\lambda). \end{aligned} \tag{3.8}$$

If we take the derivative in (1.6) over t, then we get

$$\mathcal{M}'(w,\mu)(t) = \int_0^\infty \frac{\lambda w(\lambda)}{(\lambda+t)^2} d\mu(\lambda), \quad t>0,$$

and

$$\mathcal{M}''(w,\mu)(t) = -2\int_0^\infty \frac{\lambda w(\lambda)}{(\lambda+t)^3} d\mu(\lambda), \quad t>0.$$

This gives

$$\int_0^\infty \frac{\lambda w(\lambda)}{(\lambda+\alpha)^3} d\mu(\lambda) = -\frac{1}{2}\mathcal{M}''(w,\mu)(\alpha),$$
$$\int_0^\infty \frac{\lambda w(\lambda)}{(\lambda+\beta)^3} d\mu(\lambda) = -\frac{1}{2}\mathcal{M}''(w,\mu)(\beta)$$

and by (3.2) we obtain (3.3). □

We have the following identity for the trapezoid rule:

THEOREM 3.3. For all $A, B > 0$ we have the identity

$$\begin{aligned} &\int_0^1 \mathcal{M}(w,\mu)\big((1-t)A+tB\big)dt - \frac{\mathcal{M}(w,\mu)(A)+\mathcal{M}(w,\mu)(B)}{2} \\ &= \int_0^1 t(1-t)\left[\int_0^\infty \lambda w(\lambda)\big(\lambda+(1-t)A+tB\big)^{-1}(B-A)\right. \\ &\left.\times\big(\lambda+(1-t)A+tB\big)^{-1}(B-A)\big(\lambda+(1-t)A+tB\big)^{-1} d\mu(\lambda)\right] dt. \end{aligned} \tag{3.9}$$

Proof. Using integration by parts for the Bochner integral, we have

$$\begin{aligned} &\frac{1}{2}\int_0^1 t(1-t)\frac{d^2\mathcal{M}(w,\mu)_{A,B}(t)}{dt^2}dt \\ &= \frac{1}{2}\left[t(1-t)\frac{d\mathcal{M}(w,\mu)_{A,B}(t)}{dt}\Big|_0^1 - \int_0^1 (1-2t)\frac{d\mathcal{M}(w,\mu)_{A,B}(t)}{dt}dt\right] \\ &= \int_0^1 \left(t-\frac{1}{2}\right)\frac{d\mathcal{M}(w,\mu)_{A,B}(t)}{dt}dt \\ &= \left(t-\frac{1}{2}\right)\mathcal{M}(w,\mu)_{A,B}(t)\Big|_0^1 - \int_0^1 \mathcal{M}(w,\mu)_{A,B}(t)dt \\ &= \frac{1}{2}\left[\mathcal{M}(w,\mu)_{A,B}(1)+\mathcal{M}(w,\mu)_{A,B}(0)\right] - \int_0^1 \mathcal{M}(w,\mu)_{A,B}(t)dt, \end{aligned}$$

that gives the identity

$$\frac{\mathcal{M}(w,\mu)(A)+\mathcal{M}(w,\mu)(B)}{2} - \int_0^1 \mathcal{M}(w,\mu)\big((1-t)A+tB\big)dt = \frac{1}{2}\int_0^1 t(1-t)\frac{d^2\mathcal{M}(w,\mu)_{A,B}(t)}{dt^2}dt. \tag{3.10}$$

By (2.9) we have

$$\frac{1}{2}\int_0^1 t(1-t)\frac{d^2\mathcal{M}(w,\mu)_{A,B}(t)}{dt^2}dt = -\int_0^1 t(1-t)\left[\int_0^\infty \lambda w(\lambda)\big(\lambda+(1-t)A+tB\big)^{-1}(B-A)\right. \\ \left.\times\big(\lambda+(1-t)A+tB\big)^{-1}(B-A)\big(\lambda+(1-t)A+tB\big)^{-1}d\mu(\lambda)\right]dt. \tag{3.11}$$

By making use of (3.10) and (3.11) we obtain (3.9). □

We have:

COROLLARY 3.4. Assume that $\beta \geq A, B \geq \alpha > 0$, and $0 < \delta \leq (B-A)^2 \leq \Delta$, then

$$\begin{aligned} 0 &\leq -\frac{1}{12}\delta\mathcal{M}''(w,\mu)(\beta) \\ &\leq \int_0^1 \mathcal{M}(w,\mu)\big((1-t)A+tB\big)dt - \frac{\mathcal{M}(w,\mu)(A)+\mathcal{M}(w,\mu)(B)}{2} \\ &\leq -\frac{1}{12}\Delta\mathcal{M}''(w,\mu)(\alpha). \end{aligned} \tag{3.12}$$

Proof. As in the proof of Corollary 3.2 we have

$$\begin{aligned} &\delta(\lambda+\beta)^{-3} \\ &\leq \big(\lambda+(1-t)A+tB\big)^{-1}(B-A) \\ &\times\big(\lambda+(1-t)A+tB\big)^{-1}(B-A)\big(\lambda+(1-t)A+tB\big)^{-1} \\ &\leq \Delta(\lambda+\alpha)^{-3} \end{aligned} \tag{3.13}$$

for all $\lambda \geq 0$ and $t \in [0,1]$. If we multiply by $t(1-t)\lambda w(\lambda) \geq 0$ and integrate, then we get

$$\begin{aligned} &\delta\left(\int_0^1 t(1-t)dt\right)\int_0^\infty \lambda w(\lambda)(\lambda+\beta)^{-3}d\mu(\lambda) \\ &\leq \int_0^1 t(1-t)\left[\int_0^\infty \lambda w(\lambda)\big(\lambda+(1-t)A+tB\big)^{-1}(B-A)\right. \\ &\left.\times\big(\lambda+(1-t)A+tB\big)^{-1}(B-A)\big(\lambda+(1-t)A+tB\big)^{-1}d\mu(\lambda)\right]dt \\ &\leq \Delta\left(\int_0^1 t(1-t)dt\right)\int_0^\infty \lambda w(\lambda)(\lambda+\alpha)^{-3}d\mu(\lambda). \end{aligned} \tag{3.14}$$

Since

$$\int_0^1 t(1-t)dt = \frac{1}{6},$$
$$\int_0^\infty \frac{\lambda w(\lambda)}{(\lambda+\alpha)^3}d\mu(\lambda) = -\frac{1}{2}\mathcal{M}''(w,\mu)(\alpha)$$

and

$$\int_0^\infty \frac{\lambda w(\lambda)}{(\lambda+\beta)^3}d\mu(\lambda) = -\frac{1}{2}\mathcal{M}''(w,\mu)(\beta),$$

then by (3.14) we derive (3.12). □

We have an alternative identity for the midpoint rule:

THEOREM 3.5. For all A, $B > 0$ we have the identity

$$\begin{aligned}
&\mathcal{M}(w,\mu)\left(\frac{A+B}{2}\right) - \int_0^1 \mathcal{M}(w,\mu)\big((1-t)A+tB\big)dt \\
&= \int_0^{1/2} t^2 \left[\int_0^\infty \lambda w(\lambda)\big(\lambda+(1-t)A+tB\big)^{-1}(B-A)\right. \\
&\left.\times\big(\lambda+(1-t)A+tB\big)^{-1}(B-A)\big(\lambda+(1-t)A+tB\big)^{-1}d\mu(\lambda)\right] dt \\
&+ \int_{1/2}^1 (t-1)^2 \left[\int_0^\infty \lambda w(\lambda)\big(\lambda+(1-t)A+tB\big)^{-1}(B-A)\right. \\
&\left.\times\big(\lambda+(1-t)A+tB\big)^{-1}(B-A)\big(\lambda+(1-t)A+tB\big)^{-1}d\mu(\lambda)\right] dt.
\end{aligned} \tag{3.15}$$

Proof. Using integration by parts for Bochner's integral, we have

$$\begin{aligned}
\frac{1}{2}\int_0^{1/2} t^2 \frac{d^2\mathcal{M}(w,\mu)_{A,B}(t)}{dt^2}dt &= \frac{1}{2}\left[t^2\frac{d\mathcal{M}(w,\mu)_{A,B}(t)}{dt}\bigg|_0^{1/2} - 2\int_0^{1/2} t\frac{d\mathcal{M}(w,\mu)_{A,B}(t)}{dt}dt\right] \\
&= \frac{1}{8}\frac{d\mathcal{M}(w,\mu)_{A,B}(1/2)}{dt} - \int_0^{1/2} t\frac{d\mathcal{M}(w,\mu)_{A,B}(t)}{dt}dt \\
&= \frac{1}{8}\frac{d\mathcal{M}(w,\mu)_{A,B}(1/2)}{dt} - \left[t\mathcal{M}(w,\mu)_{A,B}(t)\Big|_0^{1/2} - \int_0^{1/2}\mathcal{M}(w,\mu)_{A,B}(t)dt\right] \\
&= \frac{1}{8}\frac{d\mathcal{M}(w,\mu)_{A,B}(1/2)}{dt} - \frac{1}{2}\mathcal{M}(w,\mu)_{A,B}(1/2) + \int_0^{1/2}\mathcal{M}(w,\mu)_{A,B}(t)dt
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{2}\int_{1/2}^1 (t-1)^2 \frac{d^2\mathcal{M}(w,\mu)_{A,B}(t)}{dt^2}dt \\
&\quad= \frac{1}{2}\left[(t-1)^2\frac{d\mathcal{M}(w,\mu)_{A,B}(t)}{dt}\bigg|_{1/2}^{1} - 2\int_{1/2}^1 (t-1)\frac{d\mathcal{M}(w,\mu)_{A,B}(t)}{dt}dt\right] \\
&\quad= -\frac{1}{8}\frac{d\mathcal{M}(w,\mu)_{A,B}(1/2)}{dt} - \left[(t-1)\mathcal{M}(w,\mu)_{A,B}(t)\Big|_{1/2}^{1} - \int_{1/2}^1\mathcal{M}(w,\mu)_{A,B}(t)dt\right] \\
&\quad= -\frac{1}{8}\frac{d\mathcal{M}(w,\mu)_{A,B}(1/2)}{dt} - \frac{1}{2}\mathcal{M}(w,\mu)_{A,B}(1/2) + \int_{1/2}^1\mathcal{M}(w,\mu)_{A,B}(t)dt.
\end{aligned}$$

If we add these two equalities, then we get

$$\begin{aligned}
&\frac{1}{2}\int_0^{1/2} t^2 \frac{d^2\mathcal{M}(w,\mu)_{A,B}(t)}{dt^2}dt + \frac{1}{2}\int_{1/2}^1 (t-1)^2 \frac{d^2\mathcal{M}(w,\mu)_{A,B}(t)}{dt^2}dt \\
&= -\mathcal{M}(w,\mu)_{A,B}(1/2) + \int_0^{1/2}\mathcal{M}(w,\mu)_{A,B}(t)dt + \int_{1/2}^1\mathcal{M}(w,\mu)_{A,B}(t)dt \\
&= \int_0^1\mathcal{M}(w,\mu)_{A,B}(t)dt - \mathcal{M}(w,\mu)_{A,B}(1/2).
\end{aligned} \tag{3.16}$$

By (2.9) we obtain

$$\begin{aligned}
&\frac{1}{2}\int_0^{1/2} t^2 \frac{d^2\mathcal{M}(w,\mu)_{A,B}(t)}{dt^2}dt \\
&= -\int_0^{1/2} t^2 \left[\int_0^\infty \lambda w(\lambda)\big(\lambda+(1-t)A+tB\big)^{-1}(B-A)\right. \\
&\left.\times\big(\lambda+(1-t)A+tB\big)^{-1}(B-A)\big(\lambda+(1-t)A+tB\big)^{-1}d\mu(\lambda)\right] dt
\end{aligned} \tag{3.17}$$

and

$$\begin{aligned}&\frac{1}{2}\int_{1/2}^{1}(t-1)^2\frac{d^2\mathcal{M}(w,\mu)_{A,B}(t)}{dt^2}dt\\&=-\int_{1/2}^{1}(t-1)^2\left[\int_0^\infty \lambda w(\lambda)\big(\lambda+(1-t)A+tB\big)^{-1}(B-A)\right.\\&\left.\times\big(\lambda+(1-t)A+tB\big)^{-1}(B-A)\big(\lambda+(1-t)A+tB\big)^{-1}d\mu(\lambda)\right]dt.\end{aligned}\tag{3.18}$$

By employing (3.16)-(3.18) we derive the desired result (3.15). □

REMARK 3.6. By making use of the identity (3.15) one can obtain the same upper and lower bounds for the midpoint rule as those in Corollary 3.2.

4. SOME EXAMPLES

The case of operator monotone functions is as follows:

PROPOSITION 4.1. Assume that the function $f:[0,\infty)\longrightarrow\mathbb{R}$ is operator monotone in $[0,\infty)$ and has the representation (1.9), then for $A, B>0$,

$$\begin{aligned}f(B)&=f(A)+b(B-A)+\int_0^\infty \lambda^2(\lambda+A)^{-1}(B-A)(\lambda+A)^{-1}d\mu(\lambda)\\&-2\int_0^1(1-t)\left[\int_0^\infty \lambda^2\big(\lambda+(1-t)A+tB\big)^{-1}(B-A)\right.\\&\left.\times\big(\lambda+(1-t)A+tB\big)^{-1}(B-A)\big(\lambda+(1-t)A+tB\big)^{-1}d\mu(\lambda)\right]dt.\end{aligned}\tag{4.1}$$

Proof. From (1.9) we get

$$\mathcal{M}(\ell,\mu)(t)=f(t)-f(0)-bt,$$

where $a\in\mathbb{R}$, $\ell(\lambda)=\lambda$, $b\ge 0$ and μ is a positive measure on $(0,\infty)$.

Then

$$\mathcal{M}(\ell,\mu)(B)=f(B)-f(0)-bB,\quad \mathcal{M}(\ell,\mu)(A)=f(A)-f(0)-bA$$

and by (2.3) we derive

$$\begin{aligned}f(B)-f(0)-bB&=f(A)-f(0)-bA+\int_0^\infty \lambda^2(\lambda+A)^{-1}(B-A)(\lambda+A)^{-1}d\mu(\lambda)\\&-2\int_0^1(1-t)\left[\int_0^\infty \lambda^2\big(\lambda+(1-t)A+tB\big)^{-1}(B-A)\right.\\&\left.\times\big(\lambda+(1-t)A+tB\big)^{-1}(B-A)\big(\lambda+(1-t)A+tB\big)^{-1}d\mu(\lambda)\right]dt,\end{aligned}$$

which is equivalent to (4.1). □

The case of operator monotone functions for the Jensen's gap is as follows:

PROPOSITION 4.2. Assume that the function $f:(0,\infty)\longrightarrow\mathbb{R}$ is operator monotone in $(0,\infty)$ and has the representation (1.9). Then,

$$\begin{aligned}&f\left(\sum_{k=1}^n p_kA_k\right)-\sum_{k=1}^n p_kf(A_k)=2\int_0^\infty\lambda^2\sum_{k=1}^n p_k\left(\int_0^1(1-t)\left(\lambda+(1-t)\sum_{j=1}^n p_jA_j+tA_k\right)^{-1}\right.\\&\times\left(A_k-\sum_{j=1}^n p_jA_j\right)\left(\lambda+(1-t)\sum_{j=1}^n p_jA_j+tA_k\right)^{-1}\\&\left.\times\left(A_k-\sum_{j=1}^n p_jA_j\right)\left(\lambda+(1-t)\sum_{j=1}^n p_jA_j+tA_k\right)^{-1}dt\right)d\mu(\lambda)\ge 0\end{aligned}\tag{4.2}$$

for the n-tuple of positive operators $\mathbf{A}=(A_1,\dots,A_n)$ and probability density n-tuple $\mathbf{p}=(p_1,\dots,p_n)$.

The proof follows by Theorem 2.4 applied for

$$\mathcal{M}(\ell,\mu)(t) = f(t) - f(0) - bt,$$

where $a \in \mathbb{R}$, $\ell(\lambda) = \lambda$, $b \geq 0$ and μ is a positive measure on $(0,\infty)$.

We have the following midpoint and trapezoid inequalities for operator monotone functions:

PROPOSITION 4.3. Assume that the function $f : (0,\infty) \longrightarrow \mathbb{R}$ is operator monotone in $(0,\infty)$. If $\beta \geq A$, $B \geq \alpha > 0$ and $0 < \delta \leq (B-A)^2 \leq \Delta$, then

$$\begin{aligned} 0 &\leq -\frac{1}{24}\delta f''(\beta) \leq f\left(\frac{A+B}{2}\right) - \int_0^1 f\big((1-t)A + tB\big)\,dt \\ &\leq -\frac{1}{24}\Delta f''(\alpha) \end{aligned} \tag{4.3}$$

and

$$0 \leq -\frac{1}{12}\delta f''(\beta) \leq \int_0^1 f\big((1-t)A + tB\big)\,dt - \frac{f(A)+f(B)}{2} \leq -\frac{1}{12}\Delta f''(\alpha). \tag{4.4}$$

Proof. From (1.9) we get

$$\mathcal{M}(\ell,\mu)(t) = f(t) - f(0) - bt,$$

where $a \in \mathbb{R}$, $\ell(\lambda) = \lambda$, $b \geq 0$ and μ is a positive measure on $(0,\infty)$.

Then

$$\mathcal{M}(w,\mu)\left(\frac{A+B}{2}\right) = f\left(\frac{A+B}{2}\right) - f(0) - b\frac{A+B}{2},$$

$$\frac{\mathcal{M}(w,\mu)(A) + \mathcal{M}(w,\mu)(B)}{2} = \frac{f(A)+f(B)}{2} - f(0) - b\frac{A+B}{2},$$

$$\int_0^1 \mathcal{M}(w,\mu)\big((1-t)A+tB\big)\,dt = \int_0^1 f\big((1-t)A+tB\big)\,dt - f(0) - b\frac{A+B}{2}$$

and by Corollary 3.2 and 3.4 we derive (4.3) and (4.4). □

REMARK 4.4. If $\beta \geq A$, $B \geq \alpha > 0$ and $0 < \delta \leq (B-A)^2 \leq \Delta$, then for $r \in (0,1]$ we have the power inequalities

$$0 \leq \frac{1}{24}r(1-r)\delta\beta^{r-2} \leq \left(\frac{A+B}{2}\right)^r - \int_0^1 \big((1-t)A+tB\big)^r\,dt \leq \frac{1}{24}r(1-r)\Delta\alpha^{r-2} \tag{4.5}$$

and

$$0 \leq \frac{1}{12}r(1-r)\delta\beta^{r-2} \leq \int_0^1 \big((1-t)A+tB\big)^r\,dt - \frac{A^r+B^r}{2} \leq \frac{1}{12}r(1-r)\Delta\alpha^{r-2}. \tag{4.6}$$

We also have the logarithmic inequalities

$$0 \leq \frac{\delta}{24\beta} \leq \ln\left(\frac{A+B}{2}\right) - \int_0^1 \ln\big((1-t)A+tB\big)\,dt \leq \frac{\Delta}{24\alpha} \tag{4.7}$$

and

$$0 \leq \frac{\delta}{12\beta} \leq \int_0^1 \ln\big((1-t)A+tB\big)\,dt - \frac{\ln A + \ln B}{2} \leq \frac{\Delta}{12\alpha}, \tag{4.8}$$

if $\beta \geq A$, $B \geq \alpha > 0$ and $0 < \delta \leq (B-A)^2 \leq \Delta$.

ACKNOWLEDGEMENT

The author would like to thank the anonymous referee for valuable suggestions that have been implemented in the final version of the paper.

REFERENCES

[1] R. Bhatia. *Matrix Analysis*. Graduate Texts in Mathematics, 169. Springer-Verlag, New York, 1997. xii+347 pp. ISBN: 0-387-94846-5.

[2] R. Coleman. *Calculus on Normed Vector Spaces*. Springer, New York, 2012.

[3] J. I. Fujii and Y. Seo. On parametrized operator means dominated by power ones. *Sci. Math.*, 1:301–306, 1998.

[4] T. Furuta. Concrete examples of operator monotone functions obtained by an elementary method without appealing to Löwner integral representation. *Linear Algebra Appl.*, 429:972–980, 2008.

[5] T. Furuta. Precise lower bound of $f(A)-f(B)$ for $A>B>0$ and non-constant operator monotone function f on $[0,\infty)$. *J. Math. Inequal.*, 9(1):47–52, 2015.

[6] E. Heinz. Beiträge zur Störungsteorie der Spektralzerlegung. *Math. Ann.*, 123:415–438, 1951.

[7] K. Löwner. Über monotone MatrixFunktionen. *Math. Z.*, 38:177–216, 1934.

[8] M. S. Moslehian and H. Najafi. An extension of the Löwner-Heinz inequality. *Linear Algebra Appl.*, 437:2359–2365, 2012.

[9] H. Zuo and G. Duan. Some inequalities of operator monotone functions. *J. Math. Inequal.*, 8(4):777–781, 2014.